Solutions to Homework 1

Exercise 1. a) $\mathcal{F} = \{\emptyset, \{2\}, \{5\}, \{1,3\}, \{2,5\}, \{4,6\}, \{1,2,3\}, \{1,3,5\}, \{2,4,6\}, \{4,5,6\}, \{1,2,3,5\}, \{1,3,4,6\}, \{2,4,5,6\}, \{1,2,3,4,6\}, \{1,3,4,5,6\}, \{1,2,3,4,5,6\}\}$ ($16 = 2^4$ elements)

b) atoms of $\mathcal{F}$: $\{1,3\}, \{2\}, \{4,6\}, \{5\}$. Notice that one also has $\mathcal{F} = \sigma(\{1,3\}, \{2\}, \{4,6\}, \{5\})$, as already mentioned in the problem set.

c) Nearly by definition, $\sigma(X_1, X_2) = \sigma(\{1,2,3\}, \{1,3,5\}) = \mathcal{F}$. Besides, the random variable Y satisfies: $Y(1) = Y(3) = 2$, $Y(2) = Y(5) = 1$ and $Y(4) = Y(6) = 0$. We deduce from there that the atoms of $\sigma(Y)$ are $\{1,3\}$, $\{2,5\}$ and $\{4,6\}$, and therefore that Y contains less information than X_1, X_2, i.e., that $\sigma(Y) \subset \sigma(X_1, X_2)$ and $\sigma(Y) \neq \sigma(X_1, X_2)$.

Exercise 2. Here is a systematic but not necessarily optimal procedure, described in words.

Consider first the list of subsets $\mathcal{L} = \{A_1, \ldots, A_m, A_1^c, \ldots, A_m^c\}$. From there, generate the list $\mathcal{L}' = \{B_1, \ldots, B_p\}$ made of all possible intersections of elements of $\mathcal{L}$ (which are *subsets* of Ω). Of course, this new list is not necessarily made of atoms of $\mathcal{F}$ only. We need to browse the collection and at each item, call it G, we discard it if it is empty or if there exists another element F in the collection such that $F \neq \emptyset$, $F \subset G$ and $F \neq G$. The remaining elements are the atoms of $\mathcal{F}$.

Exercise 3. a) The atoms of $\mathcal{F}$ are the singletons $\{x\}$, with $x \in [0,1]$.

b) The answer is no. One can check indeed that the σ-field generated by the sets $\{x\}$, $x \in [0,1]$ is the list of all countable subsets of $[0,1]$, as well as all the complements of countable subsets of $[0,1]$, which is of course not equal to the list of all Borel subsets of $[0,1]$. In particular, the open intervals are not in the list.

c) $\sigma(\{x\}, x \in [0,1])$ comprises all countable unions of singletons in $[0,1]$, as well as all the complements of these sets. One can check that indeed, such a collection of sets is a σ-field, which is moreover *much* smaller than $\mathcal{B}([0,1])$.

Exercise 4. a) $\sigma(X_1) = \sigma(\{(i,j) : j \in \{1, \ldots, 6\}\}, i \in \{1, \ldots, 6\})$. Atoms of $\sigma(X_1)$ are therefore of the form $\{(i,1), (i,2), \ldots, (i,6)\}$. The σ-field generated by these atoms is simply the collection of all possible unions of them. Similarly, $\sigma(X_2) = \sigma(\{(i,j) : i \in \{1, \ldots, 6\}\}, j \in \{1, \ldots, 6\})$.

b) The answer is no. For example, take $B = \{12\}$. Then

$$\{\omega \in \Omega : X_1(\omega) + X_2(\omega) \in B\} = \{(i,j) \in \Omega : X_1(i,j) + X_2(i,j) = 12\}$$
$$= \{(i,j) \in \Omega : i + j = 12\} = \{(6,6)\},$$

which is in neither of the two lists described above.

$X_1 + X_2$ is on the contrary measurable with respect to $\sigma(X_1, X_2)$, which is nothing but $\mathcal{F} = \mathcal{P}(\Omega)$.

Exercise 5. a) Consider e.g. X_1 taking values in $\{0,1\}$ and X_2 taking values in $\{0,2\}$. Then it is possible to deduce the values of both X_1 and X_2 from the sole value of Y, so $\sigma(Y) = \sigma(X_1, X_2)$ (as an exercise, write this down formally).

b) Consider e.g. X_1 taking values in $\{3, 5\}$ and X_2 taking values in $\{7, 9\}$. When $Y(\omega) = 12$, it is impossible to tell whether $X_1(\omega) = 3, X_2(\omega) = 9$ or $X_1(\omega) = 5, X_2(\omega) = 7$. The random variable Y carries then less information than the two random variables X_1, X_2 together (again, as an exercise, write this down formally).

c) The answer is no, i.e., $\sigma(Y) \neq \sigma(X_1, X_2)$, as when $Y(\omega) = a + b$, we will not be able to tell whether $\omega = \omega_1$ or $\omega = \omega_2$.

Exercise 6*. a) Use $B = A \cup (B \backslash A)$, where A and $B \backslash A$ are disjoint, as well as $\Omega = A \cup A^c$ and $\mathbb{P}(\Omega) = 1$.

b) Use $A \cup B = A \cup (B \backslash (A \cap B))$ where A and $B \backslash (A \cap B)$ are disjoint, as well as a).

c) Use $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$, where $B_n = A_n \backslash (A_1 \cup \ldots \cup A_{n-1})$; the B_n are disjoint, so by axiom (ii) and a),

$$\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \mathbb{P}(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mathbb{P}(B_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

d) $\mathbb{P}(\cup_{n\geq 1} A_n) = \mathbb{P}(\cup_{n\geq 1}(A_n \cap A_{n-1}^c)) \overset{(*)}{=} \sum_{n=1}^{\infty} \mathbb{P}(A_n \cap A_{n-1}^c) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_i \cap A_{i-1}^c)$ $\overset{(**)}{=} \lim_{n\to\infty} \mathbb{P}(\cup_{i=1}^{n}(A_i \cap A_{i-1}^c)) = \lim_{n\to\infty} \mathbb{P}(\cup_{i=1}^{n} A_i) = \lim_{n\to\infty} \mathbb{P}(A_n)$, where $(*)$, $(**)$ follow from the fact that the sets $A_n \cap A_{n-1}^c$ are disjoint.

e) Using parts a) and d): $\mathbb{P}(\cap_{n\geq 1} A_n) = 1 - \mathbb{P}((\cap_{n\geq 1} A_n)^c) = 1 - \mathbb{P}(\cup_{n\geq 1} A_n^c) = 1 - \lim_{n\to\infty} \mathbb{P}(A_n^c) = \lim_{n\to\infty} \mathbb{P}(A_n)$.