The Moore-Penrose pseudoinverse
CS-526 Learning Theory

Consider a $M \times N$ matrix $A \in \mathbb{C}^{N\times M}$. Its transpose and complex conjugate (also called Hermitian conjugate) is the $N \times M$ matrix $\bar{A}^T$ that we denote A^*. Let $A^\dagger \in \mathbb{C}^{N\times M}$ satisfy the following four conditions:

$$AA^\dagger A = A, \quad A^\dagger A A^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A\,.$$

A theorem of Moore and Penrose states that such a matrix always exists and is unique. This matrix is called the Moore-Penrose pseudoinverse. Answer the following questions:

1) Let $\Sigma \in \mathbb{C}^{M\times N}$ be a diagonal matrix, that is, $\forall i \neq j : \Sigma_{ij} = 0$ (but you don't necessarily have $M = N$). Show that $\Sigma^\dagger$ is the $N \times M$ diagonal matrix with diagonal entries

$$\forall i \in \{1, 2, \ldots, \min\{M, N\}\} : (\Sigma^\dagger)_{ii} = \begin{cases} 1/\Sigma_{ii} & \text{if} \quad \Sigma_{ii} \neq 0\,; \\ 0 & \text{otherwise.} \end{cases}$$

2) Let $A = U\Sigma V^*$ be the singular value decomposition (SVD) of A, that is, both $U \in \mathbb{C}^{M\times M}$ and $V \in \mathbb{C}^{N\times N}$ are unitary matrices and $\Sigma \in \mathbb{R}^{M\times N}$ is a diagonal matrix with real nonnegative diagonal entries (the singular values). Give for $A^\dagger$ an expression that only involves U, V (or their inverse U^*, V^*) and $\Sigma^\dagger$.

3) Show that if A has full column rank then $A^\dagger = (A^*A)^{-1}A^*$ and $A^\dagger A = I_{N\times N}$.

4) Show that if A has full row rank then $A^\dagger = A^*(AA^*)^{-1}$ and $AA^\dagger = I_{M\times M}$.

5) Show that if A is a square matrix with full rank then $A^\dagger = A^{-1}$ is the usual inverse.

6) Let A have full column rank and B have full row rank. Check that $(AB)^\dagger = B^\dagger A^\dagger$.