Midterm Exam: Solutions

1. a1) The chain has a finite number of states, so to prove the existence of a unique stationary distribution, as well as its uniqueness, it is sufficient to show that it is also irreducible. To prove irreduciblity, we show that state 0 communicates with every state $i \in \{1, ..., N\}$ by noting that

$$p_{i0}^{(1)} = q > 0 \quad \text{and} \quad p_{0i}^{(i)} = p\alpha^{i-1}(1-q)^{i-1} > 0$$

a2) First, by using the definition of the stationary distribution for state 0 we have

$$\pi_0 = \sum_{i=0}^{N} \pi_i p_{i0} = (1-p)\pi_0 + q\sum_{i=1}^{N} \pi_i = (1-p)\pi_0 + q(1-\pi_0)$$

which implies that

$$\pi_0 = \frac{q}{p+q}$$

Next, we have

$$\pi_1 = \pi_0\ pv + \pi_1\ (1-\alpha)\ (1-q) \quad \text{so} \quad \pi_1 = \frac{p}{q+\alpha(1-q)}\,\pi_0$$

For $1 < j < N$, we obtain

$$\pi_j = \pi_{j-1}\ \alpha\ (1-q) + \pi_j\ (1-\alpha)\ (1-q) \quad \text{so} \quad \pi_j = \frac{\alpha(1-q)}{q+\alpha(1-q)}\,\pi_{j-1}$$

And for $j = N$:

$$\pi_N = \pi_{N-1}\ \alpha\ (1-q) + \pi_N\ (1-q) \quad \text{so} \quad \pi_N = \frac{\alpha(1-q)}{q}\,\pi_{N-1}$$

Finally, defining $\beta = \frac{\alpha(1-q)}{q+\alpha(1-q)}$, we have for $0 < j < N$:

$$\pi_j = \beta^{j-1}\,\frac{1}{q+\alpha(1-q)}\,\frac{pq}{p+q} \quad \text{and} \quad \pi_N = \beta^{N-1}\,\frac{p}{p+q}$$

[Note that π_0 can also be found at the end by using the normalization condition $\sum_{j=0}^{N} \pi_j = 1$.]

a3) π_0 does not depend on α (a possible intuition for this is that if we consider the set of states $\{1, \ldots, N\}$ as a "super-state", then the chain simplifies to a two-state chain with transition probabilities p in one direction and q in the other direction). On the contrary, as β is an increasing function of α, we deduce that π_N is also an increasing function of α, which is a sensible result, as the probability to move up in the chain grows with α.

1. b1) By the theorem seen in class, $\mu_{NN} = \mu_N = \frac{1}{\pi_N} = \beta^{-N+1}\,\frac{p+q}{p}$.

b2) Following the hint, we expand the expectation as

$$\mu_{NN} = \mathbb{E}(T_N|X_0 = N) = \sum_{n=1}^{\infty} n\,\mathbb{P}(T_N = n|X_0 = N)$$

which we can rewrite as (inserting the events $X_1 = j$ for $0 \le j \le N$)

$$\begin{aligned}
\mu_{NN} &= \sum_{n=1}^{\infty}\sum_{j=0}^{N} n\,\mathbb{P}(T_N = n, X_1 = j|X_0 = N) \\
&= \sum_{j=0}^{N} p_{Nj} \sum_{n=1}^{\infty} n\,\mathbb{P}(T_N = n|X_1 = j) \\
&= p_{NN} + p_{N0} \sum_{n=2}^{\infty} n\,\mathbb{P}(T_N = n|X_1 = 0) \\
&= p_{NN} + p_{N0} \sum_{m=1}^{\infty} (1+m)\,\mathbb{P}(T_N = m|X_0 = 0) \\
&= p_{NN} + p_{N0} + p_{N0}\,\mu_{0N} = 1 + q\,\mu_{0N}
\end{aligned}$$

b3) Using the two results, we obtain:

$$\mu_{0N} = \beta^{-N+1}\,\frac{p+q}{pq} - \frac{1}{q}$$

It is therefore decreasing with respect to β, and also decreasing with respect to α. Indeed, when α is the largest, then the chain has the highest chance to reach quickly N starting from state 0.

2. a) The chain has a finite number of states and is clearly irreducible and aperiodic, so the chain is ergodic. In addition, P is doubly stochastic, so the corresponding stationary distribution π is uniform. P is also symmetric, which implies in this case that detailed balance is satisfied.

b) The computation of the eigenvalues gives $\lambda_k = p\,\cos(2\pi k/5) + q\,\cos(4\pi k/5)$ for $k \in \{0, \dots, 4\}$, so $\lambda_0 = 1$,

$$\lambda_1 = \lambda_4 = p\,\cos(2\pi/5) + q\cos(4\pi/5) = p\,\frac{\sqrt{5}-1}{4} - q\,\frac{\sqrt{5}+1}{4} = (2p-1)\,\frac{\sqrt{5}}{4} - \frac{1}{4}$$

and

$$\lambda_2 = \lambda_3 = p\,\cos(4\pi/5) + q\cos(8\pi/5) = -p\,\frac{\sqrt{5}+1}{4} + q\,\frac{\sqrt{5}-1}{4} = (1-2p)\,\frac{\sqrt{5}}{4} - \frac{1}{4}$$

The spectral gap γ is therefore given by

$$\gamma = \frac{3}{4} - |2p-1|\,\frac{\sqrt{5}}{4}$$

c) From part b), it is clear that γ is maximal and takes the value $\frac{3}{4}$ when $p = q = \frac{1}{2}$ (and this is also intuitively the situation where the chain with transition matrix P is the most mixing).

d) By the bound given in class, we have: $\|P_0^n - \pi\|_{\text{TV}} \leq \frac{\sqrt{5}}{2}\exp(-3n/4)$, so

$$T_\varepsilon \leq \frac{4}{3}\log\left(\frac{\sqrt{5}}{2\varepsilon}\right)$$

3.1.a) The state space consists of $\mathcal{S}_1 \cup \{0\} \cup \mathcal{S}_2$ with $\mathcal{S}_1$ containing N_1 states and $\mathcal{S}_2$ containing N_2 states.

Irreducible equivalence classes are $\mathcal{S}_1$, $\{0\}$ and $\mathcal{S}_2$.

The class $\{0\}$ is always transient.

The class $\mathcal{S}_1$ is always irreducible, aperiodic and finite so positive-recurrent and hence also ergodic. For the class $\mathcal{S}_2$ it is always irreducible and finite hence always positive-recurrent.

If N_2 is even it is periodic of period 2 hence not ergodic.

If N_2 is odd it is aperiodic and hence also ergodic.

3.1.b) Let $\pi^{(1)}$ be the stationary distribution of P_1 (since P_1 is irreducible and finite hence pos-rec the stationary distr exists). Let $\pi^{(2)}$ the stationary distribution of P_2 (same justification for existence). We thus have the stationary distribution $\pi_0 = 0$ and $\pi_i = \alpha\pi_i^{(1)}$ for $i \in \mathcal{S}_1$ $\pi_i = (1-\alpha)\pi_i^{(2)}$ for $i \in \mathcal{S}_2$ with any $0 \leq \alpha \leq 1$. It is not unique.

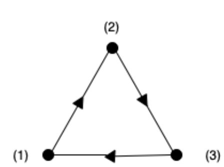
Another way to express this is to define $\tilde{\pi}_i^{(1)} = \pi_i^{(1)}$ for $i \in \mathcal{S}_1$ and $\tilde{\pi}_i^{(1)} = 0$ for $i \notin \mathcal{S}_1$, $\tilde{\pi}_i^{(2)} = \pi_i^{(2)}$ for $i \in \mathcal{S}_2$ and $\tilde{\pi}_i^{(2)} = 0$ for $i \notin \mathcal{S}_2$, and say that $\pi = \alpha\tilde{\pi}^{(1)} + (1-\alpha)\tilde{\pi}^{(2)}$ the convex combination.

3.2.a) The process $(Y_n, n \in \mathbb{N})$ is not a Markov chain. Y_n indicates the move to the left, right, or no move. Since the random walk is on a bounded set, the number of consecutive moves to the right cannot exceed a certain number.

$$P(Y_n = 1 | Y_{n-1} = 1, Y_{n-2} = 1) = 0 \neq P(Y_n = 1 | Y_{n-1} = 1, Y_{n-2} = 0)$$

3.2.b) The process $(Y_n, n \in \mathbb{N})$ is a Markov chain since Y_n can be determined only by Y_{n-1}, X_n. But, it is not an ergodic chain because the chain is not irreducible ($Y = 2$ is an absorbing state).

3.3.a) False. If the chain is periodic, the statement does not hold for all i, j. For example, in the following chain $p_{12}^{(n)}$ is positive only if n is of the form $3k+1$, but $p_{21}^{(n)}$ is positive only if n is of the form $3k+2$.



3.3.b) True. Suppose that there exists a transient state, i. Then $\sum_n p_{ii}^{(n)} < \infty$ so $p_{ii}^{(n)} \to 0$, but by the assumption made, $p_{ii}^{(n)} \to 1$.

NB: The assumption made actually *implies* that $P = I$, as $P^n P = P^{n+1}$, to taking the limit $n \to \infty$ on both sides, we get $P = I$.