Introduction to Differentiable Manifolds

EPFL – Fall 2022 F. Carocci, M. Cossarini

Exercise Series 3 - Partitions of unity, Tangent vectors 2022–10–04

Partition of Unity.

Exercise 3.1. Consider $\mathbb{R}$ with its standard smooth structure. Let $f : \mathbb{R} \to \mathbb{R}$ the sign function:

$$x \mapsto \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Let $A \subset \mathbb{R}$ a closed subset such that $f|_A$ is smooth in the sense defined in the Lecture. Find a smooth extension of $f|_A$ to all of $\mathbb{R}$ what existence is guaranteed by the Extension Lemma. Notice that $f|_{(-\infty,0)\cup(0,\infty)}$ is smooth but does not admit an extension to $\mathbb{R}$; i.e. the conclusion of the extension Lemma fails if we remove the hypothesis A closed.

Exercise 3.2. A continuous map $f : X \to Y$ is called *proper* if $f^{-1}(K)$ is compact for every compact set $K \subseteq Y$. Show that for every smooth manifold M there exists a smooth map $f : M \to [0, +\infty)$ that is proper.

Hint: Note that f must be unbounded unless M is compact. Use a function of the form $f = \sum_{i\in\mathbb{N}} c_i f_i$, where $(f_i)_{i\in\mathbb{N}}$ is a partition of unity and the c_i's are real numbers.

Exercise 3.3. Let M be a $\mathcal{C}^k$ manifold and let U be an open neighborhood of the set $M \times \{0\}$ in the space $M \times [0, +\infty)$. Show that there exists a $\mathcal{C}^k$ function $f : M \to (0, +\infty)$ whose graph is contained in U.

Tangent vectors and tangent space.

Exercise 3.4. (Derivations in $\mathbb{R}^n$)

(a) Show that $D_{v|_a} : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by $\frac{d}{dt}|_{t=0} f(a + tv)$ is a derivation, i.e. is $\mathbb{R}$-linear and satisfy the product rule.

(b) Let $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a smooth map. Prove that the linear map $DF_p : T_pU \to T_F(p)V$ is given, with respect to the standard basis $\langle \frac{\partial}{\partial x^i}|_p\rangle_{i=1,\dots n}, \langle \frac{\partial}{\partial y^j}|_{F(p)}\rangle_{j=1,\dots m}$, by the Jacobian matrix $\left(\frac{\partial F^j}{\partial x^i}\right)_{ij}$.

Exercise 3.5. Let M be a differentiable n-manifold. Show that:

(a) The differential of a smooth map $F : M \to N$ at a point $p \in M$ is a well-defined linear map $D_pF : T_pM \to T_pN$.

(b) *Chain rule*: for smooth maps $F : M \to N$, $G : N \to P$ and a point $p \in M$,

$$D_p(G \circ F) = D_{F(p)}G \circ D_pF.$$

In particular, if F is a diffeomorphism, then D_pF has inverse $(D_pF)^{-1} = D_{F(p)}(F^{-1})$.

(c) *Change of coordinates*:

Let $X \in T_pM$ be a tangent vector and let φ, $\widetilde{\varphi}$ be smooth charts of M defined at a p such that $\widetilde{\varphi} \circ \varphi^{-1} : \varphi(U \cup \widetilde{U}) \to \widetilde{\varphi}(U \cup \widetilde{U})$ is defined by $(x^1, \dots, x^n) \mapsto (z^1(x^1, \dots, x^n), \dots, z^n(x^1, \dots, x^n))$. If $X \in T_pM$ we have that in the local coordinates charts

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}|_p = \sum_{i=1}^n Z^i \frac{\partial}{\partial z^i}|_p$$

where X^i respectively Z^i are called *components* of the tangent vector in the coordinate base. Prove that

$$Z^i = \sum_{j=1}^{n} X^j \frac{\partial z^i}{\partial x^j}(\varphi(p))$$

Exercise 3.6 (Velocity vectors of curves)**.** Let M be a differentiable manifold. The *velocity vector* of a differentiable curve $\gamma : I \subseteq \mathbb{R} \to M$ at an instant $t \in I$ is the vector $\gamma'(t) := D_t\gamma(1|_t) \in T_{\gamma(t)}M$.

Show that for any tangent vector $X \in T_pM$ there exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = X$.

Exercise 3.7 (Spherical coordinates on $\mathbb{R}^3$)**.** Consider the following map defined for $(r, \varphi, \theta) \in W := \mathbb{R}^+ \times (0, 2\pi) \times (0, \pi)$:

$$\Psi(r, \varphi, \theta) = (r \cos\varphi \sin\theta, r \sin\varphi \sin\theta, r \cos\theta) \in \mathbb{R}^3.$$

Check that Ψ is a diffeomorphism[1] onto its image $\Psi(W) =: U$. We can therefore consider Ψ^{-1} as a smooth chart on $\mathbb{R}^3$ and it is common to call the component functions of Ψ^{-1} the **spherical coordinates** (r, φ, θ).

Express the coordinate vectors of this chart

$$\frac{\partial}{\partial r}\Big|_p, \frac{\partial}{\partial \varphi}\Big|_p, \frac{\partial}{\partial \theta}\Big|_p$$

at some point $p \in U$ in terms of the standard coordinate vectors $\frac{\partial}{\partial x}\big|_p, \frac{\partial}{\partial y}\big|_p, \frac{\partial}{\partial z}\big|_p$.

Exercise 3.8. (**To hand in**) Consider the inclusion $\iota : S^2 \to \mathbb{R}^3$, where we endow both spaces with the standard smooth structure. Let $p \in S^2$. What is the image of $D_p\iota : T_pS^2 \to T_p\mathbb{R}^3$? (Identify $T_p\mathbb{R}^3$ with $\mathbb{R}^3$ in the standard way, i.e. $e_i \mapsto \frac{\partial}{\partial x^i}|_p$) So the result should be the equation for a plane in $\mathbb{R}^3$.)

Hint: Use Exercise 7 on spherical coordinates.

[1] Here "diffeomorphism" is meant in the standard sense of maps between open subsets of $\mathbb{R}^3$.