Introduction to Differentiable Manifolds
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Solutions Series 3 - Partitions of unity, Tangent vectors 2021–10–20

Partition of Unity.

Exercise 3.1. Consider $\mathbb{R}$ with its standard smooth structure. Let $f : \mathbb{R} \to \mathbb{R}$ the sign function:

$$x \mapsto \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Let $A \subset \mathbb{R}$ a closed subset such that $f|_A$ is smooth in the sense defined in the Lecture. Find a smooth extension of $f|_A$ to all of $\mathbb{R}$ what existence is guaranteed by the Extension Lemma. Notice that $f|_{(-\infty,0)\cup(0,\infty)}$ is smooth but does not admit an extension to $\mathbb{R}$; i.e. the conclusion of the extension Lemma fails if we remove the hypothesis A closed.

Solution. Let $A \subseteq \mathbb{R}$ be a closed subset such that $f|_A$ is smooth. This set must exclude the point 0, where f is discontinuous (hence nonsmooth).

We have to define a smooth function $g : \mathbb{R} \to \mathbb{R}$ that is an extension $f|_A$. If A only contains positive numbers (or is empty), then we can define g as the constant function: $g(x) = 1$ for all x. Likewise, if A contains just negative numbers, we define $g(x) = -1$ for all x. Thus we may assume that A contains both positive and negative numbers. Let $a \in \mathbb{R}$ be the supremum of the closed set $A \cap (-\infty, 0]$, and let b be the infimum of the set $A \cap [0, +\infty)$. Note that $a < 0 < b$ since $0 \notin A$.

To finish the exercise, it suffices to find a smooth function g that coincides with f outside the interval (a, b). (The interval (a, b) does not intersect A, therefore the values of g in this interval are irrelevant.) In the lectures we saw that there exists a smooth function $h : \mathbb{R} \to \mathbb{R}$ that satisfies $h(x) = 1$ for $x \leq a$ and $h(x) = 0$ for $x \geq b$. Therefore the function $g(x) = 1 - 2h(x)$, which is also smooth, satisfies $g(x) = -1$ for $x \leq a$ and $g(x) = 1$ for $x \geq b$, and hence coincides with f outside the interval (a, b), and hence on the set A, as required. □

Exercise 3.2. A continuous map $f : X \to Y$ is called *proper* if $f^{-1}(K)$ is compact for every compact set $K \subseteq Y$. Show that for every smooth manifold M there exists a smooth map $f : M \to [0, +\infty)$ that is proper.

Hint: Note that f must be unbounded unless M is compact. Use a function of the form $f = \sum_{i\in\mathbb{N}} c_i f_i$, where $(f_i)_{i\in\mathbb{N}}$ is a partition of unity and the c_i's are real numbers.

Solution. Let $(U_i)_{i\in\mathbb{N}}$ be a countable topological basis for M such that $\overline{U_i}$ is compact for each i. Let (f_i) be a $\mathcal{C}^k$ partition of unity on M such that $\operatorname{supp}(f_i) \subseteq U_i$ for each i. Define the $\mathcal{C}^k$ function $f : M \to \mathbb{R}$ by the formula $f(x) = \sum_{i\in\mathbb{N}} c_i\, f_i(x)$, where $c_i \geq 0$ are numbers satisfying $\lim_{i\to\infty} c_i = +\infty$. (For instance, we may put $c_i = i$.)

We can view $f(x)$ as a weighted average of the numbers c_i, using as weights the coefficients $f_i(x) \geq 0$, which satisfy $\sum_i f_i(x) = 1$. In particular, note that if $I_x \subseteq \mathbb{N}$ is the set of indices such that U_i contains the point x, then any upper or lower bound for the numbers c_i with $i \in I_x$ is also an upper or lower bound for $f(x)$. It follows that if $f(x) < c$, then x is contained in the union of the first few U_i's which satisfy $c_i < c$.

To see that f is proper, let $K \subseteq \mathbb{R}$ be a compact set. Take any number $c \geq 0$ such that $K \subseteq (-c, c)$, and let $i_c \in \mathbb{N}$ such that $c_i \geq c$ for $i \geq i_c$. The preimage $f^{-1}(K)$ consists of points x satisfying $f(x) < c$, and is therefore contained in the compact set $\bigcup_{i<i_c} \overline{U_i}$. Since the set $f^{-1}(K)$ is closed, we conclude that it is compact. □

Exercise 3.3. Let M be a $\mathcal{C}^k$ manifold and let U be an open neighborhood of the set $M \times \{0\}$ in the space $M \times [0, +\infty)$. Show that there exists a $\mathcal{C}^k$ function $f : M \to (0, +\infty)$ whose graph is contained in U.

Solution. Every point $\{x\} \times \{0\}$ of the set $M \times \{0\}$ has a neighborhood $V \times [0, \varepsilon)$ contained in U, where $V \subseteq M$ is an open neighborhood of x and $\varepsilon > 0$. Thus there is a covering of M by open sets V_i and numbers $\varepsilon_i > 0$ such that $V_i \times [0, \varepsilon_i) \subseteq U$ for all i. Let $(f_i)_i$ be a partition of unity with $\operatorname{supp}(f_i) \subseteq V_i$. Then we can take the $\mathcal{C}^k$ function $f = \sum_i \frac{\varepsilon_i}{2} f_i$. □

Tangent vectors and tangent space.

Exercise 3.4. (Derivations in $\mathbb{R}^n$)

(a) Show that the function $D_{v|_a} : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by $f \mapsto \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(a + tv)$ is a derivation, i.e. it is $\mathbb{R}$-linear and satisfies the product rule.

Solution. Fixed the point $a \in \mathbb{R}^n$ and the vector $v \in \mathbb{R}^n$, for any function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ we define a smooth function $h_f : t \in \mathbb{R} \mapsto f(a + tv) \in \mathbb{R}$, so that $D_{v|_a}(f) = h'_f(0)$. Then for any functions $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$ and any number $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} D_{v|_a}(f + \lambda\, g) = h'_{f+\lambda\, g}(0) = (h_f + \lambda\, h_g)'(0) &= h'_f(0) + \lambda\, h'_g(0) \\ &= D_{v|_a}(f) + \lambda \cdot D_{v|_a}(g) \end{aligned}$$

and

$$\begin{aligned} D_{v|_a}(f \cdot g) = h'_{f\,g}(0) = (h_f \cdot h_g)'(0) &= h'_f(0) \cdot h_g(0) + h_f(0) \cdot h'_g(0) \\ &= D_{v|_a}(f) \cdot g(0) + f(0) \cdot D_{v|_a}(g). \end{aligned}$$

□

(b) Let $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a smooth map. Prove that the linear map $DF_p : T_pU \to T_F(p)V$ is given, with respect to the standard basis $\langle \frac{\partial}{\partial x^i}|_p \rangle_{i=1,\dots,n}, \langle \frac{\partial}{\partial y^j}|_{F(p)} \rangle_{j=1,\dots,m}$, by the Jacobian matrix $\left(\frac{\partial F^j}{\partial x^i} \right)_{ij}$.

Solution. Any vector $v \in T_p\mathbb{R}^n$ can be written in the form $v = \sum_i v^i \left.\frac{\partial}{\partial x^i}\right|_p$. Our task is to calculate the vector $DF_p(v) \in T_{F(p)}\mathbb{R}^m$ and express it as a linear combination of the vectors $\left.\frac{\partial}{\partial y^j}\right|_{F(p)}$. To determine what is the vector $DF_p(v)$, we apply it to a general function $h \in \mathcal{C}^\infty(\mathbb{R}^m)$. We apply first the definiton of $DF_p(v)$ and then the chain rule, obtaining the following:

$$\begin{aligned} DF_p(v)(h) &= v(h \circ F) \\ &= \sum_i v^i \left.\frac{\partial}{\partial x^i}\right|_p (h \circ F) \\ &= \sum_{i \in n} v^i \sum_{j \in m} \left.\frac{\partial h}{\partial y^j}\right|_{F(p)} \left.\frac{\partial F^j}{\partial x^i}\right|_p \\ &= \sum_{j \in m} \sum_{i \in n} v^i \left.\frac{\partial F^j}{\partial x^i}\right|_p \left.\frac{\partial}{\partial y^j}\right|_{F(p)} h \end{aligned}$$

Since this applies to each function $h \in C^\infty(\mathbb{R}^m)$, it means that $DF_p(v) = \sum_{j \in m} \sum_{i \in n} \left.\frac{\partial F^j}{\partial x^i}\right|_p v^i \left.\frac{\partial}{\partial y^j}\right|_{F(p)}$. Theferore the coordinates of the vector $w = DF_p(v)$ in the base $\left.\frac{\partial}{\partial y^j}\right|_{F(p)}$ are the numbers $w^j = \sum_{i \in n} \left.\frac{\partial F^j}{\partial x^i}\right|_p v^i$, i.e. the coefficents of the product of the matrix $\left(\left.\frac{\partial F^j}{\partial x^i}\right|_p \right)_{j,i}$ by the column vector $(v^i)_i$. □

Exercise 3.5. Let M be a smooth n-manifold. Show that:

(a) The differential of a smooth map $F : M \to N$ at a point $p \in M$ is a well-defined linear map $D_pF : T_pM \to T_pN$.

Solution. This was proved in Lecture 3, page 10. □

(b) *Chain rule*: for smooth maps $F : M \to N$, $G : N \to P$ and a point $p \in M$,

$$D_p(G \circ F) = D_{F(p)}G \circ D_pF.$$

In particular, if F is a diffeomorphism, then D_pF has inverse $(D_pF)^{-1} = D_{F(p)}(F^{-1})$.

Solution. Take any vector $v \in T_pM$. To determine what is the vector $(D_p(G \circ F))(v)$, we apply it to a general function $h \in C^\infty(P)$.

We get

$$\begin{aligned}
(D_p(G \circ F))(v)(h) &= v(h \circ (G \circ F)) \\
&= v((h \circ G) \circ F) \\
&= (D_pF(v))(h \circ G) \\
&= (D_{F(p)}G(D_pF(v)))(h) \\
&= ((D_{F(p)}G \circ D_pF)(v))(h)
\end{aligned}$$

Since this is valid for all functions $h \in C^\infty(P)$, it implies that $(D_p(G \circ F))(v) = (D_{F(p)}G \circ D_pF)(v)$. Since this holds for all vectors $v \in T_pM$, we conclude that $D_p(G \circ F) = D_{F(p)}G \circ D_pF$. □

(c) *Change of coordinates*:

Let $X \in T_pM$ be a tangent vector and let φ, $\widetilde{\varphi}$ be smooth charts of M defined at a p such that $\widetilde{\varphi} \circ \varphi^{-1} : \varphi(U \cup \widetilde{U}) \to \widetilde{\varphi}(U \cup \widetilde{U})$ is defined by $(x^1, \dots, x^n) \mapsto (z^1(x^1, \dots, x^n), \dots, z^n(x^1, \dots, x^n))$. If $X \in T_pM$ we have that in the local coordinates charts

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}|_p = \sum_{i=1}^n Z^i \frac{\partial}{\partial z^i}|_p$$

where X^i respectively Z^i are called *components* of the tangent vector in the coordinate base. Prove that

$$Z^i = \sum_{j=1}^n X^j \frac{\partial z^i}{\partial x^j}(\varphi(p))$$

Solution. Let $(X^i)_i$ be coordinate tuple of X with respect to the basis $\left(\frac{\partial}{\partial \varphi^i}|_p\right)_i$, and let $(\widetilde{X}^j)_j$ be the coordinate tuple of X with respect the basis $\left(\frac{\partial}{\partial \widetilde{\varphi}^j}|_p\right)_j$, so that

$$X = \sum_i X^i \frac{\partial}{\partial \varphi^i}\Big|_p = \sum_j \widetilde{X}^j \frac{\partial}{\partial \widetilde{\varphi}^j}\Big|_p.$$

Let us show that

$$\widetilde{X}^j = \sum_i X^i \frac{\partial \widetilde{\varphi}^j}{\partial \varphi^i}\Big|_{\varphi(p)},$$

where $\frac{\partial \widetilde{\varphi}^j}{\partial \varphi^i}\Big|_{\varphi(p)}$ is the partial derivative that appears in the position (j, i) of the Jacobian matrix $J_{\varphi(p)}(\widetilde{\varphi} \circ \varphi^{-1}) : \mathbb{R}^n \to \mathbb{R}^n$.

Using the equation $\frac{\partial}{\partial \varphi^i} = \sum_j \frac{\partial \widetilde{\varphi}^j}{\partial \varphi^i} \frac{\partial}{\partial \widetilde{\varphi}^j}$, we get

$$X = \sum_i X^i \frac{\partial}{\partial \varphi^i} = \sum_i X^i \sum_j \frac{\partial \widetilde{\varphi}^j}{\partial \varphi^i} \frac{\partial}{\partial \widetilde{\varphi}^j} = \sum_j \sum_i X^i \frac{\partial \widetilde{\varphi}^j}{\partial \varphi^i} \frac{\partial}{\partial \widetilde{\varphi}^j}$$

Since on the other hand we have

$$X = \sum_j \widetilde{X}^j \frac{\partial}{\partial \widetilde{\varphi}^j}$$

and the vectors $\frac{\partial}{\partial \widetilde{\varphi}^j}$ are linearly independent, we conclude that

$$\widetilde{X}^j = \sum_i X^i \frac{\partial \widetilde{\varphi}^j}{\partial \varphi^i}$$

Donc la morale c'est que nous pouvons exprimer les coefficiens des vecteurs tangents d'une base par rapport à une autre base en utilisant les coefficients (j, i) de la matrice de la transformation linéaire $D_{\varphi(p)}(\widetilde{\varphi} \circ \varphi^{-1})$. □

Exercise 3.6 (Velocity vectors of curves)**.** Let M be a differentiable manifold. The *velocity vector* of a differentiable curve $\gamma : I \subseteq \mathbb{R} \to M$ at an instant $t \in I$ is the vector $\gamma'(t) := D_t\gamma(1|_t) \in T_{\gamma(t)}M$.

Show that for any tangent vector $X \in T_pM$ there exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = X$.

Solution. Let (U, φ) be a chart of M such that $p \in U$, and let $v = (v^i)$ the n-tuple of coordinates of the vector X with respect to the basis $(\frac{\partial}{\partial \varphi^i}\big|_p)_i$ of T_pM, so that $X = \sum_i v^i \frac{\partial}{\partial \varphi^i}\big|_p$. We define the curve $\gamma = \varphi^{-1} \circ \widetilde{\gamma}$, where $\widetilde{\gamma} : (-\varepsilon, \varepsilon) \to \varphi(U) \subseteq \mathbb{R}^n$ is a curve defined by the formula $\widetilde{\gamma}(t) = \varphi(p) + t\,v$, and $\varepsilon > 0$ is small enough so that $\varphi(p) + t\,v \in \varphi(U)$ for all $t \in (-\varepsilon, \varepsilon)$.

It is clear that $\gamma(0) = p$. Furthermore, we claim that $\gamma'(0) = X$. To see this, we take and arbitrary function $f \in \mathcal{C}^\infty(M)$ and compute

$$\begin{aligned}
\gamma'(0)(f) &= D_0\gamma(1|_0)(f) \\
&= 1|_0(f \circ \gamma) \\
&= (f \circ \gamma)'(0) \\
&= (f \circ \varphi^{-1} \circ \widetilde{\gamma})'(0) \\
&= \sum_i \frac{\partial f \circ \varphi^{-1}}{\partial x^i}\bigg|_{\varphi(p)} (\widetilde{\gamma}^i)'(0) \\
&= \sum_i \left(\frac{\partial}{\partial \varphi^i}\bigg|_p f \right) v^i \\
&= X(f)
\end{aligned}$$

□

Exercise 3.7 (Spherical coordinates on $\mathbb{R}^3$)**.** Consider the following map defined for $(r, \varphi, \theta) \in W := \mathbb{R}^+ \times (0, 2\pi) \times (0, \pi)$:

$$\Psi(r, \varphi, \theta) = (r \cos\varphi \sin\theta, r \sin\varphi \sin\theta, r \cos\theta) \in \mathbb{R}^3.$$

Check that Ψ is a diffeomorphism[1] onto its image $\Psi(W) =: U$. We can therefore consider Ψ^{-1} as a smooth chart on $\mathbb{R}^3$ and it is common to call the component functions of Ψ^{-1} the **spherical coordinates** (r, φ, θ).

Express the coordinate vectors of this chart

$$\frac{\partial}{\partial r}\bigg|_p, \frac{\partial}{\partial \varphi}\bigg|_p, \frac{\partial}{\partial \theta}\bigg|_p$$

at some point $p \in U$ in terms of the standard coordinate vectors $\frac{\partial}{\partial x}\big|_p$, $\frac{\partial}{\partial y}\big|_p$, $\frac{\partial}{\partial z}\big|_p$.

[1] Here "diffeomorphism" is meant in the standard sense of maps between open subsets of $\mathbb{R}^3$.

Solution. Consider the transition from spherical coordinates (r, φ, θ) to Cartesian coordinates (x, y, z), given by the map

$$\begin{aligned} \Psi : W &\to U \\ (r, \varphi, \theta) &\mapsto (x, y, z) \end{aligned}$$

where

$$\begin{cases} x = r \cos\varphi \sin\theta \\ y = r \sin\varphi \sin\theta \\ z = r \cos\theta \end{cases}$$

Let $p \in U$. The general formula for the change of coordinates is

$$\frac{\partial}{\partial \psi^i}\Big|_p = \sum_j \frac{\partial}{\partial \psi^i}\Big|_p \widetilde{\psi}^j \frac{\partial}{\partial \widetilde{\psi}^j}\Big|_p .$$

We apply this formula in the current setting where $\psi = \Psi^{-1}$ is the given chart on U (by abuse of notation we denote its coordinate functions by (r, φ, θ)) and $\widetilde{\psi} = \mathrm{id}_{\mathbb{R}^3}$ (we denote its coordinate functions by (x, y, z)). Then for $p \in U$ we have[2]

$$\begin{aligned}
\frac{\partial}{\partial r}\Big|_p &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x}\Big|_p + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}\Big|_p + \frac{\partial z}{\partial r}\frac{\partial}{\partial z}\Big|_p \\
&= \cos\varphi \sin\theta \frac{\partial}{\partial x}\Big|_p + \sin\varphi \sin\theta \frac{\partial}{\partial y}\Big|_p + \cos\theta \frac{\partial}{\partial z}\Big|_p \\
&= \frac{1}{(x^2+y^2+z^2)^{1/2}}\left(x\frac{\partial}{\partial x}\Big|_p + y\frac{\partial}{\partial y}\Big|_p + z\frac{\partial}{\partial z}\Big|_p \right) \\
\frac{\partial}{\partial \varphi}\Big|_p &= \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x}\Big|_p + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y}\Big|_p + \frac{\partial z}{\partial \varphi}\frac{\partial}{\partial z}\Big|_p \\
&= -r\sin\varphi \sin\theta \frac{\partial}{\partial x}\Big|_p + r\cos\varphi \sin\theta \frac{\partial}{\partial y}\Big|_p \\
&= -y\frac{\partial}{\partial x}\Big|_p + x\frac{\partial}{\partial y}\Big|_p \\
\frac{\partial}{\partial \theta}\Big|_p &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x}\Big|_p + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}\Big|_p + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z}\Big|_p \\
&= r\cos\varphi \cos\theta \frac{\partial}{\partial x}\Big|_p + r\sin\varphi \cos\theta \frac{\partial}{\partial y}\Big|_p - r\sin\theta \frac{\partial}{\partial z}\Big|_p \\
&= \frac{xz}{(x^2+y^2)^{\frac{1}{2}}}\frac{\partial}{\partial x}\Big|_p + \frac{yz}{(x^2+y^2)^{\frac{1}{2}}}\frac{\partial}{\partial y}\Big|_p - (x^2+y^2)^{\frac{1}{2}}\frac{\partial}{\partial z}\Big|_p
\end{aligned}$$

□

Exercise 3.8. (**To hand in**) Consider the inclusion $\iota : S^2 \to \mathbb{R}^3$, where we endow both spaces with the standard smooth structure. Let $p \in S^2$. What is the image of $D_p\iota : T_pS^2 \to T_p\mathbb{R}^3$? (Identify $T_p\mathbb{R}^3$ with $\mathbb{R}^3$ in the standard way, i.e. $e_i \mapsto \frac{\partial}{\partial x^i}|_p$) So the result should be the equation for a plane in $\mathbb{R}^3$.)

Hint: Use Exercise 7 on spherical coordinates.

[2] There is a bit of abuse of notation going on; e.g. $\frac{\partial x}{\partial r}$ really means $\frac{\partial}{\partial r}|_p(x)$, i.e. the coordinate vector $\frac{\partial}{\partial r}$ applied to the function $x : \mathbb{R}^3 \to \mathbb{R}$ and this by definition is $\frac{\partial(r\cos\varphi\sin\theta)}{\partial r}|_{\psi(p)}$. The potentially confusing thing here is that r denotes at the same time the first component of the chart ψ (in the lecture this was φ^i) and the coordinate on the image of the chart in $\mathbb{R}^3$ (in the lecture this was x^i). But this sloppiness is common and actually helps with computations as you see above.