Exercise Series 13 - Exterior derivative, Properties of Integrals 2022–12–13

Exercise 13.1 (Properties of the integral)**.** Let M be an oriented differentiable n-manifold and let ω, η be two continuous, compactly supported n-forms on M. Prove the following:

(a) Linearity: If $a, b \in \mathbb{R}$, then

$$\int_M (a\,\omega + b\,\eta) = a\int_M \omega + b\int_M \eta.$$

(b) Positivity: If $\operatorname{sgn}(\omega|_p)$ coincides with the orientation of M at every point $p \in M$ where $\omega|_p \neq 0$, then $\int_M \omega \geq 0$, and the inequality is strict unless ω is identically zero.

(c) Diffeomorphism invariance: If $f : N \to M$ is an diffeomorphism of constant sign $\operatorname{sgn}(f) = \pm 1$ (i.e. f is either orientation preserving or orientation reversing), then

$$\int_N f^*\omega = \operatorname{sgn} f \cdot \int_M \omega.$$

(d) Orientation reversal: If $-M$ denotes M with the reversed orientation, then

$$\int_{-M} \omega = -\int_M \omega.$$

Exercise 13.2. Prove that a continuous k-form is determined by the value of its integrals (Proposition 7.3.12). ***Hint:*** Use a chart to move the problem to $\mathbb{R}^n$, then integrate on small pieces of coordinate planes.

Exercise 13.3.* Let $f : M \to N$ be a smooth map between smooth manifolds. Then for all $\omega \in \Omega^k(M)$ we have

$$f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega).$$

Exercise 13.4.* Let (x, y, z) be the standard coordinates on $\mathbb{R}^3$ and let (v, w) be the standard coordinates on $\mathbb{R}^2$. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as $\phi(x, y, z) = (x + z, xy)$. Let $\alpha = e^w\,\mathrm{d}v + v\,\mathrm{d}w$ and $\beta = v\ \mathrm{d}v \wedge \mathrm{d}w$ be 2-forms on $\mathbb{R}^2$. Compute the following differential forms:

$$\alpha \wedge \beta, \quad \phi^*(\alpha), \quad \phi^*(\beta), \quad \phi^*(\alpha) \wedge \phi^*(\beta).$$

Exercise 13.5.* Compute the exterior derivative of the following forms:

(a) on $\mathbb{R}^2 \setminus \{0\}$ $\theta = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}$.

(b) on $\mathbb{R}^3$, $\varphi = \cos(x)\,\mathrm{d}y \wedge \mathrm{d}z$.

(c) on $\mathbb{R}^3$ $\omega = A\,\mathrm{d}x + B\,\mathrm{d}y + C\,\mathrm{d}z$.