Problem Set 6 **(not graded)**
For the Exercise Sessions on Nov 18

Last name	First name	SCIPER Nr	Points

Problem 1: MMSE Estimation

Consider the scenario where $p(x|d) = de^{-dx}$, for $x \geq 0$ (and zero otherwise), that is, the observed data x is distributed according to an exponential with mean $1/d$. Moreover, the desired variable d itself is also exponentially distributed, with mean $1/\mu$.

(a) Find the MMSE estimator of d given x, and calculate the corresponding mean-squared error incurred by this estimator.

(b) Find the MAP estimator of d given x.

Problem 2: Parameter Estimation and Fisher Information

The Fisher information $J(\Theta)$ for the family $f_\theta(x), \theta \in \mathbf{R}$ is defined by

$$J(\theta) = \mathbb{E}_\theta \left(\frac{\partial f_\theta(X)/\partial\theta}{f_\theta(X)} \right)^2 = \int \frac{(f_\theta')^2}{f_\theta}$$

Find the Fisher information for the following families:

(a) $f_\theta(x) = N(0, \theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}}$

(b) $f_\theta(x) = \theta e^{-\theta x}, x \geq 0$

(c) What is the Cramèr Rao lower bound on $\mathbb{E}_\theta(\hat{\theta}(X) - \theta)^2$, where $\hat{\theta}(X)$ is an unbiased estimator of θ for (a) and (b)?

Problem 3: Conditional Independence and MMSE

For simplicity, throughout this problem, **all random variables are assumed to be zero-mean.** *Remark:* You may directly skip to Part *(d)*, taking Equation (2) for granted (as a characterization of conditional independence for Gaussians).

(a) Show that if X and Y are conditionally independent given Z, then

$$\mathbb{E}[(X - \mathbb{E}[X|Z])(Y - \mathbb{E}[Y|Z])] = 0. \tag{1}$$

(b) Recall that if X and Y are jointly Gaussian (zero-mean), then we have $Y = \alpha X + W$, for some constant α, where W is zero-mean Gaussian *independent of* X. Use this to prove the well-known fact that for jointly Gaussian X and Y, if $\mathbb{E}[XY] = 0$, then X and Y are independent. *Hint:* Simply plug in.

(c) Let X, Y, Z be jointly Gaussian (and zero-mean, as throughout this problem). Prove that if

$$\mathbb{E}[(X - \mathbb{E}[X|Z])(Y - \mathbb{E}[Y|Z])] = 0, \tag{2}$$

then X and Y are conditionally independent given Z. *Hint:* Make sure to solve Part *(b)* first. Recall that for three jointly Gaussians X, Y, Z, we can always write $Y = \gamma X + \delta Z + V$, for some constants γ and δ, where V is Gaussian and independent of X and Z.

(d) Let X, Y, Z be jointly Gaussian (and zero-mean, as throughout this problem). Recall that we can write $Z = \alpha X + \beta Y + W$, for some constants α and β, where W is Gaussian of some appropriate variance σ_W^2, independent of X and Y. Formulate a necessary and sufficient condition on the triple $(\alpha, \beta, \sigma_W^2)$ such that X and Y are conditionally independent given Z.

(e) Continuing from Part *(d),* let us now restrict to $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$, and use the notation $\rho = \mathbb{E}[XY]$. This means that we can restrict to $|\alpha| \leq 1$ and $|\beta| \leq 1$. Moreover, let us always select σ_W^2 such that $\mathbb{E}[Z^2] = 1$ (unique choice). Find the unique choice of (α, β) that attains the maximum in the estimation problem

$$\max_{\alpha,\beta} \min_f \mathbb{E}[(Z - f(X, Y))^2], \tag{3}$$

where the inner minimum is over all measurable functions $f(x, y)$.

Hint: It may be useful to introduce the notation $a = \mathbb{E}[XZ]$ and $b = \mathbb{E}[YZ]$.

Problem 4: Missing Data

We are given real-valued data with a single missing sample :

$$X_1, X_2, X_3, X_4, X_5, X_6, ?, X_8, X_9, \ldots \tag{4}$$

where we assume that the data is wide-sense stationary with autocorrelation function $R_X[k] = \alpha^{|k|}$, where $0 < \alpha < 1$. We would like to find a meaningful estimate for the missing sample X_7.

1. As a starting point, let us consider the estimate $\hat{X}_7 = wX_6$, where w is a real number. Find the value of w so as to minimize the mean-squared error $\mathbb{E}[(X_7 - \hat{X}_7)^2]$, and determine the incurred mean-squared error.

2. Now, consider the estimate $\hat{X}_7 = w_1 X_6 + w_2 X_8$. Again, find the values of w_1 and w_2 so as to minimize the mean-squared error $\mathbb{E}[(X_7 - \hat{X}_7)^2]$, and determine the incurred mean-squared error.

Problem 5: FIR Wiener Filter

Consider a (discrete-time) signal that satisfies the difference equation $d[n] = 0.5d[n-1]+v[n]$, where $v[n]$ is a sequence of uncorrelated zero-mean unit-variance random variables. We observe $x[n] = d[n] + w[n]$, where $w[n]$ is a sequence of uncorrelated zero-mean random variables with variance 0.5.

(a) (you may skip this at first and do it later — it is conceptually straightforward) Show that for this signal model, the autocorrelation function of the signal $d[n]$ is

$$\mathbb{E}[d[n]d[n+k]] = \frac{4}{3}\left(\frac{1}{2}\right)^{|k|}, \tag{5}$$

and thus the autocorrelation function of the signal $x[n]$ is

$$\mathbb{E}[x[n]x[n+k]] = \begin{cases} \frac{11}{6}, & \text{for } k=0, \\ \frac{4}{3}\left(\frac{1}{2}\right)^{|k|}, & \text{otherwise.} \end{cases} \tag{6}$$

(b) We would like to find an (approximate) linear predictor $\hat{d}[n+3]$ using only the observations $x[n], x[n-1], x[n-2], \ldots, x[n-p]$. Using the Wiener Filter framework, determine the optimal coefficients for the linear predictor. Find the corresponding mean-squared error for your predictor.

(c) We would like to find a linear denoiser $\hat{d}[n]$ using *all* of the samples $\{x[k]\}_{k=-\infty}^{\infty}$. Find the filter coefficients and give a formula for the incurred mean-squared error.

Problem 6: Tweedie's Formula

For the special case where $X = D + N$, where N is Gaussian noise of mean zero and variance σ^2, *Tweedie's formula* says that the conditional mean (that is, the MMSE estimator) can be expressed as

$$\mathbb{E}\left[D | X = x\right] = x + \sigma^2 \ell'(x), \tag{7}$$

where

$$\ell'(x) = \frac{d}{dx}\log f_X(x), \tag{8}$$

where $f_X(x)$ denotes the marginal PDF of X. In this exercise, we derive this formula.

(a) Assume that $f_{X|D}(x|d) = e^{\alpha dx - \psi(d)} f_0(x)$ for some functions $\psi(d)$ and $f_0(x)$ and some constant α (such that $f_{X|D}(x|d)$ is a valid PDF for every value of d). Define

$$\lambda(x) = \log \frac{f_X(x)}{f_0(x)}, \tag{9}$$

where $f_X(x)$ is the marginal PDF of X, i.e., $f_X(x) = \int f_{X|D}(x|\delta) f_D(\delta) d\delta$. With this, establish that

$$\mathbb{E}\left[D | X = x\right] = \frac{1}{\alpha}\frac{d}{dx}\lambda(x). \tag{10}$$

(b) Show that the case where $X = D + N$, where N is Gaussian noise of mean zero and variance σ^2, is indeed of the form required in Part *(a)* by finding the corresponding $\psi(d), f_0(x)$, and α. Show that in this case, we have

$$\frac{f_0'(x)}{f_0(x)} = -\frac{x}{\sigma^2}, \tag{11}$$

and use this fact in combination with Part *(a)* to establish Tweedie's formula.