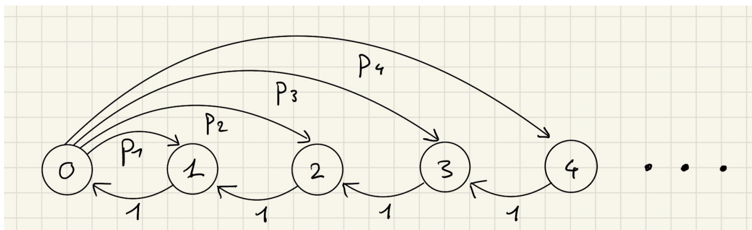
Final exam: Solutions

Exercise 1. (20 points) Let $(p_j,\, j \geq 1)$ be a sequence of non-negative numbers such that $\sum_{j\geq 1} p_j = 1$. Let also $(X_n,\, n \geq 0)$ be a time-homogeneous Markov chain with state space $S = \mathbb{N} = \{0, 1, 2, 3, \dots\}$ and transition matrix P represented by the following transition graph:



i.e., $p_{0j} = p_j$ and $p_{j,j-1} = 1$ for every $j \geq 1$ (and all other terms in the matrix P are equal to 0).

a) Under what minimal condition on the sequence $(p_j,\, j \geq 1)$ is the chain X irreducible?

Answer: It is necessary and sufficient that there is an infinite number of values of $j \geq 1$ such that $p_j > 0$.

b) Under what minimal condition on the sequence $(p_j,\, j \geq 1)$ is the chain X aperiodic?

Answer: It is necessary and sufficient that $\gcd\{k \geq 2 \,:\, p_{k-1} > 0\} = 1$.

Let us assume from now on that $p_j > 0$ for every $j \geq 1$. (*Hint for the above two questions*: Under this condition, the chain X is irreducible and aperiodic).

c) Show that under this assumption, the chain X is always recurrent.

Hint: Let $T_0 = \inf\{n \geq 1 \,:\, X_n = 0\}$ be the first return time to state 0.
Compute $f_{00}^{(n)} = \mathbb{P}(T_0 = n | X_0 = 0)$ for $n \geq 1$ and $f_{00} = \sum_{n\geq 1} f_{00}^{(n)}$.

Answer: Under the assumption made, $f_{00}^{(1)} = 0$ and $f_{00}^{(n)} = p_{n-1}$ for $n \geq 2$, so

$$f_{00} = \sum_{n\geq 1} f_{00}^{(n)} = \sum_{n\geq 2} p_{n-1} = 1$$

therefore X is recurrent.

d) Under what minimal condition on the sequence $(p_j,\, j \geq 1)$ is the chain X also positive-recurrent?

Hint: Compute $\mathbb{E}(T_0 | X_0 = 0)$.

Answer: Let us compute:

$$\mathbb{E}(T_0 | X_0 = 0) = \sum_{n\geq 1} n\, f_{00}^{(n)} = \sum_{n\geq 2} n\, p_{n-1} = \sum_{k\geq 1} (k+1)\, p_k = 1 + \sum_{k\geq 1} k\, p_k$$

so this sum should be finite in order for the chain X to be positive-recurrent (and this is the minimal condition).

e) Under the condition found in part d), compute the stationary distribution π of the chain X. Is it also a limiting distribution? Is detailed balance satisfied?

Answer: By d), $\pi_0 = 1/\mathbb{E}(T_0|X_0 = 0) = 1/\left(1 + \sum_{k\geq 1} k\, p_k\right)$, and for $j \geq 1$, we have, using the equation $\pi = \pi\, P$:

$$\pi_j = \pi_0\, p_j + \pi_{j+1} \quad \text{so} \quad \pi_j - \pi_{j+1} = \pi_0\, p_j$$

therefore (by induction on j), $\pi_j = \pi_0 \sum_{k\geq j} p_k$, so

$$\pi_j = \frac{\sum_{k\geq j} p_k}{1 + \sum_{k\geq 1} k\, p_k}, \quad \text{for } j \geq 1$$

It is also a limiting distribution, as the chain is aperiodic, but detailed balance is not satisfied, as $p_{0j} > 0$ but $p_{j0} = 0$ for $j > 1$.

f) In the particular case where $p_j = 2^{-j}$ for $j \geq 1$, compute explicitly the stationary distribution π.

Answer: Let us compute

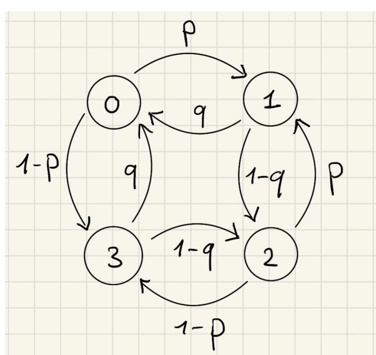
$$\sum_{k\geq 1} k\, 2^{-k} = \sum_{k\geq 1} \left(\sum_{j=1}^{k} 1\right) 2^{-k} = \sum_{j\geq 1} \sum_{k\geq j} 2^{-k} = \sum_{j\geq 1} 2^{1-j} = 2$$

so $\pi_0 = 1/3$. Besides, $\sum_{k\geq j} 2^{-k} = 2^{1-j}$, so $\pi_j = 2^{1-j}/3$ for $j \geq 1$.

Note: There are of course other ways to reach the same conclusions for these last two questions!

Exercise 2. (18+5 points)

Let $(X_n,\, n \geq 0)$ be a time-homogeneous Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition matrix P represented by the following transition graph:



where $0 < p, q < 1$.

a) Compute the stationary distribution π of the chain. Is detailed balance satisfied for all parameters $0 < p, q < 1$?

Answer: Trying to solve the detailed balance equation, we find:

$$\pi_0\, p = \pi_1\, q, \quad \pi_1\, (1-q) = \pi_2\, p, \quad \pi_2\, (1-p) = \pi_3\, (1-q), \quad \pi_0\, (1-p) = \pi_4\, q$$

which leads to

$$\pi_1 = \frac{p}{q}\, \pi_0, \quad \pi_2 = \frac{1-q}{q}\, \pi_0, \quad \pi_3 = \frac{1-p}{q}\, \pi_0$$

so using $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$, we obtain

$$\pi_0 = \frac{q}{2}, \quad \pi_1 = \frac{p}{2}, \quad \pi_2 = \frac{1-q}{2}, \quad \pi_3 = \frac{1-p}{2}$$

and yes, detailed balance is always satisfied.

b) Compute the eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3$ of the matrix P.

Hint: What is the rank of P?

Answer: The rank of P is 2 (as rows 3 and 4 are identical to rows 1 and 2), so two eigenvalues $\lambda_1 = \lambda_2 = 0$. Besides, $\lambda_0 = 1$ as P is a stochastic matrix, so $1+0+0+\lambda_3 = \mathrm{Tr}(P) = 0$, so $\lambda_3 = -1$.

c) Let $0 \leq \alpha \leq 1$ and $\widetilde{P} = \alpha\, I + (1-\alpha)\, P$ be the transition matrix of the lazy Markov chain $(\widetilde{X}_n,\, n \geq 0)$.

c1) Compute the spectral γ of $\widetilde{P}$, as a function of α.

Answer: The eigenvalues of $\widetilde{P}$ are

$$\widetilde{\lambda}_0 = 1, \quad \widetilde{\lambda}_1 = \alpha, \quad \widetilde{\lambda}_2 = \alpha, \quad \widetilde{\lambda}_3 = 2\alpha - 1$$

so $\gamma = \min(1-\alpha, 2\alpha, 2(1-\alpha)) = \min(1-\alpha, 2\alpha)$.

c2) For what value of α is γ maximal?

Answer: γ is maximal when $1-\alpha = 2\alpha$, i.e., when $\alpha = 1/3$ (and then $\gamma = 2/3$).

Let us assume from now on that α takes the value found in c2).

c3) Deduce an upper bound on $\|\widetilde{P}_0^n - \pi\|_{\mathrm{TV}}$.

Answer: Using the bound from the theorem seen in the course, we find:

$$\|\widetilde{P}_0^n - \pi\|_{\mathrm{TV}} \leq \frac{1}{2\sqrt{\pi_0}}\, \lambda_*^n = \frac{1}{\sqrt{2q}}\, \frac{1}{3^n} \left(\leq \frac{1}{\sqrt{2q}}\, \exp(-2n/3) \right)$$

BONUS d) Show that $P^3 = P$ and deduce by induction that we have the following equality for *n even*:

$$\widetilde{P}^n = \alpha^n\, I + \frac{1}{2}\, (1-\alpha^n)\, (P + P^2) \tag{1}$$

Answer: Checking that $P^3 = P$ is a direct computation, Note that it implies further that $P^4 = P^2$ and $(P+P^2)\,(P+P^2) = 2\,(P+P^2)$.

Besides, we find for $n = 2$:

$$\widetilde{P}^2 = \left(\frac{1}{3}\, I + \frac{2}{3}\, P \right)^2 = \frac{1}{9}\, I + \frac{4}{9}\, (P+P^2)$$

which corresponds to the above formula for $\alpha = 1/3$. Assume now by induction that the formula holds for n even, Then we obtain:

$$\begin{aligned}
\widetilde{P}^{n+2} &= \left(\frac{1}{3^n}\, I + \frac{1}{2} \left(1 - \frac{1}{3^n} \right) (P+P^2) \right) \left(\frac{1}{9}\, I + \frac{4}{9}\, (P+P^2) \right) \\
&= \frac{1}{3^{n+2}}\, I + \left(\frac{1}{2} \left(1 - \frac{1}{3^n} \right) \left(\frac{1}{9} + \frac{8}{9} \right) + \frac{4}{3^{n+2}} \right) (P+P^2) \\
&= \frac{1}{3^{n+2}}\, I + \frac{1}{2} \left(1 - \frac{1}{3^n} + \frac{8}{3^{n+2}} \right) (P+P^2) = \frac{1}{3^{n+2}}\, I + \frac{1}{2} \left(1 - \frac{1}{3^{n+2}} \right) (P+P^2) \quad \mathit{QED}
\end{aligned}$$

e) Use equation (1) to compute the value of $\|\widetilde{P}_0^n - \pi\|_{\text{TV}}$ for n even.

Answer: By d), the row vector $\widetilde{P}_0^n$ is given by

$$\left(\frac{1}{3^n} + \left(1 - \frac{1}{3^n}\right) \frac{q}{2}, \left(1 - \frac{1}{3^n}\right) \frac{p}{2}, \left(1 - \frac{1}{3^n}\right) \frac{1-q}{2}, \left(1 - \frac{1}{3^n}\right) \frac{1-p}{2}\right)$$

and $\pi = (\frac{q}{2}, \frac{p}{2}, \frac{1-q}{2}, \frac{1-p}{2})$, so

$$\|\widetilde{P}_0^n - \pi\|_{\text{TV}} = \frac{1}{2}\left(\frac{1}{3^n} - \frac{1}{3^n}\frac{q}{2} + \frac{1}{3^n}\frac{p}{2} + \frac{1}{3^n}\frac{1-q}{2} + \frac{1}{3^n}\frac{1-p}{2}\right) = \frac{1}{3^n}\left(1 - \frac{q}{2}\right)$$

and observe that this expression is indeed always smaller than the upper bound found in c3).

Exercise 3. (12 points)

Let $\beta_1, \beta_2 > 0$. On the set $S = \mathbb{Z}^2 = \{x = (x_1, x_2) : x_1 \in \mathbb{Z}, x_2 \in \mathbb{Z}\}$, one defines the distribution:

$$\pi_x = \frac{1}{Z} \exp(-\beta_1 x_1^2 - \beta_2 x_2^2) \quad \text{for } x = (x_1, x_2) \in \mathbb{Z}^2$$

where $Z = \sum_{x \in \mathbb{Z}^2} \exp(-\beta_1 x_1^2 - \beta_2 x_2^2)$.

Define now a base chain on S whose transition probabilities are given by

$$\psi_{xy} = \begin{cases} \frac{1}{4} & \text{if } y = x \pm e_1 \text{ or } y = x \pm e_2 \\ \\ 0 & \text{otherwise} \end{cases}$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The idea is then to use the Metropolis algorithm in order to sample from π.

a) Is this base chain irreducible? aperiodic? Does it hold that $\psi_{xy} > 0$ if and only if $\psi_{yx} > 0$?

Answer: The chain is irreducible, not a periodic and ψ is symmetric, so the last condition is satisfied.

b) Is this base chain ergodic?

Answer: No: it is not periodic and it is not positive-recurrent (but this is actually not an issue).

For the rest of this exercise, assume in all your computations that $x_1 > 0$ and $x_2 > 0$.

c) Compute the acceptance probabilities a_{xy}, as well as the resulting transition probabilities p_{xy} of the Metropolis chain (not forgetting p_{xx}).

Hint: Simplify as much as possible the expression for a_{xy}: it will help you for the next questions !

Answer:

$$a_{xy} = \min\left(1, \frac{\pi_y}{\pi_x}\right) = \begin{cases} 1 & \text{if } y = x - e_1 \quad \text{or} \quad y = x - e_2 \\ \exp(-\beta_1 (2x_1 + 1)) & \text{if } y = x + e_1 \\ \exp(-\beta_1 (2x_2 + 1)) & \text{if } y = x + e_2 \end{cases}$$

following e.g. from the computation for $y = x + e_1$:

$$a_{xy} = \frac{\exp(-\beta_1 (x_1 + 1)^2 - \beta_2 x_2^2)}{\exp(-\beta_1 x_1^2 - \beta_2 x_2^2)} = \exp(-\beta_1 (2x_1 + 1))$$

d) If $\beta_1 < \beta_2$ and $x_1 = x_2$, is a_{xy} larger when $y = x + e_1$ or when $y = x + e_2$?

Answer: a_{xy} is larger when $y = x + e_1$ in this case.

e) Is a_{xy} larger when $y = x + e_1$ and x_1 is small, or when $y = x + e_1$ and x_1 is large?

Answer: a_{xy} is larger when $y = x + e_1$ and x_1 is small.

f) Describe the shape of the set of points x (in the quadrant $x_1 \geq 0$, $x_2 \geq 0$) where the acceptance probabilities are roughly equal for both $y = x + e_1$ and $y = x + e_2$.

Answer: The acceptance probabilities are roughly equal on the axis $\beta_1 x_1 = \beta_2 x_2$ (for x_1, x_2 large).