Solutions to Homework 4

Exercise 1*. a) i) From the course, we know that if $\mathbb{E}(|X|) < +\infty$, then ϕ_X is continuously differentiable on $\mathbb{R}$. Using the contraposition, we deduce that $\mathbb{E}(|X|) = +\infty$ here.

a) ii) From the course again, the fact that ϕ_X is integrable on $\mathbb{R}$ implies that X admits a pdf p_X.

b) By the inversion formula seen in class, we have

$$\begin{aligned} p_X(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} e^{-\lambda|t|}\, dt = \frac{1}{2\pi} \left(\int_{-\infty}^{0} e^{-t(ix-\lambda)}\, dt + \int_{0}^{+\infty} e^{-t(ix+\lambda)}\, dt \right) \\ &= \frac{1}{2\pi} \left(-\frac{1}{ix-\lambda} + \frac{1}{ix+\lambda} \right) = \frac{1}{\pi}\, \frac{\lambda}{x^2+\lambda^2} \end{aligned}$$

This pdf is the that of a (centered) Cauchy distribution with parameter λ (also known as Lorentz distribution in physics). The word "centered" is a bit misleading here, as we have seen in part a)i) that $E(|X|) = +\infty$ (which can also been checked directly from the expression of p_X), so that $\mathbb{E}(X)$ is ill-defined. Nevertheless, the pdf appears to have a peak clearly centered in $x = 0$ here, and writing $\mathbb{E}(X) = 0$ can actually be justified via a more general definition of expectation. Besides, the parameter $\lambda > 0$ is connected to the width of the peak, but is by no means connected to the standard deviation of the random variable X, which is truly infinite.

c) Using the change of variable formula, we obtain

$$\begin{aligned} p_Y(x) = p_{1/X}(x) &= p_X\left(\frac{1}{x}\right) \cdot \left| -\frac{1}{x^2} \right| = \frac{1}{\pi}\, \frac{\lambda}{x^{-2}+\lambda^2}\, \frac{1}{x^2} \\ &= \frac{1}{\pi}\, \frac{\lambda}{1+\lambda^2\, x^2} = \frac{1}{\pi}\, \frac{\lambda^{-1}}{\lambda^{-2}+x^2} \end{aligned}$$

so we see that Y is again a Cauchy random variable, with parameter $1/\lambda$.

d) By the factorization property of characteristic functions, we obtain

$$\phi_{X_1+\ldots+X_n}(t) = \textstyle\prod_{i=1}^{n} \phi_{X_i}(t) = (\phi_X(t))^n = \exp(-\lambda n|t|)$$

so $X_1 + \ldots + X_n$ is also a Cauchy random variable with parameter λn, and $Z_n = \frac{X_1+\ldots+X_n}{n}$ is a Cauchy random variable with parameter λ, for every $n \geq 1$. Similarly, we obtain, using part b),

$$\phi_{1/X_1+\ldots+1/X_n}(t) = (\phi_{1/X}(t))^n = \exp(-n|t|/\lambda)$$

so $1/X_1+\ldots+1/X_n$ is a Cauchy random variable with parameter n/λ. Therefore, again by part b), $\frac{1}{1/X_1+\ldots+1/X_n}$ is a Cauchy random variable with parameter λ/n and $W_n = \frac{n}{1/X_1+\ldots+1/X_n}$ is (again) a Cauchy random variable with parameter λ.

e) The first oddity of the above results is that the empirical average $Z_n = \frac{X_1+\ldots+X_n}{n}$ does not converge to a limit as n goes to infinity. One reason for this is that $\mathbb{E}(|X|) = +\infty$, so the law of large numbers does not hold, as we shall see later in the course. The second oddity is that the sum of an arbitrary number of Cauchy random variables is still a Cauchy random variable. The other well known distribution sharing this property is the Gaussian distribution, but that's basically it, as this property is an exception among probability distributions. The third oddity is that the *arithmetic mean* Z_n of the random variables $X_1, \ldots, X_n$ has the same distribution as their *harmonic mean* W_n. However, as we deal here with random variables taking positive and negative values, the classical inequality "arithmetic mean $\geq$ harmonic mean" does not hold, so there is no contradiction.

Exercise 2. a) Option 1: by the assumptions made, $\mathrm{Cov}(X_1 + X_2, X_1 - X_2) = \mathrm{Var}(X_1) + \mathrm{Cov}(X_2, X_1) - \mathrm{Cov}(X_1, X_2) - \mathrm{Var}(X_2) = \mathrm{Var}(X_1) - \mathrm{Var}(X_2) = 0$. Besides, as X_1, X_2 are independent Gaussian random variables, $X = (X_1, X_2)$ is a Gaussian random vector, so $(X_1 + X_2, X_1 - X_2)$ is also a Gaussian random vector whose components are uncorrelated, and therefore independent, by Proposition 6.8 of the course.

Option 2 is to show directly that

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{it_1(X_1+X_2)})\,\mathbb{E}(e^{it_2(X_1-X_2)}) \quad \forall t_1, t_2 \in \mathbb{R}$$

as this would imply independence of $X_1 + X_2$ and $X_1 - X_2$. We check indeed that

$$\begin{aligned}
&\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{i(t_1+t_2)X_1+i(t_1-t_2)X_2}) \\
&= \mathbb{E}(e^{i(t_1+t_2)X_1})\,\mathbb{E}(e^{i(t_1-t_2)X_2}) = e^{i\mu_1(t_1+t_2)-\sigma_1^2(t_1+t_2)^2/2}\,e^{i\mu_2(t_1-t_2)-\sigma_2^2(t_1-t_2)^2/2}
\end{aligned}$$

Because of the assumption made ($\sigma_1^2 = \sigma_2^2 = \sigma^2$), the above expression is further equal to

$$\begin{aligned}
&= e^{i(\mu_1+\mu_2)t_1+i(\mu_1-\mu_2)t_2-\sigma^2(t_1^2+t_2^2)} = e^{i(\mu_1+\mu_2)t_1-\sigma^2t_1^2}\,e^{i(\mu_1-\mu_2)t_2-\sigma^2t_2^2} \\
&= \mathbb{E}(e^{it_1(X_1+X_2)})\,\mathbb{E}(e^{it_2(X_1-X_2)})
\end{aligned}$$

which proves the claim.

b) 1. Skipped. Just note that closing our eyes, we could compute

$$\phi_X'(t) = i\,\mathbb{E}(X\,e^{itX}) \quad \text{and} \quad \phi_X''(t) = -\mathbb{E}(X^2e^{itX}), \quad t \in \mathbb{R}$$

and deduce from there that indeed, if $\mathbb{E}(X^2) < +\infty$, then ϕ_X is twice continuously differentiable. As a by-product, we obtain the relation

$$\phi_X''(0) = -\mathbb{E}(X^2)$$

from the second formula evaluated in $t = 0$.

2. Skipped.

3. By the assumptions made, we obtain

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{it_1(X_1+X_2)})\,\mathbb{E}(e^{it_2(X_1-X_2)})$$

and also

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{i(t_1+t_2)X_1+i(t_1-t_2)X_2}) = \phi_{X_1}(t_1+t_2)\,\phi_{X_2}(t_1-t_2)$$

so

$$\log\phi_{X_1}(t_1+t_2) + \log\phi_{X_2}(t_1-t_2) = \log\mathbb{E}(e^{it_1(X_1+X_2)}) + \log\mathbb{E}(e^{it_2(X_1-X_2)}) = g_1(t_1) + g_2(t_2)$$

proving the claim.

4. Differentiating first the equality with respect to t_1, we obtain

$$f_1'(t_1+t_2) + f_2'(t_1-t_2) = g_1'(t_1)$$

and then with respect to t_2:

$$f_1''(t_1+t_2) - f_2''(t_1-t_2) = 0$$

Setting $t_1 = t_2 = \frac{t}{2}$ leads to $f_1''(t) = f_2''(0)$, and setting $t_1 = -t_2 = \frac{t}{2}$ leads to $f_2''(t) = f_1''(0)$. As these equalities are satisfied for arbitrary $t \in \mathbb{R}$, this says that the second derivatives of both f_1 and f_2 are constant functions, therefore that both f_1 and f_2 are polynomials of degree less than or equal to 2.

5. The assumption is that $\log \phi_X(t) = at^2 + bt + c$ for $t \in \mathbb{R}$. Using the hint and writing $\mu = \mathbb{E}(X)$, $\sigma^2 = \mathrm{Var}(X)$, we obtain successively:

$$\begin{aligned} e^c &= \phi_X(0) = 1 && \text{so } c = 0 \\ b &= \phi_X'(0) = i\mu && \text{so } b = i\mu \\ 2a + b^2 &= \phi_X''(0) = -\mathbb{E}(X^2) = -(\mu^2 + \sigma^2) && \text{so } a = -\sigma^2/2 \end{aligned}$$

Therefore, $\phi_X(t) = e^{i\mu t - \sigma^2 t^2/2}$, which is the characteristic function of a Gaussian.

6. As X_1, X_2 are independent and Gaussian, this implies that (X_1, X_2) is a Gaussian vector, i.e., that X_1, X_2 are jointly Gaussian. By the assumptions made, we also have

$$0 = \mathrm{Cov}(X_1 + X_2, X_1 - X_2) = \mathrm{Var}(X_1) + \mathrm{Cov}(X_2, X_1) - \mathrm{Cov}(X_1, X_2) - \mathrm{Var}(X_2) = \mathrm{Var}(X_1) - \mathrm{Var}(X_2)$$

so $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$ [note in passing that we did not use here the assumption that X_1 and X_2 are uncorrelated]. This finally completes the proof of the result stated in part b).