# Introduction to Differentiable Manifolds

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**Exercise Series 4 - More on Tangent vectors and smooth embeddings 2022–10–12**

**Tangent spaces and Tangent bundles.**

**Exercise 4.1** (A little heads-up regarding coordinate vectors)**.** Let $\varphi$ and $\psi$ be smooth charts on a smooth manifold $M$ defined on the same domain $U$. Let call $Cx^1, \dots, x^n)$ the coordinates induced by $\varphi$ and $Cz^1, \dots, z^n)$ the coordinates induced by $\psi$. If the first coordinate functions $x^1$ and $z^1$ agree ($x^1 = z^1$ on $U$), this does *not* imply $\frac{\partial}{\partial x^1}|_p = \frac{\partial}{\partial z^1}|_p$ for $p \in U$.

Work out a simple example of this fact e.g. on $M = \mathbb{R}^2$ by considering on the one hand the Cartesian coordinates $(x, y)$ and on the other hand the chart $(u, v)$ given by $u = x$, $v = x + y$.
*This shows that* $\frac{\partial}{\partial x^i}|_p$ *depends on the whole system* $(x^1, \dots, x^n)$*, not only on* $x^i$.

**Exercise 4.2** (The tangent space of a vector space)**.** Let $V$ be an $n$-dimensional vector space, endowed with the natural smooth structure given by picking an isomorphism $\mathbb{R}^n \to V$ (via the Smooth Charts Lemma)

(a) Fix $a \in V$. To every $v \in V$ we associate the curve passing through $a$

$$\gamma_v : \mathbb{R} \to V : t \mapsto a + tv$$

Show that the map $\Phi_a : V \to T_aV : v \mapsto \gamma_v'(0)$ is an isomorphism of vector spaces.

(b) Let $f : V \to W$ be a *linear* map between vector spaces $V, W$. Consider the differential $D_af : T_aV \to T_{F(a)}W$ at any point $a \in V$. Identifying $T_aV \cong V$ and $T_{f(a)}W \cong W$ via the isomorphisms $\Phi_a$, $\Phi_{f(a)}$, show that $D_af$ is identified with $f$. That is, show that the following diagram commutes:

$$\begin{array}{ccc} T_aV & \xrightarrow{D_af} & T_{f(a)}W \\ \Phi_a \uparrow & & \uparrow \Phi_{f(a)} \\ V & \xrightarrow[f]{} & W \end{array}$$

**Exercise 4.3** (Differential of the determinant function)**.** Consider the determinant function $\det : M_n(\mathbb{R}) \to \mathbb{R}$, where $M_n(\mathbb{R}) \simeq \mathbb{R}^{n\times n}$ is the vector space of real $n \times n$, with its natural smooth structure. We want to compute its differential transformation $D_A \det$ at any matrix $A \in \mathrm{GL}_n(\mathbb{R})$ (i.e. at any invertible matrix),

$$D_A \det : T_AM_n(\mathbb{R}) \to T_{\det(A)}\mathbb{R}$$

(Note that we may identify $T_AM_n(\mathbb{R})$ with $M_n(\mathbb{R})$ and $T_{\det(A)}\mathbb{R}$ with $\mathbb{R}$.)

(a) Verify that det is a smooth function.
*Hint:* Write the determinant as a sum over all $n$-permutations.

(b) Show that the differential of det at the identity matrix $I \in M_n(\mathbb{R})$ is

$$D_I \det(B) = \mathrm{tr}(B).$$

where tr denotes the trace.

(c) Show that for arbitrary $A \in \mathrm{GL}_n(\mathbb{R}), B \in M_n(\mathbb{R})$.

$$D_A \det(B) = (\det A)\,\mathrm{tr}(A^{-1}B)$$

*Hint:* Write $\det(A + tB) = (\det A)(\det(I + tA^{-1}B))$.

(d) Show that $D_A \det$ is the null linear transformation if $A = 0$ and $n \geq 2$.

**Exercise 4.4** (Tangent Bundles). (a) Show that $T_{(m_1,m_2)}M_1 \times M_2 \cong T_{m_1}M_1 \oplus T_{m_2}M_2$. Show that if fact this extends to the tangent bundles, i.e. there is a diffeomorphism $T(M_1 \times M_2) \cong TM_1 \times TM_2$.

(b) Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

**Immersions and smooth Embeddings.**

**Exercise 4.5.** Consider the map

$$f : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (2 + \tanh t) \cdot (\cos t, \sin t).$$

Show that $f$ is an injective immersion. Is it a smooth embedding?

**Exercise 4.6.** Consider the following subsets of $\mathbb{R}^2$. Which is an embedded submanifold ? Which is the image of an immersion ?

(a) The "cross" $S := \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$.

(b) The "corner" $C := \{(x, y) \in \mathbb{R}^2 \mid xy = 0, x \geq 0, y \geq 0\}$

**Exercise 4.7.** Let $N$ be a embedded $n$-submanifold of some $m$-manifold $M$. Show that there exists an open set $U \subseteq M$ that contains $N$ as a closed subset.

**Exercise 4.8 (To hand in).** Let $f : M \to N$ be an injective immersion of smooth manifolds. Show that there exists a closed embedding $M \to N \times \mathbb{R}$.

***Hint:*** Recall that there exists a proper map $g : M \to \mathbb{R}$ (Exercise 3.2)