Introduction to Differentiable Manifolds	
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Exercise 2.1 (Manifolds from vector spaces)**.** Let V be a vector space and $|\cdot| : V \to \mathbb{R}$ a norm. Then V is endowed with a distance function and thus has the structure of a topological space. Let $E_1, \dots, E_n$ be a basis for V, then $E : \mathbb{R}^n \to V$ defined by $(x_1, \dots, x_n) \to \sum_i x_i E_i$.

(a) Show that (V, E^{-1}) is a chart for V;

Solution. We recall the following fact from analysis: *Any two norms N_0, N_1 on $\mathbb{R}^n$ are equivalent.* That is, there exist numbers α, $\beta > 0$ such that $N_0(x) \leq \alpha N_1(v)$ and $N_1(x) \leq \beta N_0(v)$ for all $v \in \mathbb{R}^n$. (A proof of this can be found e.g. in https://math.stackexchange.com/q/2890032.) This fact implies that the two topologies induced by the two distances $d_i(x, y) = N_i(y, x)$ (for $i = 0, 1$) coincide.

Now, let us solve the exercise. On the space V we define the distance function $d(x, y) = \|y - x\|$, which gives the topology to V, and on the space $\mathbb{R}^n$ we define the distance function $d'(x', y') = d(E(x'), E(y'))$, so that the map $E : (\mathbb{R}^n, d') \to (V, d)$ is an isometry, hence an homeomorphism. More precisely, E^{-1} is a homeomorphism from V (which is clearly an open subset of V) to $\mathbb{R}^n$ (which is an open subset of $\mathbb{R}^n$). We conclude that the pair (V, E^{-1}) is a chart of V. ☐

(b) Show that given a different base $\widetilde{E}_1, \dots, \widetilde{E}_n$, the charts (V, E^{-1}), $(V, \widetilde{E}^{-1})$ of V are smoothly compatible. We say that the collection of charts of this form define the *standard smooth structure* on V.

Solution. Each of the charts E^{-1}, $\widetilde{E}^{-1}$ is a linear isomorphism, therefore the transition map $\tau = E^{-1} \circ \widetilde{E}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism as well. (Note that in this case it is was easy to compute the domain and image of the transition map because both charts E^{-1}, $\widetilde{E}^{-1}$ are defined on the whole space V.)

Since $\tau : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism, there exists an invertible matrix A such that $\tau(v) = Av$ and $\tau^{-1}v = A^{-1}v$ for all v. We conclude that τ is a smooth function with smooth inverse, therefore the charts E^{-1}, $\widetilde{E}^{-1}$ are smoothly compatible. ☐

Use the previous part of the exercise to show that:

(a) The space $M(n \times m, \mathbb{R})$ of $n \times m$ matrices has a natural smooth manifold structure

Solution. This space is a vector space of dimension $n \cdot m$, hence it has a natural smooth structure by the first part of the exercise. ☐

(b) The general linear group $\mathrm{Gl}(n, \mathbb{R})$ has a natural smooth manifold structure

Solution. The set $GL(n, \mathbb{R})$ is an open subset of $M(n, \mathbb{R})$ therefore it is naturally a smooth manifold. ☐

(c) Let V, W two vector spaces and $L(V; W)$ the space of linear maps from V to W has a natural smooth manifold structure.

Solution. The space of linear maps $L(V; W)$ is a vector space, hence it has a natural smooth structure. ☐

Exercise 2.2 (Stereographic projection.)**.** Let $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ be the *north pole* and $S = -N$ the *south pole* of the sphere $\mathbb{S}^n$. Define stereographic projection

$\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x_0, \dots, x_n) = \frac{1}{1 - x_n}(x_0, \dots, x_{n-1}).$$

Let $\widetilde{\sigma}(x) = \sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

(a) Show that σ is bijective, and

$$\sigma^{-1}(u_0, \dots, u_{n-1}) = \frac{1}{|u|^2 + 1}(2u_0, \dots, 2u_{n-1}, |u|^2 - 1).$$

Solution. To show that σ is bijective and σ^{-1} is its inverse, it is sufficient to verify that

$$\sigma^{-1} \circ \sigma = \mathrm{id},$$
$$\sigma \circ \sigma^{-1} = \mathrm{id}.$$

Let us show first how one could find the formulas for σ and σ^{-1}. We use the following notation: if $u \in \mathbb{R}^n$ and $a \in \mathbb{R}$, we denote (u, a) the point of $\mathbb{R}^{n+1}$ whose first n coordinates are the u_i's and whose last coordinate is a. Thus the hyperplane $\Pi = \mathbb{R}^n \times \{0\} \equiv \mathbb{R}^n$ contains the points of the form $(u, 0)$.

Every non-horizontal line r containing the point N intersects the sphere $\mathbb{S}^n$ at one point x (other than N) and intersects the plane Π at a point $(u, 0)$. We want the formulas for the maps $\sigma : x \mapsto u$ and $\sigma^{-1} : u \mapsto x$. If we know x, then r is the image of the map

$$t \in \mathbb{R} \mapsto N + t(x - N) = (tx_0, \dots, tx_{n-1}, 1 + t(x_n - 1))$$

The intersection with the plane Π occurs when the last coordinate is 0, that is, when $t = \frac{1}{1-x_n}$. The point of intersection is $(u, 0)$, where

$$u = (tx_0, \dots, tx_{n-1}) = \frac{1}{1 - x_n}(x_0, \dots, x_{n-1}).$$

This gives the formula for σ.

To compute σ^{-1} suppose we know the point u. Then r is the image of the map

$$t \in \mathbb{R} \mapsto N + t((u, 0) - N)) = (tu_0, \dots, tu_n, 1 - t)$$

This point is contained in $\mathbb{S}^n$ if and only if $t^2|u|^2 + (1-t)^2 = 1$. We rewrite the equation as $(|u|^2 + 1)t^2 - 2t = 0$ and find the solutions $t = 0$ (corresponding to the north pole) and $t = \frac{2}{|u|^2+1}$, corresponding to the point

$$x = \frac{1}{|u|^2 + 1}(2u_0, \dots, 2u_{n-1}, |u|^2 - 1).$$

This gives the formula for σ^{-1}. □

(b) Verify that $\{\sigma, \widetilde{\sigma}\}$ is a smooth atlas for $\mathbb{S}^n$.

Solution. The domains of σ and $\widetilde{\sigma}$ cover $\mathbb{S}^n$. The transition map $\widetilde{\sigma} \circ \sigma^{-1}$, defined on $\mathbb{R}^n \setminus \{0\}$ by the formula

$$\widetilde{\sigma} \circ \sigma^{-1}(u) = \sigma\left(-\frac{2u_0, \dots, 2u_{n-1}, |u|^2 - 1}{|u|^2 + 1}\right) = -\frac{(u_0, \dots, u_{n-1})}{|u|^2},$$

is smooth. □

(c) Show that the smooth structure defined by the atlas $\{\sigma, \widetilde{\sigma}\}$ is the same as the one defined via graph coordinates in the lecture.

Solution. It suffices to show that each chart in $\{\sigma, \widetilde{\sigma}\}$ is compatible with all the graph charts $\phi_i^{\pm}$. The transition function is given by the formula

$$\begin{aligned}\phi_i^+ \circ \sigma^{-1}(y) &= \phi_i^+ \left(\frac{2y_0, \dots, 2y_{n-1}, |y|^2 - 1}{|y|^2 + 1}\right) \\ &= \left(\frac{2y_0, \dots, 2y_{i-1}, 2y_{i+1}, \dots, 2y_{n-1}, |y|^2 - 1}{|y|^2 + 1}\right)\end{aligned}$$

and is therefore a smooth map. The inverse map $\sigma \circ (\phi_i^+)^{-1}$ is defined on $\sigma^{-1}(\mathbb{S}^n \cap U_i^+)$ and is [...] □

Exercise 2.3. Let N be an open subset of a smooth n-manifold $(M, \mathcal{A})$, endowed with the smooth structure described in Exercise 1.4. Prove that:

(a) The inclusion map $\iota : N \hookrightarrow M$ is a smooth map of manifolds.

Solution. We just need to check that the local expressions of ι are $\mathcal{C}^k$. These local expressions are of the form

$$\iota_\psi^\varphi = \varphi \circ \iota \circ \psi^{-1} = \varphi \circ \psi^{-1}$$

with $\varphi \in \mathcal{B}$ and $\psi \in \mathcal{A}$. Now, both charts φ, ψ belong to $\mathcal{A}$ since $\mathcal{B} \subseteq \mathcal{A}$. Therefore the map $\varphi \circ \psi^{-1}$ is a transition map of $\mathcal{A}$, therefore it is $\mathcal{C}^k$. □

(b) A function $f : L \to N$ from a smooth manifold l is smooth f and only if the composite $\iota \circ f$ is smooth.

Solution. If f is $\mathcal{C}^k$, the composite $\iota \circ f$ is $\mathcal{C}^k$ because ι is $\mathcal{C}^k$. Reciprocally, suppose ι is $\mathcal{C}^k$. Then f is $\mathcal{C}^k$, because any local expression f_ξ^ψ (with ξ a chart of L and $\psi \in \mathcal{B}$) is also a local expression of $\iota \circ f$. Indeed,

$$f_\xi^\psi = \psi \circ f \circ \xi^{-1} = \psi \circ \iota \circ f \circ \xi^{-1}.$$

□

Exercise 2.4 (Properties of manifolds)**.** Show that:

(a) Let $\mathcal{A}, \mathcal{A}'$ be smooth atlases on a topological manifold M. Then $\mathcal{A}$ and $\mathcal{A}'$ determine the same smooth structure on M if and only if their union is a smooth atlas.

Solution. Let $\bar{\mathcal{A}}$ denote the maximal atlas determined by $\mathcal{A}$: the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. We have to show that $\bar{\mathcal{A}} = \bar{\mathcal{A}}'$ if and only if $\mathcal{A} \cup \mathcal{A}'$ is an atlas. We can define the equivalence relation between charts $\phi \sim \psi$ iff ϕ is smoothly compatible with ψ. Then the argument follow from the transitivity property of the smoothly compatible relationship. If $\mathcal{A} \cup \mathcal{A}'$ is an atlas then, $\phi \sim \psi$ for every $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{A}'$, This implies $\bar{\mathcal{A}} = \bar{\mathcal{A}}'$. Conversely, $\bar{\mathcal{A}} = \bar{\mathcal{A}}'$ implies that for $\xi \in \bar{\mathcal{A}}$, $\xi \sim \phi$ for every $(U, \phi) \in \mathcal{A}$ then $\xi \sim \psi$ for every $(V, \psi) \in \mathcal{A}'$. Therefore, $\phi \sim \psi$ for every $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{A}'$. So $\mathcal{A} \cup \mathcal{A}'$ is an atlas. □

Solution. We know that $\phi : U \to \phi(U)$ is an homeomorphism. Moreover ϕ is a smooth map iff the composition $\mathrm{id}_{\mathbb{R}^n} \circ \phi \circ \phi^{-1}$ is smooth. Since $\mathrm{id}_{\mathbb{R}^n} \circ \phi \circ \phi^{-1} = \mathrm{id}_{\mathbb{R}^n}$ is a smooth map so ϕ is a smooth map. One can show that ϕ^{-1} is smooth as well by a similar argument. □

Solution. Let $f : M \to N$ and $g : N \to P$ be smooth maps. Then by definitions the maps $\phi \circ f \circ \psi^{-1}$ and $\psi \circ g \circ \xi^{-1}$ are smooth for every smooth local chart ϕ, ψ, and ξ on M, N, and P respectively. Then the composition $h = f \circ g$ is smooth since the function

$$\phi \circ h \circ \xi^{-1} = \phi \circ f \circ g \circ \xi^{-1} = \phi \circ f \circ \psi^{-1} \circ \psi \circ g \circ \xi^{-1}$$

is smooth for every local chart. □

(b) Two smooth atlases $\mathcal{A}_1, \mathcal{A}_2$ on M are equivalent iff the following holds:
For every function $f : N \to M$ (where N is a smooth manifold), the function f is smooth as a map $N \to M_0$ if and only if it is $\mathcal{C}^k$ as a map $N \to M_1$.

Solution. ($\Rightarrow$) is clear. Let us prove prove ($\Leftarrow$). Thus assuming the last property holds, let us show that the atlases $\mathcal{A}_0$, $\mathcal{A}_1$ are equivalent. For this, it suffices to prove the following:

Claim: *For every $\varphi \in \mathcal{A}_0$, $\psi \in \mathcal{A}_1$ the transition maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are $\mathcal{C}^k$.*

Proof of claim: The function φ is a $\mathcal{C}^k$ isomorphism $U \to V$, where $U \subseteq M_0$ and $V \subseteq \mathbb{R}^n$ are open sets. Therefore its inverse $\varphi^{-1} : V \to M_0$ is $\mathcal{C}^k$. Then, by the hypothesis (applied to $f = \varphi^{-1}$), $\varphi^{-1} : V \to M_1$ is $\mathcal{C}^k$. Therefore $\psi \circ \varphi^{-1}$ is $\mathcal{C}^k$ for each $\psi \in \mathcal{A}_1$, as claimed. An analogous argument shows that $\varphi \circ \psi^{-1}$ is $\mathcal{C}^k$. □

Exercise 2.5 (to hand in). Prove the following

(a) Let $c : M \to N$ the constant map between two smooth manifolds; c is smooth
(b) Every smooth chart $\varphi : U \to \varphi(U)$ of M is a diffeomorphism; here U and $\varphi(U)$ are given the open subspace smooth structure defined in Exercise 1.4.
(c) The composite $g \circ f$ of two smooth maps $f : M \to N$, $g : N \to P$ is smooth map.
(d) Show that the quotient map $\pi : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ is a smooth map of manifolds where on $R\mathbb{P}^n$ we considered the smooth structure defined in Exercise 1.7.

Exercise 2.6. On the real line $\mathbb{R}$ (with the standard topology) we define two atlases $\mathcal{A} = \{\mathrm{id}_{\mathbb{R}}\}$, $\mathcal{B} = \{\varphi\}$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x) = x^3$.

(a) Find a smooth diffeomorphism $(\mathbb{R}, \overline{\mathcal{A}}) \to (\mathbb{R}, \overline{\mathcal{B}})$.

Solution. The homeomorphism $f : (\mathbb{R}, \overline{\mathcal{A}}) \to (\mathbb{R}, \overline{\mathcal{B}})$ defined by $f(x) = x^3$ is a diffeomorphism since the local expressions

$$f^{\varphi}_{\mathrm{id}_{\mathbb{R}}} = \mathrm{id}_{\mathbb{R}} \circ f \circ \varphi^{-1} = \mathrm{id}_{\mathbb{R}}$$

and

$$f^{-1}|^{\mathrm{id}_{\mathbb{R}}}_{\varphi} = \varphi \circ f^{-1} \circ \mathrm{id}_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}}$$

are smooth. □