Solutions to Homework 3

Exercise 1. a) In this case,

$$\mathbb{P}^{(1)}(\{X_1 \in B_1, X_2 \in B_2\}) = \mu(B_1) \cdot \mu(B_2) = \mathbb{P}^{(1)}(\{X_1 \in B_1\}) \cdot \mathbb{P}^{(1)}(\{X_2 \in B_2\})$$

The random variables X_1 and X_2 are therefore independent and identically distributed (i.i.d.).

b) In this case,

$$\mathbb{P}^{(2)}(\{X_1 \in B_1, X_2 \in B_2\}) = \mu(B_1 \cap B_2)$$

Note first that whenever $B_1 \cap B_2 = \emptyset$, the above probability is zero, so it can never be the case that X_1, X_2 take values simultaneously in disjoint sets B_1, B_2. As this must hold for *any* disjoint sets B_1, B_2, it holds in particular for non-intersecting intervals $]a_1, b_1[$, $]a_2, b_2[$. This is to say that $\mathbb{P}^{(2)}(\{(X_1, X_2) \in R\}) = 0$ for any open rectangle $R \subset \mathbb{R}^2$ not touching the diagonal $\Delta = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$. From this, one deduces that $\mathbb{P}^{(2)}(\{(X_1, X_2) \in B\}) = 0$ for any open set B not touching the diagonal, which further implies that $\mathbb{P}^{(2)}(\{(X_1, X_2) \in \Delta\}) = 1$, i.e., that $\mathbb{P}^{(2)}(\{X_1 = X_2\}) = 1$.

NB: Please note that in both cases, the two random variables X_1, X_2 have the same *distribution*, but in one case, they are independent, while in the other, they are the same random variable.

Exercise 2. By the formula seen in class, we have:

$$\begin{aligned}
p_{X_1+X_2}(t) &= \int_{\mathbb{R}} dx_1\, p_{X_1}(x_1)\, p_{X_2}(t - x_1) = \int_{\mathbb{R}} dx_1\, \frac{1}{\sqrt{2\pi}} \exp(-x_1^2/2)\, \frac{1}{\sqrt{2\pi}} \exp(-(t - x_1)^2/2) \\
&= \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) \int_{\mathbb{R}} dx_1\, \frac{1}{\sqrt{2\pi}} \exp(tx_1 - x_1^2) \\
&= \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) \int_{\mathbb{R}} dx_1\, \frac{1}{\sqrt{2\pi}} \exp(-(x_1 - t/2)^2) \exp(t^2/4) \\
&= \frac{1}{\sqrt{4\pi}} \exp(-t^2/4) \int_{\mathbb{R}} dx_1\, \frac{1}{\sqrt{\pi}} \exp(-(x_1 - t/2)^2)
\end{aligned}$$

The integral on the right-hand side is equal to 1, as the integrand is the pdf of a $\mathcal{N}(t/2, 1/2)$ random variable, so we remain with

$$p_{X_1+X_2}(t) = \frac{1}{\sqrt{4\pi}} \exp(-t^2/4), \quad t \in \mathbb{R}$$

which shows that $X_1 + X_2$ is a $\mathcal{N}(0, 2)$ random variable.

Exercise 3*. a) We have

$$\mathbb{E}(Y) = \mathbb{E}(X^a) = \int_0^{+\infty} x^a\, \lambda\, \exp(-\lambda x)\, dx < +\infty \quad \text{if and only if} \quad a > -1$$

b) Likewise:

$$\mathbb{E}(Y^2) = \mathbb{E}(X^{2a}) = \int_0^{+\infty} x^{2a}\, \lambda\, \exp(-\lambda x)\, dx < +\infty \quad \text{if and only if} \quad a > -\frac{1}{2}$$

c) Therefore, c1) $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$ is well defined and finite $\forall a > -\frac{1}{2}$; c2) $\mathrm{Var}(Y)$ is well defined but takes the value $+\infty$ for $-\frac{1}{2} \geq a > -1$, and c3) $\mathrm{Var}(Y)$ is ill-defined (indetermination of the type $\infty - \infty$) for $a \leq -1$.

d) The only integer values of a for which $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$ are well-defined are non-negative values. For $a = 0$, we have $Y = X^0 = 1$, so $\mathbb{E}(Y) = 1$ and $\mathrm{Var}(Y) = 0$. For $a \geq 1$, we obtain by integration by parts:

$$\mathbb{E}(Y) = \mathbb{E}(X^a) = \int_0^{+\infty} x^a \, \lambda \exp(-\lambda x) \, dx$$
$$= \int_0^{+\infty} \frac{a}{\lambda} \, x^{a-1} \, \lambda \exp(-\lambda x) \, dx = \ldots = \frac{a!}{\lambda^a} \cdot 1$$

so

$$\mathbb{E}(Y^2) = \mathbb{E}(X^{2a}) = \frac{(2a)!}{\lambda^{2a}} \quad \text{and} \quad \mathrm{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = \frac{(2a)! - (a!)^2}{\lambda^{2a}}$$

Exercise 4. First note that as $X \sim -X$, it holds that $\mathbb{P}(\{X \geq 0\}) \geq \frac{1}{2}$ and $\mathbb{E}(X) = 0$.

a) $\mathrm{Cov}(X, Y) = \mathbb{E}(X \, 1_{\{X \geq 0\}}) \geq 0$ as $X \, 1_{\{X \geq 0\}}$ is a non-negative random variable.

b) Using the suggested inequality, we find

$$\mathrm{Cov}(X, Y) \leq \sqrt{\mathrm{Var}(X)} \, \sqrt{\mathrm{Var}(Y)} = \sqrt{1} \, \sqrt{\mathbb{P}(\{X \geq 0\}) - \mathbb{P}(\{(X \geq 0\})^2} \leq \sqrt{\frac{1}{4}} = \frac{1}{2} = C$$

as $\mathbb{P}(\{X \geq 0\}) - \mathbb{P}(\{(X \geq 0\})^2 \leq \frac{1}{4}$ (which is maximized when $\mathbb{P}(\{X \geq 0\}) = \frac{1}{2}$).

c) The computation gives

$$\mathrm{Cov}(X, Y) = \mathbb{E}(X \, 1_{\{X \geq 0\}}) = \int_0^{+\infty} x \, \frac{1}{\sqrt{2\pi}} \, \exp(-x^2/2) \, dx = \frac{1}{\sqrt{2\pi}} \left(-\exp(-x^2/2)\right) \Big|_{x=0}^{x=+\infty} = \frac{1}{\sqrt{2\pi}}$$

(clearly satisfying the above two inequalities)

d) The answer to the first question is yes: take X such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$ (verifying $X \sim -X$, $\mathrm{Var}(X) = 1$ and $\mathrm{Cov}(X, Y) = \frac{1}{2}$).

e) The answer to the first question is no, but the one to the second is yes: consider X_n such that $\mathbb{P}(\{X_n = n\}) = \mathbb{P}(\{X_n = -n\}) = \frac{1}{2n^2}$ and $\mathbb{P}(\{X_n = 0\}) = 1 - \frac{1}{n^2}$. Then $X_n \sim -X_n$ and $\mathrm{Var}(X_n) = 1$ for every n, and $\mathrm{Cov}(X_n, Y_n) = \mathbb{E}(X_n \, 1_{\{X_n \geq 0\}}) = n \, \frac{1}{2n^2} = \frac{1}{2n} \underset{n \to \infty}{\to} 0$.

Exercise 5. a) Using the formula given in the problem set, we obtain:

$$\mathbb{E}(X) = \int_0^{+\infty} \exp(-\lambda t) \, dt = \frac{1}{\lambda}$$

b) Using the formula given in the problem set together with the fact that X is integer-valued, we obtain:

$$\mathbb{E}(X) = \sum_{k \geq 0} \int_k^{k+1} \mathbb{P}(\{X \geq t\}) \, dt = \sum_{k \geq 0} \int_k^{k+1} \mathbb{P}(\{X \geq k+1\}) \, dt = \sum_{k \geq 0} \mathbb{P}(\{X \geq k+1\}) = \sum_{k \geq 1} \mathbb{P}(\{X \geq k\})$$

c) Applying the above formula, we obtain in the first case ($X \sim \mathrm{Bern}(p)$):

$$\mathbb{E}(X) = \mathbb{P}(\{X \geq 1\}) = p$$

In the second case ($X \sim \mathrm{Geom}(p)$), we obtain:

$$\mathbb{E}(X) = \sum_{k \geq 1} \sum_{m \geq k} p^m \, (1-p) = \sum_{k \geq 1} p^k \sum_{m \geq k} p^{m-k} \, (1-p) = \sum_{k \geq 1} p^k \, \frac{1}{1-p} \, 1 - p = \frac{1}{1-p} - 1 = \frac{p}{1-p}$$