Homework 1

Exercise 1. Let $(S_n,\, n \in \mathbb{N})$ be the simple asymmetric random walk on $\mathbb{Z}$, defined as

$$S_0 = 0, \quad S_n = \xi_1 + \ldots + \xi_n, \quad n \geq 1,$$

where the random variables $(\xi_n,\, n \geq 1)$ are i.i.d. with $\mathbb{P}(\xi_n = +1) = p \in]0,1[$ and $\mathbb{P}(\xi_n = -1) = q = 1 - p$. Using Stirling's formula (valid for large values of n):

$$n! \sim \sqrt{2\pi n}\, \left(\frac{n}{\mathrm{e}}\right)^n,$$

show that

$$p_{0,0}^{(2n)} = \mathbb{P}(S_{2n} = 0 \,|\, S_0 = 0) \sim \frac{(4pq)^n}{\sqrt{\pi n}}.$$

NB: The notation $a_n \sim b_n$ means precisely

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Exercise 2. Let $\left(\overrightarrow{S_n},\, n \in \mathbb{N}\right)$ be the simple symmetric random walk in two dimensions, that is,

$$\overrightarrow{S_0} = (0,0), \quad \overrightarrow{S_n} = \overrightarrow{\xi_1} + \ldots + \overrightarrow{\xi_n}, \quad n \geq 1,$$

where $\left(\overrightarrow{\xi_n},\, n \geq 1\right)$ are i.i.d random variables such that

$$\mathbb{P}\left(\overrightarrow{\xi_n} = (+1,0)\right) = \mathbb{P}\left(\overrightarrow{\xi_n} = (-1,0)\right) = \mathbb{P}\left(\overrightarrow{\xi_n} = (0,+1)\right) = \mathbb{P}\left(\overrightarrow{\xi_n} = (0,-1)\right) = \frac{1}{4}.$$

Let us write $\overrightarrow{S_n} = (X_n, Y_n)$.

a) Compute the transition matrices of the random walks $(X_n,\, n \in \mathbb{N})$ and $(Y_n,\, n \in \mathbb{N})$.

b) Are these two random walks independent?

Define now $U_n = X_n + Y_n$ and $V_n = X_n - Y_n$, $n \in \mathbb{N}$. Again the same questions:

c) Again, compute the transition matrices of the random walks $(U_n,\, n \in \mathbb{N})$ and $(V_n,\, n \in \mathbb{N})$.

d) Are these two random walks independent?

e) Deduce from this the value of $\mathbb{P}\left(\overrightarrow{S_{2n}} = (0,0) \,|\, \overrightarrow{S_0} = (0,0)\right)$. How does it behave for large n?

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Exercise 3. Prove that the intercommunicating states of a Markov chain have the same period.

Hint 1: Consider two intercommunicating states, i and j. Then, find a lower bound for $p_{ii}^{(m+n+r)}$ as a function of $p_{ij}^{(m)}$, $p_{ji}^{(n)}$, and $p_{jj}^{(r)}$.

Hint 2: Show that $p_{jj}^{(r)}$ can be non-zero only if $d(i)|r$. Then, find an argument to conclude that $d(i) = d(j)$.

Note: $d|r$ is the notation for "d divides r".