Problem Set 1 —*Due Friday, September 30, before class starts*

For the Exercise Sessions on September 23

Last name	First name	SCIPER Nr	Points

Problem 1: Review of Random Variables

Let X and Y be discrete random variables defined on some probability space with a joint pmf $p_{XY}(x, y)$. Let $a, b \in \mathbb{R}$ be fixed.

(a) Prove that $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$. Do not assume independence.

(b) Prove that if X and Y are independent random variables, then $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

(c) Assume that X and Y are not independent. Find an example where $\mathbb{E}[X \cdot Y] \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$, and another example where $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

(d) Prove that if X and Y are independent, then they are also uncorrelated, i.e.,

$$Cov(X, Y) := \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = 0. \tag{1}$$

(e) Find an example where X and Y are uncorrelated but dependent.

(f) Assume that X and Y are uncorrelated and let σ_X^2 and σ_Y^2 be the variances of X and Y, respectively. Find the variance of $aX + bY$ and express it in terms of $\sigma_X^2, \sigma_Y^2, a, b$.
Hint: First show that $Cov(X, Y) = \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

Solution 1. *(a)*

$$\begin{aligned}
\mathbb{E}[aX + bY] &= \sum_x \sum_y (ax + by) p_{XY}(x, y) \\
&= \sum_x ax \sum_y p_{XY}(x, y) + \sum_y by \sum_x p_{XY}(x, y) \\
&= a \sum_x x\, p_X(x) + b \sum_y y\, p_Y(y) \\
&= a\mathbb{E}[X] + b\mathbb{E}[Y].
\end{aligned}$$

(b) If X and Y are independent, we have $p_{XY}(x, y) = p_X(x) p_Y(y)$, then

$$\begin{aligned}
\mathbb{E}[X \cdot Y] &= \sum_X \sum_Y xy p_{XY}(x, y) \\
&= \sum_X \sum_Y x p_X(x) y p_Y(y) \\
&= \sum_X x p_X(x) \sum_Y y p_Y(y) \\
&= \mathbb{E}[X] \cdot \mathbb{E}[Y]
\end{aligned}$$

(c) For the first example, suppose $Pr(X = 0, Y = 1) = Pr(X = 1, Y = 0) = \frac{1}{2}$, and $Pr(X = 0, Y = 0) = Pr(X = 1, Y = 1) = 0$. X, Y are dependent, and we have $\mathbb{E}[X \cdot Y] = 0$ while $\mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{4}$

For the second example, suppose $Pr(X = -1, Y = 0) = Pr(X = 0, Y = 1) = Pr(X = 1, Y = 0) = \frac{1}{3}$. X, Y are dependent. Obviously we have $\mathbb{E}[X \cdot Y] = 0$, and furthermore $\mathbb{E}[X] = 0$, hence $\mathbb{E}[X]\mathbb{E}[Y] = 0$.

(d) If X and Y are independent, we have $p_{XY}(x, y) = p_X(x)p_Y(y)$, then

$$\begin{aligned}
\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] &= \sum_x \sum_y (x - \mathbb{E}[X])(y - \mathbb{E}[Y])\, p_{XY}(x, y) \\
&= \sum_x \sum_y (x - \mathbb{E}[X])(y - \mathbb{E}[Y])\, p_X(x)p_Y(y) \\
&= \sum_x (x - \mathbb{E}[X])\, p_X(x) \sum_y (y - \mathbb{E}[Y])\, p_Y(y) \\
&= (\mathbb{E}[X] - \mathbb{E}[X])(\mathbb{E}[Y] - \mathbb{E}[Y]) = 0.
\end{aligned}$$

Thus, X and Y are uncorrelated.

(e) One example where X and Y are uncorrelated but dependent is

$$\mathbb{P}_{XY}(x, y) = \begin{cases} \frac{1}{3} & \text{if } (x, y) \in \{(-1, 0), (1, 0), (0, 1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

First, it can be easily checked that $\mathbb{E}[X \cdot Y] = 0 = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ (note that $\mathbb{E}[X] = 0$). Second, X and Y are dependent since $\mathbb{P}_{XY}(1, 0) = \frac{1}{3}$ but $\mathbb{P}_X(1)\mathbb{P}_Y(0) = \frac{1}{3} \times \frac{2}{3}$.

(f) First, we have

$$\begin{aligned}
Cov(X, Y) &= \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] \\
&= \mathbb{E}\left[XY - X\mathbb{E}[Y] - \mathbb{E}[X]Y + \mathbb{E}[X]\mathbb{E}[Y]\right] \\
&= \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y].
\end{aligned}$$

Thus, $Cov(X, Y) = 0$ if and only if $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

Then,

$$\begin{aligned}
\sigma^2_{aX+bY} &= \mathbb{E}[aX + bY - \mathbb{E}[aX + bY]]^2 \\
&= \mathbb{E}[(aX + bY)^2] - (\mathbb{E}[aX + bY])^2 \\
&= a^2\mathbb{E}[X^2] + 2ab\mathbb{E}[X \cdot Y] + b^2\mathbb{E}[Y^2] - a^2\mathbb{E}[X]^2 - 2ab\mathbb{E}[X]\mathbb{E}[Y] - b^2\mathbb{E}[Y]^2 \\
&= a^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2) + b^2(\mathbb{E}[Y^2] - \mathbb{E}[Y]^2) \\
&= a^2\sigma^2_X + b^2\sigma^2_Y.
\end{aligned}$$

We remark that since the independence of X and Y implies $Cov(X, Y) = 0$, we also have $\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y$ if X and Y are independent.

Problem 2: Review of Gaussian Random Variables

A random variable X with probability density function

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \tag{2}$$

is called a *Gaussian* random variable.

(a) Explicitly calculate the mean $\mathbb{E}[X]$, the second moment $\mathbb{E}[X^2]$, and the variance $Var[X]$ of the random variable X.

(b) Let us now consider events of the following kind:

$$\Pr(X < \alpha). \tag{3}$$

Unfortunately for Gaussian random variables this cannot be calculated in closed form. Instead, we will rewrite it in terms of the standard Q-function:

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \tag{4}$$

Express $\Pr(X < \alpha)$ in terms of the Q-function and the parameters m and σ^2 of the Gaussian pdf.

Like we said, the Q-function cannot be calculated in closed form. Therefore, it is important to have *bounds* on the Q-function. In the next 3 subproblems, you derive the most important of these bounds, learning some very general and powerful tools along the way:

(c) Derive the Markov inequality, which says that for any non-negative random variable X and positive a, we have

$$\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}. \tag{5}$$

(d) Use the Markov inequality to derive the Chernoff bound: the probability that a real random variable Z exceeds b is given by

$$\Pr(Z \geq b) \leq \mathbb{E}\big[e^{s(Z-b)}\big], \qquad s \geq 0. \tag{6}$$

(e) Use the Chernoff bound to show that

$$Q(x) \leq e^{-\frac{x^2}{2}} \quad \text{for } x \geq 0. \tag{7}$$

Solution 2. *(a)* First,

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x p_X(x)\, dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx \\
&\overset{(*)}{=} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} u e^{-\frac{u^2}{2\sigma^2}}\, du + m \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}}\, du \\
&\overset{(\dagger)}{=} 0 + m \\
&= m,
\end{aligned} \tag{8}$$

where $(*)$ follows by a change of variable $u = x - m$ and $(\dagger)$ follows since the first integrand in (**??**) is an odd function and the second integrand in (**??**) is a probability density function. We remark that the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx$$

known as *Gaussian integral*, can be evaluated explicitly to be $\sqrt{\pi}$. Second,

$$
\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 p_X(x)\,dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \\
&\overset{(*)}{=} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} u^2 e^{-\frac{u^2}{2\sigma^2}}\,du + \frac{2m}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} u e^{-\frac{u^2}{2\sigma^2}}\,du + m^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}}\,du \qquad (9) \\
&\overset{(\dagger)}{=} \sigma^2 + 0 + m^2 \\
&= \sigma^2 + m^2,
\end{aligned}
$$

where $(*)$ follows by a change of variable $u = x - m$ and $(\dagger)$ follows from the same arguments in the evaluation of $\mathbb{E}[X]$ and an integration by parts to the first integral in (**??**):

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} u^2 e^{-\frac{u^2}{2\sigma^2}}\,du &= -\frac{\sigma^2}{\sqrt{2\pi\sigma^2}} \left(u e^{-\frac{u^2}{2\sigma^2}} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}}\,du \right) \\
&= 0 + \sigma^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Var[X] &= \mathbb{E}[X - \mathbb{E}[X]]^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \sigma^2 + m^2 - m^2 \\
&= \sigma^2.
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbb{P}(X < \alpha) &= \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \\
&\overset{(*)}{=} \int_{-\infty}^{\frac{\alpha-m}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du \\
&= 1 - Q\left(\frac{\alpha - m}{\sigma}\right),
\end{aligned}
$$

where $(*)$ follows by a change of variable $u = \frac{x-m}{\sigma}$.

(c)

$$
\begin{aligned}
\mathbb{E}[X] &= \int_0^a x p_X(x)\,dx + \int_a^{\infty} x p_X(x)\,dx \\
&\geq 0 + a \int_a^{\infty} p_X(x)\,dx \\
&= a\mathbb{P}(X \geq a).
\end{aligned}
$$

(d) Fix $s \geq 0$, then we have

$$
\begin{aligned}
\mathbb{P}(Z \geq b) &\leq \mathbb{P}(s(Z-b) \geq 0) \\
&= \mathbb{P}(e^{s(Z-b)} \geq e^0) \\
&\overset{(*)}{\leq} \mathbb{E}\big[e^{s(Z-b)}\big],
\end{aligned}
$$

where $(*)$ follows from the Markov inequality.

(e) Let X be a Gaussian random variable with mean zero and unit variance, then we have

$$\begin{aligned} Q(x) &= \mathbb{P}(X \geq x) \\ &\stackrel{(*)}{\leq} \mathbb{E}\left[e^{s(X-x)}\right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{s(u-x)} e^{-\frac{u^2}{2}}\, du \\ &= e^{-sx+\frac{s^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(u-s)^2}{2}}\, du \\ &= e^{-sx+\frac{s^2}{2}}, \end{aligned}$$

where $(*)$ follows from the Chernoff bound. In order to get the tightest bound, we need to minimize $-sx + s^2/2$ which gives $s = x$ and then the desired bound is established.

Problem 3: Moment Generating Function

In the class we had considered the logarithmic moment generating function

$$\phi(s) := \ln \mathbb{E}[\exp(sX)] = \ln \sum_x p(x) \exp(sx)$$

of a real-valued random variable X taking values on a finite set, and showed that $\phi'(s) = \mathbb{E}[X_s]$ where X_s is a random variable taking the same values as X but with probabilities $p_s(x) := p(x)\exp(sx)\exp(-\phi(s))$.

(a) Show that

$$\phi''(s) = \mathrm{Var}(X_s) := \mathbb{E}[X_s^2] - \mathbb{E}[X_s]^2$$

and conclude that $\phi''(s) \geq 0$ and the inequality is strict except when X is deterministic.

(b) Let $x_{\min} := \min\{x : p(x) > 0\}$ and $x_{\max} := \max\{x : p(x) > 0\}$ be the smallest and largest values X takes. Show that

$$\lim_{s\to-\infty} \phi'(s) = x_{\min}, \quad \text{and} \quad \lim_{s\to\infty} \phi'(s) = x_{\max}.$$

Solution 3. *(a)* As $\phi(s) := \ln \mathbb{E}[\exp(sX)]$, we have

$$\phi'(s) = \frac{\mathbb{E}[X\exp(sX)]}{\mathbb{E}[\exp(sX)]} = \mathbb{E}[X\exp(sX)\exp(-\phi(s))] = \mathbb{E}[X_s] \tag{10}$$

$$\phi''(s) = \frac{\mathbb{E}[X^2\exp(sX)]}{\mathbb{E}[\exp(sX)]} - \frac{\mathbb{E}[X\exp(sX)]\mathbb{E}[X\exp(sX)]}{\mathbb{E}[\exp(sX)]^2} \tag{11}$$

The second term is $\mathbb{E}[X_s]^2$ and the first term equals $\sum_x x^2\exp(sx)/\exp(\phi(s)) = \mathbb{E}[X_s^2]$. So $\phi''(s) = \mathrm{Var}(X_s)$. Moreover, $\mathrm{Var}(X_s) \geq 0$ with equality only when X_s is deterministic. But X_s is deterministic only when X is.

(b) Observe that

$$\begin{aligned} \phi'(s) &= \frac{\mathbb{E}[X\exp(sX)]}{\mathbb{E}[\exp(sX)]} = \frac{\mathbb{E}[X\exp(sX)]\exp(-sx_{max})}{\mathbb{E}[\exp(sX)]\exp(-sx_{max})} \qquad (12) \\ &= \frac{\sum_x p(x)x\exp(-s(x_{max}-x))}{\sum_x p(x)\exp(-s(x_{max}-x))} \qquad (13) \end{aligned}$$

In the sums above, as $s \to \infty$, all terms vanish except the ones for $x = x_{max}$. Hence we have

$$\lim_{s\to\infty} \phi'(s) = \frac{p(x_{max})x_{max}}{p(x_{max})} = x_{max} \tag{14}$$

Similarly, we can show that $\lim_{s\to-\infty} \phi'(s) = x_{min}$.

Problem 4: Hoeffding's Lemma

Prove Lemma 2.3 in the lecture notes. In other words, prove that if X is a zero-mean random variable taking values in $[a,b]$ then

$$\mathbb{E}[e^{\lambda X}] \leq e^{\frac{\lambda^2}{2}[(a-b)^2/4]}.$$

Expressed differently, X is $[(a-b)^2/4]$-subgaussian.

Hint: You can use the following steps to prove the lemma:

1. Let $\lambda > 0$. Let X be a random variable such that $a \leq X \leq b$ and $\mathbb{E}[X] = 0$. By considering the convex function $x \to e^{\lambda x}$, show that
$$\mathbb{E}[e^{\lambda X}] \leq \frac{b}{b-a}e^{\lambda a} - \frac{a}{b-a}e^{\lambda b}. \tag{15}$$

2. Let $p = -a/(b-a)$ and $h = \lambda(b-a)$. Verify that the right-hand side of (8) equals $e^{L(h)}$ where
$$L(h) = -hp + \log(1 - p + pe^h).$$

3. By Taylor's theorem, there exists $\xi \in (0,h)$ such that
$$L(h) = L(0) + hL'(0) + \frac{h^2}{2}L''(\xi).$$
Show that $L(h) \leq h^2/8$ and hence $\mathbb{E}[e^{\lambda X}] \leq e^{\lambda^2(b-a)^2/8}$.

Solution 4. Since $e^{\lambda x}$ is convex in x we have for all $a \leq x \leq b$,

$$e^{\lambda x} \leq \frac{b-x}{b-a}e^{\lambda a} + \frac{x-a}{b-a}e^{\lambda b}.$$

If we take the expected value of this wrt X and recall that $\mathbb{E}[X] = 0$ then it follows that

$$\mathbb{E}[e^{\lambda X}] \leq \frac{b}{b-a}e^{\lambda a} - \frac{a}{b-a}e^{\lambda b}.$$

Consider the right-hand side. Note that we must have $a < 0$ and $b > 0$ since $\mathbb{E}[X] = 0$. Set $p = -a/(b-a)$, $0 \leq p \leq 1$, and $\lambda' = \lambda(b-a)$. The right-hand side can then be written as

$$(1-p)e^{-\lambda' p} + pe^{\lambda'(1-p)} \leq e^{\frac{\lambda'^2}{8}} = e^{\frac{\lambda^2}{2}[(b-a)^2/4]},$$

where in the first step we have used the inequality we have seen in class for the Bernoulli random variable with parameter p.

An alternative way to solve this problem could be define $\phi(\lambda) = \ln \mathbb{E}[e^{\lambda X}]$.

$$\phi'(\lambda) = \frac{d}{d\lambda}\ln \mathbb{E}[e^{\lambda X}] = \frac{\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]}$$

So $\phi(0) = \frac{0}{1} = 0$.

$$\phi''(\lambda) = \frac{d}{d\lambda}\phi'(\lambda) = \frac{d}{d\lambda}\frac{\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} = \frac{\mathbb{E}[X^2e^{\lambda X}]\mathbb{E}[e^{\lambda X}] - \mathbb{E}[Xe^{\lambda X}]\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]^2}$$

For $\lambda = 0$, we have

$$\phi''(0) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \text{Var}(X)$$

Also, we have $\phi(\lambda) \leq \phi(0) + \phi'(0)\lambda + \phi''(0)\frac{\lambda^2}{2} = \frac{\lambda^2}{2}\text{Var}(X)$ As X is random variable taking values in $[a, b]$. The largest variance is achieved when $\Pr\{X = a\} = \frac{b}{b-a}$ $\Pr\{X = b\} = \frac{-a}{b-a}$.

$$\text{Var}(X) \leq \frac{(b-a)^2}{4} \tag{16}$$

Therefore we have

$$\mathbb{E}[e^{\lambda X}] \leq e^{\frac{\lambda^2}{2}\frac{(b-a)^2}{4}}$$

X is $[(b-a)^2/4]$-subgaussian.

Problem 5: Expected Maximum of Subgaussians

Let $\{X_i\}_{i=1}^n$ be a collection of n σ^2-subgaussian random variables, not necessarily independent of each other. Let $Y = \max_{i\in\{1,2,\cdots,n\}} X_i$. Prove that $\mathbb{E}[Y] \leq \sqrt{2\sigma^2 \log n}$. *Hint:* Recall that by Jensen, $e^{\lambda\mathbb{E}[X]} \leq \mathbb{E}[e^{\lambda X}]$.

Solution 5. Consider the MGF of Y, we have the following relations for all $\lambda \geq 0$

$$\mathbb{E}[e^{\lambda Y}] = \mathbb{E}[\exp(\lambda \max_{i\in\{1,2,\ldots,n\}} X_i)] \leq \mathbb{E}[\sum_{i\in\{1,2,\ldots,n\}} e^{\lambda X_i}].$$

Note that by the linearity of expectation (this does not require independence) and the assumptions that $\{X_i\}_{i=1}^n$ are σ^2-subgaussian random variables, we have

$$\mathbb{E}[e^{\lambda Y}] \leq ne^{\lambda^2\sigma/2}.$$

Using the hints, we have

$$e^{\lambda E[Y]} \leq e^{\lambda^2\sigma/2+\log n},$$

which implies that

$$E[Y] \leq \lambda\frac{\sigma^2}{2} + \frac{1}{\lambda}\log n.$$

Optimizing over λ, we have the optimal $\lambda^* = \sqrt{\frac{2\log n}{\sigma^2}}$, which gives us the desired inequality.