

Homework 6**Exercise 1.** (extended law of large numbers)Let $(\mu_n, n \geq 1)$ be a sequence of real numbers such that

$$\lim_{n \rightarrow \infty} \frac{\mu_1 + \dots + \mu_n}{n} = \mu \in \mathbb{R}$$

Let $(X_n, n \geq 1)$ be a sequence of square-integrable random variables such that

$$\mathbb{E}(X_n) = \mu_n, \quad \forall n \geq 1 \quad \text{and} \quad \text{Cov}(X_n, X_m) \leq C_1 \exp(-C_2 |m - n|), \quad \forall m, n \geq 1.$$

for some constants $C_1, C_2 > 0$ (the random variables X_n are said to be *weakly* correlated). Let finally $S_n = X_1 + \dots + X_n$.

a) Show that

$$\frac{S_n}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \mu$$

b) Is it also true that

$$\frac{S_n}{n} \xrightarrow[n \rightarrow \infty]{} \mu \quad \text{almost surely?}$$

In order to check this, you need to go through the proof of the strong law of large numbers made in class. Does that proof need the fact that the random variables X_n are independent?c) Application to auto-regressive processes: Let $(Z_n, n \geq 1)$ be a sequence of i.i.d. $\sim \mathcal{N}(0, 1)$ random variables, $x, a \in \mathbb{R}$ and $(X_n, n \geq 1)$ be the sequence of random variables defined recursively as

$$X_1 = x, \quad X_{n+1} = aX_n + Z_{n+1}, \quad n \geq 1$$

For what values of $x, a \in \mathbb{R}$ does the sequence $(X_n, n \geq 1)$ satisfy the assumptions made in a)? Compute μ in this case.**Exercise 2*.** a) Let $(X_n, n \geq 1)$ be a sequence of bounded i.i.d. random variables such that $\mathbb{E}(X_1) = 0$ and $\text{Var}(X_1) = 1$, and let $S_n = X_1 + \dots + X_n$ for $n \geq 1$. Show that the event

$$A = \left\{ \frac{S_n}{n} \text{ converges} \right\}$$

belongs to the tail σ -field $\mathcal{T} = \bigcap_{n \geq 1} \sigma(X_n, X_{n+1}, \dots)$ (implying that $\mathbb{P}(A) \in \{0, 1\}$ by Kolmogorov's 0-1 law; but the law of large numbers tells you more in this case, namely that $\mathbb{P}(A) = 1$).b) Assume now that $(X_n, n \geq 1)$ is a sequence of bounded, *uncorrelated* and identically distributed random variables such that $\mathbb{E}(X_1) = 0$ and $\text{Var}(X_1) = 1$. The answer to Exercise 1.b) above tells you what happens to $\mathbb{P}(A)$ in this case.That said, under this more general assumption, Kolmogorov's 0-1 law may not necessarily hold. Prove it by exhibiting a sequence of random variables $(X_n, n \geq 1)$ satisfying these assumptions and an event $B \in \mathcal{T}$ such that $0 < \mathbb{P}(B) < 1$.