Homework 5

Exercise 1. a) Let X be a square-integrable random variable such that $\mathbb{E}(X) = 0$ and $\mathrm{Var}(X) = \sigma^2$. Show that

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{\sigma^2 + t^2} \quad \text{for } t > 0$$

Hint: You may try various versions of Chebyshev's inequality here, but not all of them work. A possibility is to use the function $\psi(x) = (x+b)^2$, where b is a free parameter to optimize (but watch out that only some values of $b \in \mathbb{R}$ lead to a function ψ that satisfies the required hypotheses).

b) Let X be a square-integrable random variable such that $\mathbb{E}(X) > 0$. Show that

$$\mathbb{P}(\{X > t\}) \geq \frac{(\mathbb{E}(X) - t)^2}{\mathbb{E}(X^2)} \quad \forall 0 \leq t \leq \mathbb{E}(X)$$

Hint: Use first Cauchy-Schwarz' inequality with the random variables X and $Y = 1_{\{X>t\}}$.

Exercise 2*. Let $(X_n,\, n \geq 1)$ be independent random variables such that $X_n \sim \mathrm{Bern}(1 - \frac{1}{(n+1)^\alpha})$, where $\alpha > 0$.

Let us also define $Y_n = \prod_{j=1}^n X_j$ for $n \geq 1$.

a) What minimal condition on the parameter $\alpha > 0$ ensures that $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$?

Hint: Use the approximation $1 - x \simeq \exp(-x)$ for x small.

b) Under the same condition as that found in a), does it also hold that $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$?

c) Under the same condition as that found in a), does it also hold that $Y_n \underset{n\to\infty}{\to} 0$ almost surely ?

Hint: If $Y_n = 0$, what can you deduce on Y_m for $m \geq n$?

Exercise 3. a) Show that if $(A_n,\, n \geq 1)$ are *independent* events in $\mathcal{F}$ and $\sum_{n\geq 1} \mathbb{P}(A_n) = \infty$, then

$$\mathbb{P}\Big(\bigcup_{n\geq 1} A_n\Big) = 1$$

Hints: - Start by observing that the statement is equivalent to $\mathbb{P}\left(\bigcap_{n\geq 1} A_n^c\right) = 0$.
- Use the inequality $1 - x \leq e^{-x}$, valid for all $x \in \mathbb{R}$.

b) From the same set of assumptions, reach the following stronger conclusion with a little extra effort:

$$\mathbb{P}(\{\omega \in \Omega : \omega \in A_n \text{ infinitely often}\}) = \mathbb{P}\Big(\bigcap_{N\geq 1} \bigcup_{n\geq N} A_n\Big) = 1$$

which is actually the statement of the *second Borel-Cantelli lemma.*

c) Let $(X_n,\, n \geq 1)$ be a sequence of *independent* random variables such that for some $\varepsilon > 0$, $\sum_{n\geq 1} \mathbb{P}(\{|X_n| \geq \varepsilon\}) = +\infty$. What can you conclude on the almost sure convergence of the sequence X_n towards the limiting value 0?

d) Let $(X_n,\, n \geq 1)$ be a sequence of independent random variables such that $\mathbb{P}(\{X_n = n\}) = p_n = 1 - \mathbb{P}(\{X_n = 0\})$ for $n \geq 1$. What minimal condition on the sequence $(p_n, n \geq 1)$ ensures that

d1) $X_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$? d2) $X_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$? d3) $X_n \underset{n\to\infty}{\to} 0$ almost surely?

e) Let $(Y_n,\, n \geq 1)$ be a sequence of independent random variables such that $Y_n \sim \text{Cauchy}(\lambda_n)$ for $n \geq 1$. What minimal condition on the sequence $(\lambda_n,\, n \geq 1)$ ensures that

e1) $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$? e2) $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$? e3) $Y_n \underset{n\to\infty}{\to} 0$ almost surely?