

### Homework 9

[Please note that there is no Homework 8!]

**Exercise 1.** [Barker's algorithm]

Let  $\pi = (\pi_i, i \in S)$  be a distribution on a finite state space  $S$  such that  $\pi_i > 0$  for all  $i \in S$  and let us consider the base chain with transition probabilities  $\psi_{ij}$ , which is assumed to be irreducible, aperiodic and such that  $\psi_{ij} > 0$  if and only if  $\psi_{ji} > 0$ . Define the following acceptance probabilities:

$$a_{ij} = \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij} + \pi_j \psi_{ji}}$$

as well as a new chain with transition probabilities  $p_{ij} = \psi_{ij} a_{ij}$  if  $j \neq i$ . Show that this new chain is ergodic and that it satisfies the detailed balance equation:

$$\pi_i p_{ij} = \pi_j p_{ji}, \quad \forall i, j \in S$$

**Exercise 2.** Let  $\pi$  be a probability distribution  $S = \mathbb{N}^* = \{1, 2, 3, \dots\}$ . We assume that  $\pi_i > 0$  for every  $i \in S$  and moreover that  $\pi_i \geq \pi_{i+1}$  for every  $i \in S$ . In order to sample from  $\pi$ , let us consider the base chain with transition probabilities:

$$\psi_{1,2} = 1, \quad \psi_{i,i\pm 1} = \frac{1}{2}, \quad \text{for } i \geq 2$$

and  $\psi_{ij} = 0$  for all other values of  $i, j$  (*NB*: Does this chain satisfy the required assumptions?).

a) Compute the general expression for the acceptance probabilities  $a_{ij}$  and the transition probabilities  $p_{ij}$  of the corresponding Metropolis chain.

b) Consider then the following three particular cases (where the constants  $C_1, C_2, C_3$  are appropriate normalization constants):

1.  $\pi_i = C_1/i^2, i \geq 1$
2.  $\pi_i = C_2 \exp(-i), i \geq 1$
3.  $\pi_i = C_3 \exp(-i^2), i \geq 1$

In each case, compute the acceptance probabilities  $a_{ij}$ , as well as the limit  $\lim_{i \rightarrow \infty} a_{i,i+1}$ .

**Exercise 3.** Let  $n \geq 1, 0 < p < 1$ , and consider the binomial distribution on  $S = \{0, 1, \dots, n\}$  defined as

$$\pi_k = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{for } k \in S$$

Construct a base chain on  $S$ , as well as the corresponding Metropolis chain whose stationary and limiting distribution is  $\pi$  (simplifying as much as you can the expression for the acceptance probabilities).