Problem Set 3 **(Graded)** —*Due Friday, October 28, before class starts*

For the Exercise Sessions on Oct 14 and Oct 21

Last name	First name	SCIPER Nr	Points

Problem 1: Some review problems on linear algebra

(a) (Frobenius norm) Prove that $\|A\|_F^2 = \mathrm{trace}(A^H A)$.

(b) (Singular Value Decomposition) Let $\sigma_i(A)$ denote the i^{th} singular value of an $m \times n$ matrix A. Prove that $\|A\|_F^2 = \sum_{i=1}^{\min\{m,n\}} \sigma_i^2(A)$

(c) (Projection Matrices) Consider a set of k orthonormal vectors in $\mathbb{C}^n$, denoted by $\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}$. The projection matrix (that projects an arbitrary vector into the subspace spanned by these orthonormal vectors) is given by

$$P = \sum_{i=1}^{k} \mathbf{u}_i \mathbf{u}_i^H. \tag{1}$$

- Prove that this matrix is *Hermitian,* i.e., $P^H = P$.
- Prove that this matrix is *idempotent,* i.e., $P^2 = P$. (In words, projecting twice into the same subspace is the same as projecting only once.)
- Prove that $\mathrm{trace}(P) = k$, i.e., equal to the dimension of the subspace.
- Prove that the diagonal entries of P must be real-valued and non-negative. Then, prove that the diagonal entries of P cannot be larger than 1 (this is a little more tricky).

Problem 2: Eckart–Young Theorem

In class, we proved the converse part of the Eckart–Young theorem for the spectral norm. Here, you do the same for the case of the Frobenius norm.

(a) For any matrix A of dimension $m \times n$ and an arbitrary orthonormal basis $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ of $\mathbb{C}^n$, prove that

$$\|A\|_F^2 = \sum_{k=1}^{n} \|A\mathbf{x}_k\|^2. \tag{2}$$

(b) Consider any $m \times n$ matrix B with $\mathrm{rank}(B) \leq p$. Clearly, its null space has dimension no smaller than $n - p$. Therefore, we can find an orthonormal set $\{\mathbf{x}_1, \cdots, \mathbf{x}_{n-p}\}$ in the null space of B. Prove that for such vectors, we have

$$\|A - B\|_F^2 \geq \sum_{k=1}^{n-p} \|A\mathbf{x}_k\|^2. \tag{3}$$

(c) (This requires slightly more subtle manipulations.) For any matrix A of dimension $m \times n$ and any *orthonormal* set of $n-p$ vectors in $\mathbb{C}^n$, denoted by $\{\mathbf{x}_1, \cdots, \mathbf{x}_{n-p}\}$, prove that

$$\sum_{k=1}^{n-p} \|A\mathbf{x}_k\|^2 \quad \geq \quad \sum_{j=p+1}^{r} \sigma_j^2. \tag{4}$$

Hint: Consider the case $m \geq n$ and the set of vectors $\{\mathbf{z}_1, \cdots, \mathbf{z}_{n-p}\}$, where $\mathbf{z}_k = V^H \mathbf{x}_k$. Express your formulas in terms of these and the SVD representation $A = U\Sigma V^H$.

(d) Briefly explain how *(a)-(c)* imply the desired statement.

Problem 3: A Hilbert space of matrices

In this problem, we consider the set of matrices $A \in \mathbb{R}^{m \times n}$ with standard matrix addition and multiplication by scalar.

(a) Briefly argue that this is indeed a vector space, using the definition given in class.

(b) Show that $\langle A, B \rangle = \text{trace}(B^H A)$ is a valid inner product.

(c) Explicitly state the norm induced by this inner product. Is this a norm that you have encountered before?

(d) Consider as a further inner product candidate the form $\langle A, B \rangle = \text{trace}(B^H W A)$, where W is a square ($m \times m$) matrix. Give conditions on W such that this is a valid inner product. Explicit and detailed arguments are required for full credit.

Problem 4: Haar Wavelet

This problem is taken from Vetterli/Kovacevic, p. 295.

Consider the wavelet series expansion of continuous-time signals $f(t)$ and assume that $\psi(t)$ is the Haar wavelet.

(a) Give the expansion coefficients for $f(t) = 1, t \in [0, 1]$, and 0 otherwise.

(b) Verify that for $f(t)$ as in Part *(a)*, $\sum_m \sum_n \|\langle \psi_{m,n}, f \rangle\|^2 = 1$ (i.e., Parseval's identity).

(c) Consider $f_1(t) = f(t - 2^{-i})$, where i is a positive integer. Give the range of scales over which expansion coefficients are non-zero. (Take $f(t)$ as in Part *(a)*.)

(d) Same as above, but now for $f_2(t) = f(t - 1/\sqrt{2})$. (Take $f(t)$ as in Part *(a)*.)

Problem 5: Dual Representation of Norm

(a) Assume that $p > 0$ and $q > 0$ fulfils $1/p + 1/q = 1$- Show that the following inequality holds for all $a \geq 0$ and $b \geq 0$.

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{5}$$

Show that the equality holds if $a^p = b^q$. [Hint: Use the concavity of log function]

(b) Given vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$ show that,

$$\frac{\sum_{i=1}^{n} |x_i y_i|}{||\mathbf{x}||_p ||\mathbf{y}||_q} \leq 1 \tag{6}$$

What is the condition for equality?

(c) Show that

$$||\mathbf{x}||_p = sup < \mathbf{y}, \mathbf{x} >: \mathbf{y} \in \mathbb{R}^n, ||\mathbf{y}||_{p^*} = 1. \tag{7}$$

where $1/p + 1/p^* = 1$