Introduction to Differentiable Manifolds	
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Exercise Series 1 - Topological and smooth manifolds	**2022–09–20**

Convention: We understand a subset/product/quotient of topological space(s) to be automatically endowed with the subspace/product/quotient topology unless we state otherwise.

Exercise 1.1. Which of the following spaces are locally Euclidean? Which are (globally) homeomorphic to some Euclidean space?

(a) an open ball in $\mathbb{R}^n$

$B_R(0) = \{x \in \mathbb{R}^n : \|x\| < R\}$ is globally homeomorphic to $\mathbb{R}^n$. And the homeomorphism $\varphi(x) = R\frac{x}{1+\|x\|}$ maps $\mathbb{R}^n$ into $B_R(0)$. Observe that $\varphi^{-1}(x) = \frac{x}{R-\|x\|}$.

(b) the closed interval $[0, 1] \subset \mathbb{R}$

The interval $[0, 1]$ is neither locally nor globally homeomorphic to $\mathbb{R}$. Global homeomorphism is excluded since $[0, 1]$ is compact but $\mathbb{R}$ is not. A continuous map will map a compact set to a compact set. Next, suppose, for a contradiction, that $[0, 1]$ is locally homeomorphic to $\mathbb{R}$ and denote by φ the homeomorphism. Take one of the extrema (e.g. 0 or 1) of the interval and consider an open neighborhood in the subspace topology: $U = [0, \varepsilon)$ for example. U is connected and open hence $\varphi(U)$ is connected and open as well. Furthermore $(0, \varepsilon)$ is still open and connected but its image through φ is not connected because we remove $\varphi(0)$.

(c) the circle $\mathbb{S}^1 \subset \mathbb{R}^2$

S^1 is locally homeomorphic to $\mathbb{R}$. In fact denote $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and define the nord and south stereographic projections as

$$\begin{aligned} p_\pm : S^1 \setminus \{(0, \pm 1)\} &\to \mathbb{R} \\ (x, y) &\mapsto \frac{x}{1 \mp y} \end{aligned}$$

It is not difficult to verify that for every point $p \in S^1$ there exists an open set U containing p, such that the image of U via one of the two stereographic projections is an open set in $\mathbb{R}$.

(d) the zero set of the function $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = xy$

The set $E = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ is not locally Euclidean because no neighborhood U of the origin in E is homeomorphic to $\mathbb{R}$. To prove this last statement argue by contradiction: suppose that there exist an homeomorphism $\varphi : U \to \mathbb{R}$. Then $U' = U \setminus \{(0, 0)\}$ has at least 4 connected components while $\varphi(U')$ has just 2 connected components. The contradiction arises from the fact that a homeomorphism preserves connected components.

(e) the "bent line" $\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, xy = 0\}$.

The set $E\{(x, y) \in \mathbb{R}^2_+ : xy = 0\}$ is globally homeomorphic to $\mathbb{R}$ via the homeomorphism

$$\begin{aligned} \varphi : E &\to \mathbb{R} \\ (x, y) &\mapsto \begin{cases} x & \text{if } y = 0 \\ -y & \text{if } x = 0 \end{cases} \end{aligned}$$

Exercise 1.2. If a space M is locally Euclidean of dimension n at some point p, show that p has an open neighborhood that is homeomorphic to the whole space $\mathbb{R}^n$, or to a open ball $B_r(x)$.

Deduce the equivalent definitions of topological n-manifold.

We know that the there is an open neighborhood U of p and a homeomorpshism φ from U to an open subset $\varphi(U)$ of $\mathbb{R}^n$. Then we can find a ball $B(\varphi(p), r) \subseteq \varphi(U) \subseteq \mathbb{R}^n$ for some $r > 0$. Let us consider the map $\psi : B(\varphi(p), r) \to \mathbb{R}^n$ given by $\psi(x) := \frac{x-\varphi(p)}{r-\|x-\varphi(p)\|}$. One can verify that ψ is a homeomorphism with inverse $\psi^{-1}(y) := \varphi(p) + \frac{y}{1+\|y\|}$. Set $U' := \varphi^{-1}(B(\varphi(p), r)) \subseteq M$, which is a neighborhood of p in M and the map $\theta := \psi \circ \varphi : U' \to \mathbb{R}^n$. We showed that θ is a homeomorphism since ψ and φ are both homeomorphisms.

Exercise 1.3. The **line with two origins** is the space M obtained as quotient of the space $X = \{\pm 1\} \times \mathbb{R}$ by the equivalence relation $(i, x) \sim (j, y)$ iff $x = y \neq 0$.

(a) Show that M is locally Euclidean and second countable, but not Hausdorff.

Denote $\pi : X \to M$ the quotient map $(i, x) \mapsto [(i, x)]$.

The two "origins" are the equivalence classes of the points $(i, 0) \in X$ (for $i = \pm 1$); these classes have just one element each and we denote them $0_i = [(i, 0)] = \{(i, 0)\} \in M$. In contrast, the equivalence class of any other point $(i, x) \in X$ with $x \neq 0$ is the two-point set $\widetilde{x} = [(i, x)] = \{(1, x), (-1, x)\} \in M$. Therefore M is the set of equivalence classes

$$M = X/_{\sim} = \{0_+\} \cup \{0_-\} \cup \{\widetilde{x}\}_{x \neq 0}.$$

The space M is locally Euclidean of dimension 1 because it is the union of two open sets $\mathbb{R}_i = \{[(i, x)] \in M : x \in \mathbb{R}\}$ (for $i = \pm 1$), each of which is homeomorphic to $\mathbb{R}$ via the map

$$\mathbb{R} \to \mathbb{R}_i : x \mapsto [(i, x)].$$

To see that the sets $\mathbb{R}_i$ are open in the quotient topology, note that $\pi^{-1}(\mathbb{R}_i) = X \setminus 0_{-i}$, which is open in X.

Moreover, M is second countable because it is the union of two second countable open subsets, namely, the sets $\mathbb{R}_i$.

Finally M is not Hausdorff since every pair of open subsets containing 0_- and 0_+ respectively have non-empty intersection.

(b) Find a sequence of points in M that converges to two different points, and show that this cannot happen in a Hausdorff space. Recall that given a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X, x is a limit point if for each neighbourhood U_x of x there exist a N such that $x_n \in U_x$ for each $n \geq N$.

Take as a sequence in M $\pi((-1)^n, \frac{1}{n})$; 0_+ and 0_- are both limit point.

If Y is a Housdorff space if the limit exist is unique: suppose otherwise that x, y are two limits of a given sequence $\{x_n\}_{n \in \mathbb{N}}$. Then for every neighbourhood U_x of x and U_y of y we have N, M such that $x_n \in U_x$ for each $n \geq N$ and $x_n \in U_y$ for each $n \geq M$, in particular $x_n \in U_x \cup U_y$ for each $n \geq \max(N, M)$. This implies that any any neighbourhood of x and y intersect, contradicting the Hausdorff hypothesis.

Exercise 1.4. Let N be an open subset of a topological n-manifold M.

(a) Show that N is a topological n-manifold.

First, we note that a subset of a Hausdorff (resp. second countable) space is a Hausdorff (resp. secound countable) space. Indeed, let S be a subset of a topological space X, endowed with the subpspace topology. If X is X is Hausdorff, to show that S is Hausdorff as well, take two distinct points $p, q \in S$. Let U, V be disjoint neighborhoods of p, q in X. Then the sets $U' = U \cap S$, $V' = V \cap S$ are disjoint open neighborhoods of p, q in S. If X is second countable, let $\{U_i\}_{i \in I}$ be a countable basis for the topology of X. Then $\{U_i \cap S\}_{i \in I}$ is a countable basis for S. Thus S is second-countable.

Now let M be a topological n-manifold, and let $U \subset M$ be an open set, endowed with the subspace topology.

By the results mentioned above, U is Hausdorff and secound countable.

Let us show that N is locally Euclidean. For each point $\forall p \in N$, there exists an open neighborhood $U \subset M$ that is homeomorphic to an open set $V \subseteq \mathbb{R}^n$. Let $\varphi : U \to V$ be a homeomorphism. Since N is open in M, the set $U \cap N$ is open in U. Therefore $\varphi(U \cap N)$ is open in V (and thus, in $\mathbb{R}^n$), and the restricted map $\varphi : U \cap N \to \varphi(U \cap N)$ is a homeomorphism. This shows that N is locally Euclidean.

(b) Show that any smooth structure $\mathcal{A}$ on M determines a smooth structure $\mathcal{B}$ on N, consisting of the charts $(U, \varphi) \in \mathcal{A}$ such that $U \subseteq N$.

We will use the following fact: For any chart $(U, \varphi) \in \mathcal{A}$, the map $\varphi|_V$ is also a chart of $\mathcal{A}$ for any open set $V \subseteq U$.

Proof. Note that $\varphi|_V : V \to \varphi(V)$ is a homeomorphism from an open subset of M to an open subset of $\varphi(U) \subseteq \mathbb{R}^n$, therefore $\varphi|_V$ is a topological chart of M.

To show that $\varphi|_V \in \mathcal{A}$, since $\mathcal{A}$ is a maximal smooth atlas, it suffices to show that the chart $\varphi|_V$ is smooth compatible with all charts $\psi \in \mathcal{A}$. And indeed, it is compatible because the transition maps

$$\varphi|_V \circ \psi^{-1} = (\varphi \circ \psi^{-1})|_{\psi(V)}$$

$$\psi \circ (\varphi|_V)^{-1} = (\psi \circ \varphi^{-1})|_{\varphi(V)}$$

are smooth since they are restrictions of transition maps of $\mathcal{A}$. □

Now let's go back to N. N is a topological n-manifold because it is an open subset of M. Each element of $\mathcal{B}$ is a topological chart of N, because it is an homeomorphism $\varphi : U \to V$, where $U \subseteq N$ is an open subset of M (hence of N) and V is an open subset of $\mathbb{R}^n$. These charts are smooth compatible because they are taken from the smooth structure of M. The domains of the charts of $\mathcal{B}$ cover N, because for each point $p \in N$ there is a chart $\varphi \in \mathcal{A}$ of M with domain $U \ni p$, and then the restriction $\varphi|_{U \cap N} : U \cap N \to \varphi(U \cap N)$ is a chart in $\mathcal{B}$ that is defined at p. All this proves that $\mathcal{B}$ is a smooth atlas for N.

(Here we used the fact that $\varphi \in \mathcal{A}$, then the chart $\varphi|_V \in \mathcal{A}$ for any open set $V \subseteq \operatorname{Dom} \varphi$. To see this, since $\mathcal{A}$ is maximal, it suffices to show that the chart $\varphi|_V$ is smooth compatible with all charts $\psi \in \mathcal{A}$. And indeed, it is compatible because the transition maps

$$\varphi|_V \circ \psi^{-1} = (\varphi \circ \psi^{-1})|_{\psi(V)}$$

$$\psi \circ (\varphi|_V)^{-1} = (\psi \circ \varphi^{-1})|_{\varphi(V)}$$

are smooth since they are restrictions of transition maps of $\mathcal{A}$.)

Finally, let us prove that $\mathcal{B}$ is maximal as a smooth atlas. Let ψ be a chart of N that is smooth-compatible with all charts of $\mathcal{B}$. We have to show that ψ belongs to $\mathcal{B}$. It suffices to show that ψ is compatible with all charts of $\mathcal{A}$, because this implies by maximality of $\mathcal{A}$ that $\psi \in \mathcal{A}$, and it follows that $\psi \in \mathcal{B}$ since the domain of ψ is contained in N. Thus we just have to show that ψ is compatible with every chart $\varphi \in \mathcal{A}$, i.e. that the transition maps $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are smooth. To see that these maps are smooth we rewrite them as

$$\varphi \circ \psi^{-1} = \widetilde{\varphi} \circ \psi^{-1}$$

$$\psi \circ \varphi^{-1} = \psi \circ \widetilde{\varphi}^{-1}$$

where $\widetilde{\varphi} = \varphi|_{N \cap \operatorname{Dom} \varphi}$ is the restriction of φ. The chart $\widetilde{\varphi}$ is a chart of $\mathcal{B}$, since it is a chart of $\mathcal{A}$ and its domain is contained in N. Therefore ψ is smooth compatible with $\widetilde{\varphi}$, hence the transition maps are smooth.

Exercise 1.5. Show that the product of two topological manifolds is a topological manifold. What is its dimension?

Let M, N be topological manifolds of respective dimensions m, n. Let us show that the product $M \times N$ is a topological manifold of dimension $m \times n$. The space $M \times N$ is Hausdorff and second countable, since a product of Hausdorff (resp. second countable) spaces is a Hausdorff (resp. second countable) space.

Let us show that $M \times N$ is locally Euclidean of dimension $m + n$. For every $(p, q) \in M \times N$, we can find open neighborhoods $U \subset M$, $V \subseteq N$ of p and q that are respectively homeomorphic to $\mathbb{R}^m$ and $\mathbb{R}^n$. It follows that the set $U \times V$ (which is an open neighborhood of (p, q) in the product topology) is homeomorphic to $\mathbb{R}^{m+n}$.

Exercise 1.6. We have seen in the lecture that $\mathbb{S}^n$ is a topological n-manifold. Show that the charts $(U_i^{+,-}, \varphi_i^{+,-})_{i=1,\ldots,n}$ form a smooth atlas for $\mathbb{S}^n$.

We only need to verify that the transition functions are smooth. For i, j different indices, we can assume $i < j$ we can explicitly compute the transition functions to be:

$$\varphi_i^{+,-} \circ \varphi_j^{+,-}(u^1, \ldots, u^n) = (u^1, \ldots, \hat{u^i}, \ldots, +, -(1 - |u|^2)^{\frac{1}{2}}, \ldots, u^n).$$

A similar formula holds for $i > j$ and for $i = j$ one can compute that $\varphi_i^+ \circ (varphi_i^-)^{-1} = \varphi_i^- \circ (varphi_i^+)^{-1} = Id_{\mathbb{B}^n}$.

Exercise 1.7 (To hand in). Show that the **projective space** $\mathbb{P}^n$, defined as the quotient of $\mathbb{R}^{n+1} \backslash \{0\}$ by the equivalence relation $x \sim y$ iff $x = \lambda\, y$ for some $\lambda \in \mathbb{R} \backslash \{0\}$, is a smooth n-manifold with atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i=0,\ldots,n}$ given by

$$U_i := \{[x] \in \mathbb{P}^n \mid x_i \neq 0\}, \qquad \varphi_i([x]) = \left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right),$$

where $[x] \in \mathbb{P}^n$ denotes the equivalence class of a point $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \backslash \{0\}$.

Exercise 1.8. Show that the **n-torus** $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, defined as the quotient of $\mathbb{R}^n$ by the equivalence relation $x \sim y$ iff $y - x \in \mathbb{Z}^n$, is a topological n-manifold.

Let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the quotient map

$$x \mapsto [x] = \{x + z : z \in \mathbb{Z}^n\}.$$

Note that two points x, y of $\mathbb{R}^n$ are in the same equivalence class if and only if the coordinates x_i, y_i coincide modulo 1 for each i. (In other words, the real numbers x_i, y_i have the same integer part.)

(a) π *is an open map.* Indeed, let $U \subseteq \mathbb{R}^n$ be an open set. To see that $\pi(U)$ is open in the quotient topology, we verify that its preimage $\pi^{-1}(\pi(U)) = \bigcup_{z \in \mathbb{Z}^n} U + \{z\}$ is open, being a union of translate copies of U.

(b) $\mathbb{T}^n$ *is second countable.* Indeed, the image of any (countable) topological basis by a surjective open map is a (countable) topological basis.

(c) To prove that $\mathbb{T}^n$ is locally Euclidean, we show that:
The quotient map π is locally injective, i.e., each point $x \in X$ has an open neighborhood U where the quotient map π is injective. Indeed, let U be an open neighborhood of x with diameter < 1. Then there are no two different points $x', x'' \in U$ such that $x'' - x' \in \mathbb{Z}^n$. Therefore π is injective on U. Furthermore, the set $\pi(U)$ is open in $\mathbb{T}^n$, and the restricted quotient map $\pi : U \to \pi(U)$ is a homeomorphism because it is bijective and open. This proves that the $\mathbb{T}^n$ is locally Euclidean of dimension n.

(d) *$\mathbb{T}^n$ is Hausdorff.* Take two different points $\pi(x), \pi(y) \in \mathbb{T}^n$. Then there is some i such that the coordinates x_i, y_i are different modulo 1. Let $\varepsilon > 0$ be the distance between the numbers x_i, y_i taken modulo 1, that is, $\varepsilon = \min_{z \in \mathbb{Z}} y_i - (x_i + z)$. This number is the least we would have to move y_i so that it coincides with x_i modulo 1. Then the Euclidean open balls $U = B(x, \frac{\varepsilon}{2})$, $V = B(y, \frac{\varepsilon}{2})$ satisfy $\pi(U) \cap \pi(V) = \emptyset$ because for every pair of points in U and V, their i-th coordinates do not coincide modulo 1. The sets $\pi(U)$, $\pi(V)$ are disjoint open neighborhoods of $\pi(x)$, $\pi(y)$, as needed to show that $\mathbb{T}^n$ is Hausdorff.

Exercise 1.9. Show that $(\mathbb{R}, \mathrm{id}_\mathbb{R})$ and $(\mathbb{R}, \psi : x \mapsto x^3)$ define to different smooth structures on the real line.

The charts ϕ and $\mathrm{id}_\mathbb{R}$ are not smoothly compatible since the transition map $\mathrm{id}_\mathbb{R} \circ \phi^{-1} : y \mapsto y^{1/3}$ is not smooth. Therefore, the atlases $\mathcal{A}$ and $\mathcal{B}$ define different smooth structures.