Introduction to Differentiable Manifolds	
EPFL – Fall 2022	F. Carocci, M. Cossarini
Solutions Series 5 - Submanifolds	**2021–11–03**

Warning Both notations T_pF and D_pF for the differential of a smooth map at a point are used in the file. This is due to the fact that solutions have been written in different moments. Hopefully, this will not cause any confusion.

Exercise 5.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^3 + y^3 + 1$.

(a) What are the regular values of f? For which $c \in \mathbb{R}$ is the level set $f^{-1}(\{c\})$ an embedded submanifold of $\mathbb{R}^2$?

Solution. The gradient of f,

$$\nabla f(x, y) = (3x^2, 3y^2),$$

vanishes precisely at the origin $(x, y) = (0, 0)$. Thus $D_pf : D_p\mathbb{R}^2 \to T_{f(p)}\mathbb{R}$ has rank 0 if and only if $p = (x, y) = (0, 0)$. Thus every $c \in \mathbb{R}$ is a regular value except $c = 1$.

By the regular preimage theorem, each level set $f^{-1}(\{c\})$ with $c \neq 1$ is a smooth embedded submanifold in $\mathbb{R}^2$. As for the level set $f^{-1}(\{1\})$ we have to argue differently. The theorem does not say that $f^{-1}(\{1\})$ is not a smooth submanifold. We have to study this case separately. Observe that in this case one has

$$f^{-1}(\{1\}) = \{x^3 + y^3 = 0\} = \{x = -y\}$$

i.e., $f^{-1}(\{1\})$ is a line going through the origin. Thus, also $f^{-1}(\{1\})$ is a smooth submanifold of $\mathbb{R}^2$. Summing up, all level sets of this function are smooth submanifolds. $\square$

(b) In the case where $S = f^{-1}(\{c\})$ is an embedded submanifold, $p \in S$, write down an equation for the tangent space $\iota_*(\mathrm{T}_pS) \subset \mathrm{T}_p\mathbb{R}^2$ where as usual we identify $T_p\mathbb{R}^2 \cong \mathbb{R}^2$ (i.e. you are expected to write down the equation for a line in $\mathbb{R}^2$).

Solution. By the regular preimage theorem, if $c \neq 1$ we have $\mathrm{T}_pS = \mathrm{Ker}\ \mathrm{T}_pf$ for all $p \in S = f^{-1}(c)$.

Let us compute T_pf. If $V = (V_x, V_y) \in \mathrm{T}_p\mathbb{R}^2 \equiv \mathbb{R}^2$, then $\mathrm{T}_pf(V) = 3\,p_x^2\,V_x + 3\,p_y^2\,V_y$, where $p = (p_x, p_y)$. Hence

$$\mathrm{Ker}\ \mathrm{T}_pf = \{V \in \mathrm{T}_p\mathbb{R}^2 : p_x^2\,V_x + p_y^2\,V_y = 0\}.$$

When $c = 1$ we notice that $S = \{x = -y\}$, thus $\mathrm{T}_pS = \{V \in \mathrm{T}_p\mathbb{R}^2 : V_x = -V_y\}$. $\square$

Exercise 5.2. Let $S = F^{-1}(c)$ for c a regular value of a smooth function $F : M \to N$. Let us fix $p \in S$. Prove that $T_pS = \mathrm{Ker}(D_pF : T_pM \to T_F(p)N)$.

Hint: Use the Slice chart Lemma and the fact that $T_pM \cong T_pU \cong T_{\varphi(p)}\varphi(U)$ for every open neighbourhood U of p and for any smooth chart φ

Solution. By Theorem 5.12 in Lee's book (proved in Lecture 5) since c is a regular value $S = F^{-1}(c)$ is a smoothly embedded submanifold of M. Let us denote by $\iota\colon S \hookrightarrow M$ the embedding. Fix a point p and charts (U, φ) around p, (V, ψ) around $c = F(p)$ such that

$$F|_{\varphi}^{\psi} : \varphi(U) \subseteq R^m \to \psi(V) \subseteq \mathbb{R}^n$$

is the standard submersion. These exist by the constant rank theorem and the definition of regular point. Then $\varphi(S \cap U) = \left\{x^{n+1} = \cdots = x^m = 0\right\}$ and $(S \cap U, \varphi|_S)$ is a slice chart for S and $\varphi(S \cap U) \xrightarrow{\varphi\circ\iota} \varphi(U)$ is the standard $m - n$ embedding. Using the hint,

$$T_pS \cong T_{\varphi(p)}\varphi(S \cap U) \cong \mathrm{Im}(D_{\varphi(p)}\iota)$$

where the last identification immediately follows from the local coordinate expression $\iota|^{\varphi}_{\varphi|_S}$ of ι. Now, since

$$F|^{\psi}_{\varphi} \circ \iota|\varphi|_S{}^{\varphi} = 0$$

it follows that

$$D_{\varphi(p)}F|^{\psi}_{\varphi} \circ D\iota_{\varphi(p)}|\varphi|_S{}^{\varphi} = 0$$

which implies

$$\mathrm{Im}(D_{\varphi(p)}\iota) \hookrightarrow \mathrm{Ker}(D_{\varphi(p)}F|^{\psi}_{\varphi}).$$

On the other hand, since $D_{\varphi(p)}F|^{\psi}_{\varphi}$ is surjective by hypothesis, the dimension of $\mathrm{Ker}(D_{\varphi(p)}F|^{\psi}_{\varphi}) = m - n$ which is also the dimension of S and thus of its tangent space. But an injective linear map between vector spaces of the same dimension is an isomorphism, so we are done.

Notice that also without choosing slice charts, assuming we already new that $S = F^{-1}(C)$ is an embedded submanifold of M with embedding $\iota \colon S \hookrightarrow M$, we would already know that $D_pF \circ D_p\iota = 0$ since by definition of S the composition $F \circ \iota = c$. □

Exercise 5.3. Show that the map $g : \mathbb{T}^2 \to \mathbb{R}^3$ given by

$$g([s,t]) = ((2 + \cos s)\cos t, (2 + \cos s)\sin t, \sin s)$$

is a smooth embedding of the 2-torus in $\mathbb{R}^3$.
(In this case the torus is defined as $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$.)

Solution. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the quotient map. We define the composite map $f = g \circ \pi : \mathbb{R}^2 \to \mathbb{R}^3$. Note that

$$f(s,t) = ((2 + \cos s)\cos t, (2 + \cos s)\sin t, \sin s).$$

Clearly f is smooth, therefore (by the previous exercise) g is smooth.

Let us show that g is an embedding. We first show that f is an immersion. This follows because for any point $p = (s,t)$, the vectors

$$D_pf(e_0) = \left.\frac{\partial f(s,t)}{\partial s}\right|_p = (-\sin(s)\cos t, -\sin s \sin t, \cos s)$$
$$D_pf(e_1) = \left.\frac{\partial f(s,t)}{\partial t}\right|_p = (-(2 + \cos s)\sin(t), (2 + \cos s)\sin t, 0)$$

are linearly independent. Since π is a surjective local diffeomorphism, it follows that g is an immersion. (Indeed, each point $q \in \mathbb{T}^2$ is of the form $q = \pi(p)$, with $p \in \mathbb{R}^2$. Differentiating the composite map $f = g \circ \pi$ at p we get

$$\mathrm{T}_pf = \mathrm{T}_qg \circ \mathrm{T}_p\pi,$$

and since T_pf is injective and $\mathrm{T}_p\pi$ is an isomorphism, we conclude that T_qg is injective as well.)

Finally, $g : \mathbb{T}^2 \to \mathbb{R}^3$ is a closed map because its domain is compact and its codomain is Hausdorff. Since g is injective, we conclude that g is a a topological embedding. □

Exercise 5.4 (To hand in). Show that the following subgroups of $GL_n(\mathbb{R})$ are closed submanifolds. Compute their dimension and their tangent space at the identity.

(a) The *special linear group* $\mathrm{SL}_n(\mathbb{R})$, consisting of matrices with determinant equal to 1.

(b) The *orthogonal group* $O_n(\mathbb{R})$, consiting of the orthogonal matrices A (which satisfy $A^\top A = I_n$).

Hint: Consider the map $f : M_n \to M_n^{sym}$ that sends $A \mapsto A^\top A$, there M_n^{sym} is the vector space of *symmetric* $n \times n$ matrices.

Exercise 5.5. If S_0, S_1 are smooth embedded submanifolds of M_0, M_1 respectively, then $S_0 \times S_1$ is a smooth embedded submanifold of $M_0 \times M_1$.

Solution. By hypothesis there exists embeddings $f_0 : L_0 \to M_0$ and $f_1 : L_1 \to M_1$ whose images are S_0 and S_1 respectively. The set $S_0 \times S_1$ is the image of the $\mathcal{C}^r$ map $f_0 \times f_1 : L_0 \times L_1 \to M_0 \times M_1$ that sends $(p_0, p_1) \mapsto (f_0(p_0), f_1(p_1))$. Thus it suffices to prove that $f_0 \times f_1$ is a $\mathcal{C}^r$ embedding.

We check first that $f_0 \times f_1$ is an immersion. For this, note that for each point $p = (p_0, p_1)$ the tangent transformation $D_p f_0 \times f_1$ sends $(v_0, v_1) \mapsto (D_{p_0} f_0(v_0), D_{p_1} f_1(v_1))$. (Here we are using the identification $T_{p_0,p_1}(L_0 \times L_1) \equiv D_{p_0} L_0 \times D_{p_1} L_1$.) This transformation is injective since both $D_{p_0} f_0$ and $D_{p_1} f_1$ are injective.

Finally, let us check that $f_0 \times f_1$ is a topological embedding. We know that each map $f_i|^{S_i}$ has a topological inverse $g_i : S_i \to L_i$. Thus the map $g_0 \times g_1 : S_0 \times S_1 \to L_0 \to L_1$ is an inverse of $(f_0 \times f_1)|^{S_0 \times S_1}$. This proves that $f_0 \times f_1$ is an homeomorphism onto its image $S_0 \times S_1$. We conclude that $f_0 \times f_1$ is a $\mathcal{C}^r$ embedding. $\square$

Exercise 5.6. (a) Show that a subset $S \subseteq \mathbb{R}^n$ is a smooth-embedded k-submanifold if each point $x \in S$ has an open neighborhood W such that the set $S \cap W$ is the graph of a smooth function that expresses some $n-k$ coordinates in terms of the remaining k coordinates. (More precisely, the function is of the form $f : U \subseteq \mathbb{R}^I \to \mathbb{R}^{I'}$, where I is a k-element subset of $n := \{0, \dots, n-1\}$, I' is its complement, and $U \subseteq \mathbb{R}^I$ is an open set.)

Solution. Let $x \in S$. By hypothesis there exists a k-element set $I \subseteq \{0, \dots, n-1\}$ (we assume w.l.o.g. $I = \{0, \dots, k-1\}$), an open set $W \subseteq \mathbb{R}^n$, an open set $U \subseteq \mathbb{R}^I$ and a $\mathcal{C}^r$ function $f : U \to \mathbb{R}^{I'}$ such that $S \cap W = \mathrm{Gra}_f$. Instead of the open set W, it is better to use the smaller open set $W' = W \cap (U \times \mathbb{R}^{I'})$. Note that this set contains the graph of f, therefore we still have $S \cap W' = \mathrm{Gra}_f$.

The set $S \cap W'$ is the image of the map $g : U \to W' : x \mapsto (x, f(x))$. This map is a $\mathcal{C}^r$ embedding because it is $\mathcal{C}^r$ and it admits a $\mathcal{C}^r$ retraction $W' \to U : (x, y) \mapsto x$. Therefore its image $\mathrm{Img}(g) = \mathrm{Gra}_f = S \cap W'$ is an embedded submanifold of W'. This proves that S fulfills the condition of being locally an embedded k-submanifold of $\mathbb{R}^n$. By a theorem of the course, we conclude that S is an embedded submanifold of $\mathbb{R}^n$. $\square$

(b) Let S be the set of real $m \times n$ matrices of rank k. Show that S is a smooth submanifold of $\mathbb{R}^{m \times n}$. What is its dimension ?

Hint: A rank-k matrix $A \in \mathbb{R}^{m \times n}$ has an invertible $k \times k$ submatrix $A|_{I \times J}$ (where $I \subseteq m$, $J \subseteq n$ are k-element sets). Show that the coefficients $A_{i',j'}$ with $i' \notin I$ and $j' \notin J$ can be expressed as a smooth function of the other coefficients of A.

Solution. For any pair of k-element sets $I \subseteq m$, $J \subseteq n$ we define an open set $U_{I,J} \subseteq \mathbb{R}^{m \times n}$ by

$$U_{I,J} = \{A \in \mathbb{R}^{m \times n} \mid \text{the } k \times k \text{ matrix } A|_{I \times J} \text{ is invertible}\},$$

where $A|_{I \times J} = (a_{i,j})_{i \in I, j \in J}$. Note that the sets $U_{I,J}$ cover S because every matrix of rank k has an invertible $k \times k$ submatrix.

Let us show that $S_{I,J} = S \cap U_{I,J}$ is the graph of a smooth function. For a matrix $A \in S_{I,J}$ we will show that the part $A|_{I' \times J'}$ of the matrix can be expressed as a function of the remaining coefficients. (Recall that $I' \subseteq m$ and $J' \subseteq n$ are the complements of I and J).

Since the column space of A has dimension k, and the k columns $A_{*,j}$ with $j \in J$ are linearly independent (because the block $A|_{I \times J}$ is invertible), these columns form a base of the column space. Hence any other column $A_{*,j'}$, with $j' \in J'$, is a linear combination of the columns $A_{*,j}$ with $j \in J$. That is, we can write

$$A_{*,j'} = \sum_{j\in J} A_{*,j}\, x_{j,j'},$$

using some real coefficients $(x_{j,j'})_{j\in J, j'\in J'}$. Thus we have

$$A_{i,j'} = \sum_{j\in J} A_{i,j}\, x_{j,j'} \quad \text{for } i \in n. \tag{1}$$

Using these equations just for $i \in I$ we can find out the coefficients $x_{j,j'}$ because the matrix $A|_{I\times J}$ is invertible. Denote its inverse by $B = (B_{l,i})_{l\in J, i\in I}$. (Note that B depends smoothly on A, this can be seen using the formula for the inverse matrix in terms of cofactors.) Multiplying equations (1) for $i \in I$ by the matrix B (that is, multiplying by the coefficient $B_{l,i}$ and summing over $i \in I$), we get

$$\sum_{i\in I} B_{l,i}\, A_{i,j'} = \sum_{i\in I}\sum_{j\in J} B_{l,i}\, A_{i,j}\, x_{j,j'} = \sum_{j\in J} \delta_{l,j}\, x_{j,j'} = x_{l,j'} \quad \text{for } l \in J,$$

or, renaming,

$$\sum_{i\in I} B_{j,i}\, A_{i,j'} = x_{j,j'} \quad \text{for } j \in J,\ j \in J'$$

Now that we know the value of the coefficients $x_{j,j'}$ we can replace in equation (1), this time restricted to the remaining values of i, that is, for $i \in I'$. Renaming the index i by i', we get

$$A_{i',j'} = \sum_{j\in J} A_{i',j}\, x_{j,j'} = \sum_{j\in J} A_{i',j} \sum_{i\in I} B_{j,i}\, A_{i,j'} \quad \text{for } i' \in I',\ j' \in J'.$$

Since B depends smoothly on A, this last equation shows that the $(m-k)(n-k)$ coefficients $A_{i',j'}$ with $i' \in I'$, $j' \in J'$ can be expressed as a smooth function of the remaining $k^2 + (m-k)k + k(n-k) = k(m+n-k)$ coefficients. In summary, for each open set $U_{I,J}$, the set $S \cap U_{I,J}$ is the graph of the smooth function

$$\begin{array}{ccc} GL_k \times \mathbb{R}^{(m-k)\times k} \times \mathbb{R}^{k\times(n-k)} & \to & \mathbb{R}^{(m-k)\times(n-k)} \\ (A|_{I\times J}, A|_{I'\times J}, A|_{I\times J'}) & \mapsto & (\sum_{j\in J}\sum_{i\in I} A_{i',j} B_{j,i}\, A_{i,j'})_{i'\in I',\, j'\in J'} \end{array}$$

where B is the inverse of $A|_{I\times J}$. Therefore S is a smoothly embedded $k(m+n-k)$-submanifold of $\mathbb{R}^{m\times n}$. □

Exercise 5.7. If M is connected and $f : M \to M$ is an idempotent smooth map ("idempotent" means that $f \circ f = f$), then $f(M)$ is an embedded submanifold of M.
Hint: Show that f has constant rank. Use what you know about a linear projector $P : V \to V$ and the complementary projector $\mathrm{id}_V - P$.

Solution. Unfortunately the place where this exercise was taken from has an incomplete solution, thus we will not follow the hint. We will give a more complicated solution that is suggested in `https://mathoverflow.net/questions/162552/idempotents-split-in-category-of-smooth-manifolds/162556#162556`.

We first record some facts that do not involve differentiability.

Lemma. If X is a topological space and $f : X \to X$ is an idempotent continuous map, then:

(a) The image $f(X)$ is the set of fixed points $\mathrm{fix}(X) = \{x \in X : f(x) = x\}$.
(b) In consequence, the image $f(X)$ is a closed subset of X.
(c) If X is connected, then $f(X)$ is connected.
(d) Every open neighborhood U of a point $p \in f(X)$ contains a smaller open neighborhood U' of p that is invariant by f, i.e. $f(U') \subseteq U'$.

Proof. (a) If $y \in f(X)$, we can write $y = f(x)$ for some $x \in X$, therefore $f(y) = f(f(x)) = f(x) = y$, thus $y \in \operatorname{fix}(f)$. Reciprocally, if $x \in \operatorname{fix}(f)$, then $f(x) = x$ and it is clear that $x \in f(X)$.

(b) follows from (a) since the equation $f(x) = x$ define a closed subset of M.

(c) is a general property of continuous maps.

(d) We define $U' = U \cap f^{-1}(U)$. We claim that U' is invariant by f. Indeed, take any point $x \in U'$. This means that both x and $f(x)$ are in U. Then the point $y = f(x)$ is in U' because both y and $f(y) = f(f(x)) = f(x) = y$ are in U. □

Now we solve the following local version of the problem.

Proposition. If $M \subseteq \mathbb{R}^n$ is an open set and $f : M \to M$ is an idempotent $\mathcal{C}^r$ map, then each point $p \in f(M)$ has an open neighborhood U such that $f(M) \cap U$ is a $\mathcal{C}^r$-embedded submanifold of U of dimension $k = \operatorname{rank}_p(f)$.

Proof. For a point $p \in f(M)$, the tangent operator $D_p f$ is a linear endomorphism of $D_p M = \mathbb{R}^n$ which satisfies

$$D_p f = D_p(f \circ f) = D_p f \circ D_p f.$$

Thus $D_p f$ is a linear projector in $\mathbb{R}^n$, and its image and kernel are complementary subspaces of $\mathbb{R}^n$ of dimensions k and $k' = n - k$.

Let $\pi = \operatorname{id}_{\mathbb{R}}^n - D_p f$ be the complementary projector of $D_p f$. (Check that π is also a linear projector and has $\operatorname{Ker}(\pi) = \operatorname{Img}(D_p f)$ and $\operatorname{Img}(\pi) = \operatorname{Ker}(D_p f)$.)

We may assume w.l.o.g that $\operatorname{Ker} D_p f = \mathbb{R}^{k'}$ and we consider π as a map $\mathbb{R}^n \to \mathbb{R}^{k'}$.

We define a map $g : M \to \mathbb{R}^{k'}$ that sends $x \mapsto \pi(x - f(x))$.

Note that $D_p g = \pi \circ (D_p f - \operatorname{id}_{\mathbb{R}^n}) = \pi \circ \pi = \pi$. Therefore g has rank k' and hence there is an open neighborhood W of p such that $g|_W : W \to \mathbb{R}^{k'}$ is a submersion. By the Lemma, we may assume that W is invariant by f.

By the regular preimage theorem, the set

$$S = \{q \in W : g(q) = 0\} = (g|_W)^{-1}(0),$$

is a k-submanifold of W.

Note that $f(M) \cap W = \operatorname{fix}(f|_W)$ is contained in S. However, it is not clear that all points of S are in $f(M)$.

Now, consider the $\mathcal{C}^r$ map $f|_W^S : W \to S$. Since f has rank k at p, and $\dim S = k$, we see that $f(W)$ contains an open neighborhood V' of p in S. We write $V' = V \cap S$, where V is an open set of M.

Let $U = W \cap V$. We claim that $f(M) \cap U$ is an embedded k-submanifold of U. In fact $f(M) \cap U = V'$. Indeed, if $x \in V' = V \cap S$, then $x \in f(W)$ (by definition of V') and it follows that $x \in W$, thus $x \in U = V \cap W$. We conclude that $x \in f(W) \cap U$. Reciprocally, if $x \in f(M) \cap U = f(M) \cap W \cap V$, we see that x is fixed by f, and also $x \in W$, so it follows that $x \in \operatorname{fix}(f|_W) \subseteq S$, thus $x \in S \cap V = V'$. This shows that $f(M) \cap U$ coincides with V', which is an open subset of S, which in turn is an embedded k-submanifold of W. Thus $f(M) \cap U$ is an embedded k-submanifold of U. □

Now we can solve the original problem. Let $f : M \to M$ be an idempotent $\mathcal{C}^r$ map, where M is a connected $\mathcal{C}^r$ manifold. We will show that $f(M) = \operatorname{fix}(f)$ is an embedded submanifold of M.

We first note that the Proposition holds for the manifold M even though M is not an open subset of $\mathbb{R}^n$.

Claim. Each point $p \in \operatorname{fix}(f) = f(M)$ has an open neighborhood U in M such that $\operatorname{fix}(f) \cap U$ is an embedded submanifold of U of dimension $k_p = \operatorname{rank}_p f$.

Proof. Proof: Take a chart (V, ϕ) that is defined at p. By the Lemma, we may assume that its domain V is f-invariant. Therefore the map $f|_V^V$ is an idempotent map $V \to V$. It follows that the local expression $\widetilde{f} = \phi \circ f \circ \phi^{-1}$ is an idempotent $\mathcal{C}^r$ map of the open set $\widetilde{V} = \phi(V) \subseteq \mathbb{R}^n$. In addition, the point $\widetilde{p} = \phi(p)$ is fixed by $\widetilde{f}$. By the Proposition, there is an open neighborhood $\widetilde{U}$ of $\widetilde{p}$ in $\widetilde{V}$ such that $\operatorname{fix}(\widetilde{f}) \cap \widetilde{U}$ is a k_p-submanifold of $\widetilde{U}$. Applying the diffeomorphism ϕ^{-1}, we get an open subset $U = \phi^{-1}(\widetilde{U})$ of M such that $\operatorname{fix}(f) \cap U$ is a k_p-submanifold of U. □

To finish showing that $\operatorname{fix}(f)$ is an embedded submanifold of M, we must show that the function $p \mapsto k_p = \operatorname{rank}_p f$ is constant throughout $f(M)$. Since $\operatorname{fix}(f) = f(M)$ is connected, it suffices to show that k_p is locally constant. But this follows from the claim. Indeed, if $p \in \operatorname{fix}(f)$ and U_p is an open neighborhood of p such that $\operatorname{fix}(f) \cap U_p$ is a k_p-submanifold of U_p, then for any point $q \in \operatorname{fix}(f) \cap U_p$ we have $k_q = k_p$, because applying the claim again we get an open set U_q such that $\operatorname{fix}(f) \cap U_q$ is a k_q submanifold, and then $\operatorname{fix}(f) \cap U_q \cap U_p$ is a submanifold of dimensions k_p and k_q at the same time. This manifold is nonempty because it contains the point q, therefore $k_p = k_q$.

□