Homework 3

Exercise 1. Let $\Omega = \mathbb{R}^2$ and $\mathcal{F} = \mathcal{B}(\mathbb{R}^2)$. Let also $X_1(\omega) = \omega_1$ and $X_2(\omega) = \omega_2$ for $\omega = (\omega_1, \omega_2) \in \Omega$ and let finally μ be a probability distribution on $\mathbb{R}$. We consider below two different probability measures defined on $(\Omega, \mathcal{F})$, defined on the "rectangles" $B_1 \times B_2$ (Caratheodory's extension theorem then guarantees that these probability measures can be extended uniquely to $\mathcal{B}(\mathbb{R}^2)$).

a) $\mathbb{P}^{(1)}(B_1 \times B_2) = \mu(B_1) \cdot \mu(B_2)$

b) $\mathbb{P}^{(2)}(B_1 \times B_2) = \mu(B_1 \cap B_2)$

In each case, describe what is the relation between the random variables X_1 and X_2.

Exercise 2. Let X_1, X_2 be two independent and identically distributed (i.i.d.) $\mathcal{N}(0,1)$ random variables. Compute the pdf of $X_1 + X_2$ (using convolution).

Exercise 3*. Let $\lambda > 0$ and $X \sim \mathcal{E}(\lambda)$, and let us define $Y = X^a$, where $a \in \mathbb{R}$.

a) For what values of $a \in \mathbb{R}$ does it hold that $\mathbb{E}(Y) < +\infty$?

b) For what values of $a \in \mathbb{R}$ does it hold that $\mathbb{E}(Y^2) < +\infty$?

c) For what values of $a \in \mathbb{R}$ is $\mathrm{Var}(Y)$:

c1) well-defined and finite? c2) well-defined but infinite? c3) ill-defined?

d) Compute $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$ for the values of $a \in \mathbb{Z}$ such that these quantities are well-defined.

Hint: Use integration by parts, recursively.

Exercice 4. Let X be a random variable that is symmetrically distributed (i.e. $X \sim -X$) and square-integrable with $\mathrm{Var}(X) = 1$. Let also $Y = 1_{\{X \geq 0\}}$.

a) Show that for any distribution of the random variable X, $\mathrm{Cov}(X, Y) \geq 0$.

b) Using the inequality $\mathrm{Cov}(X,Y) \leq \sqrt{\mathrm{Var}(X)}\,\sqrt{\mathrm{Var}(Y)}$ (whose proof is to come in the sequel of the course), find the least value $C > 0$ such that $\mathrm{Cov}(X, Y) \leq C$ for every distribution of X.

c) Compute $\mathrm{Cov}(X, Y)$ for $X \sim \mathcal{N}(0,1)$.

d) Is it possible to find a distribution for X such that $\mathrm{Cov}(X,Y) = C$? If not, is it possible to find a sequence of random variables $(X_n,\, n \geq 1)$ with varying distributions (all respecting the above constraints) and $Y_n = 1_{\{X_n \geq 0\}}$, such that $\mathrm{Cov}(X_n, Y_n) \underset{n\to\infty}{\to} C$?

e) Is it possible to find a distribution for X such that $\mathrm{Cov}(X,Y) = 0$? If not, is it possible to find a sequence of random variables $(X_n,\, n \geq 1)$ with varying distributions (all respecting the above constraints) and $Y_n = 1_{\{X_n \geq 0\}}$, such that $\mathrm{Cov}(X_n, Y_n) \underset{n\to\infty}{\to} 0$?

Exercise 5. For a generic *non-negative* random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, it holds that (the exchange of expectation and integral sign is valid here):

$$\mathbb{E}(X) = \mathbb{E}\left(\int_0^X dt\right) = \mathbb{E}\left(\int_0^{+\infty} 1_{\{X\geq t\}}\, dt\right) = \int_0^{+\infty} \mathbb{E}(1_{\{X\geq t\}})\, dt = \int_0^{+\infty} \mathbb{P}(\{X \geq t\})\, dt$$

a) Use this formula to compute $\mathbb{E}(X)$ for $X \sim \mathcal{E}(\lambda)$.

b) Particularize the above formula for $\mathbb{E}(X)$ to the case where X takes values in $\mathbb{N}$ only.

c) Use this new formula to compute $\mathbb{E}(X)$ for $X \sim \text{Bern}(p)$ and $X \sim \text{Geom}(p)$ for some $0 < p < 1$.