Solutions to Homework 5

Exercise 1. a) Using $\psi(x) = x^2$ or $\psi(x) = \sigma^2 + x^2$ in Chebyshev's inequality leads to respectively

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{t^2} \quad \text{and} \quad \mathbb{P}(\{X \geq t\}) \leq \frac{2\sigma^2}{\sigma^2 + t^2}$$

which is not what we want. Using the hint (with $b \geq 0$ in order to satisfy the hypotheses), we obtain

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\mathbb{E}((X+b)^2)}{(t+b)^2} = \frac{\sigma^2 + b^2}{(t+b)^2}$$

Optimizing over the parameter b, we find that best possible bound is obtained by setting $b^* = \frac{\sigma^2}{t}$ (which is non-negative), leading to

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{\sigma^2 + t^2}$$

b) Using Cauchy-Schwarz's inequality with the random variables X and $Y = 1_{\{X>t\}}$, we obtain

$$\mathbb{E}(X\, 1_{\{X>t\}})^2 \leq \mathbb{E}(X^2)\, \mathbb{P}(\{X > t\})$$

On the other hand, we have $\mathbb{E}(X\, 1_{\{X>t\}}) = \mathbb{E}(X) - \mathbb{E}(X\, 1_{\{X \leq t\}}) \geq \mathbb{E}(X) - t$, therefore the result.

Exercise 2*. a) Let us compute for $\varepsilon > 0$:

$$\mathbb{P}(\{|Y_n - 0| > \varepsilon\}) \leq \mathbb{P}(\{Y_n > 0\}) = \mathbb{P}(\{Y_n = 1\}) = \prod_{j=1}^{n} \mathbb{P}(\{X_j = 1\})$$

$$= \prod_{j=1}^{n} \left(1 - \frac{1}{(j+1)^\alpha}\right) \simeq \exp\left(-\sum_{j=1}^{n} \frac{1}{(j+1)^\alpha}\right)$$

where the hint was used in the last (approximate) equality. If $\alpha > 1$, then $\sum_{j=1}^{n} \frac{1}{(j+1)^\alpha}$ converges to a fixed value $< +\infty$ as $n \to \infty$, so $\mathbb{P}(\{Y_n > 0\})$ does not converge to 0 as $n \to \infty$.

On the contrary, if $0 < \alpha \leq 1$, then $\sum_{j=1}^{n} \frac{1}{(j+1)^\alpha} \underset{n\to\infty}{\to} +\infty$, in which case $\mathbb{P}(\{Y_n > 0\}) \underset{n\to\infty}{\to} 0$, so $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$ in this case.

b) The answer is yes. Indeed, we have $\mathbb{E}((Y_n - 0)^2) = \mathbb{E}(Y_n^2) = \mathbb{P}(\{Y_n = 1\})$, so $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$ if and only if $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$.

c) The answer is again yes. Indeed, if for a given realization ω, $Y_n(\omega) = 0$, then $Y_m(\omega) = 0$ for every $m \geq n$, and therefore $\lim_{n\to\infty} Y_n(\omega) = 0$. This implies that

$$\mathbb{P}(\{\lim_{n\to\infty} Y_n = 0\}) \geq \mathbb{P}(\{Y_n = 0\})$$

for any fixed value of $n \geq 1$. If $0 < \alpha \leq 1$, we have seen in question a) that $\mathbb{P}(\{Y_n = 0\}) \underset{n\to\infty}{\to} 1$. So the above inequality implies that $Y_n \underset{n\to\infty}{\to} 0$ almost surely in this case.

Remark. Please note finally that when $\alpha > 1$, convergence in probability does not hold, so automatically in this case, quadratic convergence and almost sure convergence do not hold either.

Exercise 3. a) By independence, we obtain

$$\mathbb{P}\left(\bigcap_{n\geq 1} A_n^c\right) = \prod_{n\geq 1} \mathbb{P}(A_n^c) = \prod_{n\geq 1} (1 - \mathbb{P}(A_n)) \leq \prod_{n\geq 1} \exp(-\mathbb{P}(A_n)) = \exp\left(-\sum_{n\geq 1} \mathbb{P}(A_n)\right) = 0$$

where we have used the fact that $1 - x \leq \exp(-x)$ for $0 \leq x \leq 1$. Therefore, $\mathbb{P}\left(\bigcup_{n\geq 1} A_n\right) = 1$.

Note: The first equality above is "obviously true", but actually needs a proof (not required in the homework): if $(A_n,\, n \geq 1)$ is a countable sequence of independent events, then it holds that $\mathbb{P}(\cap_{n\geq 1} A_n) = \prod_{n\geq 1} \mathbb{P}(A_n)$. Here is why: define $B_n = \cap_{k=1}^n A_k$. Observe that $\cap_{n\geq 1} A_n = \cap_{n\geq 1} B_n$ and $B_n \supset B_{n+1}$ for every $n \geq 1$, so by the continuity property of $\mathbb{P}$,

$$\mathbb{P}(\cap_{n\geq 1} A_n) = \mathbb{P}(\cap_{n\geq 1} B_n) = \lim_{n\to\infty} \mathbb{P}(B_n) = \lim_{n\to\infty} \prod_{k=1}^{n} \mathbb{P}(A_k) = \prod_{n\geq 1} \mathbb{P}(A_n)$$

b) By exactly the same argument as above, we can prove $\mathbb{P}\left(\bigcap_{n\geq N} A_n^c\right) = 0$, $\forall N \geq 1$, and we have seen in class that this holds true if and only if $\mathbb{P}\left(\bigcup_{N\geq 1} \bigcap_{n\geq N} A_n^c\right) = 0$, i.e. $\mathbb{P}\left(\bigcap_{N\geq 1} \bigcup_{n\geq N} A_n\right) = 1$.

c) If for some $\varepsilon > 0$, $\sum_{n\geq 1} \mathbb{P}(\{|X_n| \geq \varepsilon\}) = +\infty$, then by part b), $\mathbb{P}(\{|X_n| \geq \varepsilon \text{ infinitely often}\}) = 1$. This says that almost sure convergence (towards the limiting value 0) of the sequence X_n does not hold, as for this convergence to hold, we would need exactly the opposite, namely that for every $\varepsilon > 0$, $\mathbb{P}(\{|X_n| \geq \varepsilon \text{ infinitely often}\}) = 0$.

d1) For any fixed $\varepsilon > 0$, $\mathbb{P}(\{|X_n| \geq \varepsilon\}) = p_n$ for sufficiently large n, so the minimal condition ensuring convergence in probability is simply $p_n \underset{n\to\infty}{\to} 0$ (said otherwise, $p_n = o(1)$).

d2) $\mathbb{E}((X_n - 0)^2) = n^2\, p_n$, so the minimal condition for L^2 convergence is $p_n = o(\frac{1}{n^2})$.

d3) Using the two Borel-Cantelli lemmas (both applicable here as the X_n are independent), we see that the minimal condition for almost sure convergence is $\sum_{n\geq 1} p_n < +\infty$, satisfied in particular if $p_n = O(n^{-1-\delta})$.

e1) We have in this case, for any fixed $\varepsilon > 0$:

$$\mathbb{P}(\{|Y_n| \geq \varepsilon\}) = 2\int_{\varepsilon}^{+\infty} dx\, \frac{1}{\pi}\, \frac{\lambda_n}{\lambda_n^2 + x^2} = \frac{2}{\pi}\left(\frac{\pi}{2} - \arctan\left(\frac{\varepsilon}{\lambda_n}\right)\right) \underset{n\to 0}{\to} 0$$

if and only if $\lambda_n \underset{n\to\infty}{\to} 0$.

e2) $\mathbb{E}(Y_n^2) = +\infty$ in all cases, so L^2 convergence does not hold.

e3) Observe first that by the change of variable $y = \lambda_n x$,

$$\mathbb{P}(\{|Y_n| \geq \varepsilon\}) = 2\int_{\varepsilon}^{+\infty} dy\, \frac{\lambda_n}{\pi\,(\lambda_n^2 + y^2)} = 2\int_{\varepsilon/\lambda_n}^{+\infty} dx\, \frac{1}{\pi(1 + x^2)} \simeq 2\int_{\varepsilon/\lambda_n}^{+\infty} dx\, \frac{\lambda_n}{\pi x^2} = \frac{2\lambda_n}{\pi\,\varepsilon}$$

when λ_n is small. So the minimal condition for almost sure convergence is $\sum_{n\geq 1} \lambda_n < +\infty$, satisfied in particular if $\lambda_n = O(n^{-1-\delta})$.