Midterm Exam

Surname: .... First name: .... Section: ....

PLEASE JUSTIFY ALL YOUR ANSWERS !!!

Exercise 1. (25 points)

Let $0 < p, q, \alpha < 1$, and let X be a Markov chain with state space $\mathcal{S} = \{0, ..., N\}$ and transition matrix P given by

$$p_{01} = 1 - p_{00} = p \quad , \quad p_{N0} = 1 - p_{NN} = q$$

and

$$p_{ij} = \begin{cases} q & \text{if } j = 0 \\ \alpha(1-q) & \text{if } j = i+1 \\ (1-\alpha)(1-q) & \text{if } j = i \end{cases}$$

for $0 < i < N$.

a) Stationary distribution:

a1) Prove that the chain X admits a unique stationary distribution (without computing it).

a2) Compute this stationary distribution π as a function of p, q, and α.

Hint: Defining $\beta = \frac{\alpha(1-q)}{q+\alpha(1-q)}$ allows simplifying the notations.

a3) Are π_0 and π_N decreasing, increasing or constant with respect to α? (for p and q fixed)

Note: For this last question, you might try intuitive arguments if you did not complete the former computations.

b) Expected arrival time: For $i, j \in \mathcal{S}$, let us define

$$T_j = \inf\{n \geq 1 : X_n = j\} \quad \text{and} \quad \mu_{ij} = \mathbb{E}(T_j | X_0 = i)$$

b1) Compute μ_{NN}.

b2) Express a relation between μ_{NN} and μ_{0N}, and deduce the value of μ_{0N}.

Hint: Start by writing $\mu_{NN} = \mathbb{E}(T_N | X_0 = N) = \sum_{n \geq 1} n\, \mathbb{P}(T_N = n | X_0 = N) = \ldots$ and then follow a procedure similar to what was already done in the course / in some exercises.

b3) Is μ_{0N} decreasing, increasing or constant with respect to α? (for p and q fixed)

Note: Again, for this last question, you might try intuitive arguments if you did not complete the former computations.

Exercise 2. (15 points)

Let $0 < p, q < 1$ be such that $p + q = 1$ and let X be a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3, 4\}$ and transition matrix P given by

$$P = \frac{1}{2} \begin{pmatrix} 0 & p & q & q & p \\ p & 0 & p & q & q \\ q & p & 0 & p & q \\ q & q & p & 0 & p \\ p & q & q & p & 0 \end{pmatrix}$$

a) Prove that the chain X is ergodic and compute its stationary distribution π. Is detailed balance satisfied?

b) Compute the spectral gap γ of the chain X as a function of the parameter $0 < p < 1$.

c) For what value(s) of p is the spectral gap maximal? What is then the value of γ? (please provide a numerical value!)

d) For the value of γ found in c), deduce an upper bound on $T_\varepsilon = \inf\{n \geq 1 : \|P_0^n - \pi\|_{\text{TV}} \leq \varepsilon\}$.

Hints for part b):

- If $A = \text{circ}(c_0, c_1, \ldots, c_{N-1})$ is an $N \times N$ circulant matrix, then its eigenvalues are given by

$$\lambda_k = \sum_{j=0}^{N-1} c_j \exp(2\pi i j k / N) \quad k = 0, \ldots, N-1$$

(please note that with this notation, the eigenvalues λ_0, λ_1, etc. are not ordered.)

- We also have the following trigonometric equalities:

$$\cos(-x) = \cos(-x) \qquad \cos(\pi - x) = -\cos(x) \qquad \cos(\pi/5) = \frac{\sqrt{5}+1}{4} \qquad \cos(2\pi/5) = \frac{\sqrt{5}-1}{4}$$

Exercise 3. (20 points) The following are short "quiz problems" that do not require calculations, but only short answers (*with* justifications).

Quiz 3.1: Let P_1 and P_2 be $N_1 \times N_1$ and $N_2 \times N_2$ stochastic matrices (we assume $N_1, N_2 \geq 3$). Let a_M denote the M dimensional column vector with all components equal to $a \geq 0$ and $0_{P\times Q}$ the $P \times Q$ all-zero matrix. Consider the following transition matrix:

$$P = \begin{pmatrix} P_1 & 0_{N_1} & 0_{N_1\times N_2} \\ \frac{1}{4N_1}1_{N_1}^T & \frac{1}{2} & \frac{1}{4N_2}1_{N_2}^T \\ 0_{N_2\times N_1} & 0_{N_2} & P_2 \end{pmatrix} \tag{1}$$

We will assume throughout that the matrix P_1 defines an irreducible, aperiodic chain.

Hint: It is a good idea to picture the state graph and to separate the cases N_2 even and odd.

a) Let the matrix P_2 define the *circular* symmetric random walk with N_2 states (in particular there are no self-loops). Give all equivalence classes of the chain with transition matrix P. Fully characterize each equivalence class: say if it is transient, null-recurrent or positive-recurrent / periodic or aperiodic / ergodic.

b) Does there exist a stationary distribution for the chain defined by P? If yes, is it unique? If it is not unique, describe the structure of the whole set of stationary distributions.

Quiz 3.2: For the following two processes, justify whether the process $(Y_n, n \in \mathbb{N})$ is Markov or not, and if it is a Markov chain, determine if it is ergodic or not.

a) Consider a random walk on the set $\{-1, 0, 1\}$ with transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Let X_n be the position of this random walk at time n. The process $(Y_n, n \in \mathbb{N})$ is defined as $Y_0 = X_0$ and $Y_n = X_n - X_{n-1}$ for $n \geq 1$.

b) Consider a sequence of i.i.d. random variables $X_1, \ldots, X_n$ such that

$$\mathbb{P}(X_1 = 0) = \frac{1}{4},\ \mathbb{P}(X_1 = 1) = \frac{1}{2},\ \mathbb{P}(X_1 = 2) = \frac{1}{4}$$

The process $(Y_n, n \in \mathbb{N})$ is defined as $Y_0 = 0$, $Y_n = \max\{X_1, \ldots, X_n\}$ for $n \geq 1$.

Quiz 3.3: For each statement below, tell whether it is true or false, and provide a justification if the answer is "true / a counter-example if the answer is "false.

a) Consider an irreducible Markov chain and let $i \neq j$ be two states in this chain. Then there exists $n \in \mathbb{N}$ such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(n)} > 0$.

b) Let P be the transition matrix of a Markov chain. If $P^n \to I$, then all states are recurrent.

Note: A sequence $\{A_n\}_{n\in\mathbb{N}}$ of matrices converges to the matrix A, which is denoted by $A_n \to A$, if $(A_n)_{ij} \to A_{ij}$ as $n \to \infty$, for all i, j.