Midterm exam

Exercise 1. Quiz. (18 points)
Answer each yes/no question below (1 pt) and provide a short justification for your answer (2 pts).

a) Let $\mathcal{F}$ be a σ-field on Ω and $\mathcal{G}, \mathcal{H}$ be two sub-σ-fields of $\mathcal{F}$.

a1) Does it always hold that $\mathcal{G} \cap \mathcal{H}$ is a σ-field ?

Note: $\mathcal{G} \cap \mathcal{H}$ is by definition the list of subsets of Ω belonging to both $\mathcal{G}$ and $\mathcal{H}$.

a2) Does it always hold that $\mathcal{G} \cup \mathcal{H}$ is a σ-field ?

Note: $\mathcal{G} \cup \mathcal{H}$ is by definition the list of subsets of Ω belonging to either $\mathcal{G}$ or $\mathcal{H}$.

b) Let X, Y, Z be three random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

b1) Does it always hold that $\sigma(X, Y, Z) = \sigma(X+Y, Y+Z, Z+X)$?

b2) If $\sigma(X+Y+Z) = \sigma(X, Y, Z)$, does that necessarily imply that X, Y, Z are independent ?

c) Let X be a continuous random variable and Y be a discrete random variable which is independent of X and also such that $Y(\omega) \neq 0$ for all $\omega \in \Omega$. Let finally $Z = X \cdot Y$.

c1) Is it always the case that Z is a continuous random variable ?

c2) Assume now that $X \sim \mathcal{N}(0,1)$ and $\mathbb{P}(\{Y = +1\}) = \mathbb{P}(\{Y = +2\}) = \frac{1}{2}$. Is Z a Gaussian random variable in this case ?

Exercise 2. (15 points)
a) Let $\lambda > 0$ and $X \sim \mathcal{E}(\lambda)$, i.e., $p_X(x) = \lambda\, e^{-\lambda x}$ for $x \geq 0$. What is the distribution of $Y = \mu X$, where $\mu > 0$?

b) Let X be a discrete random variable taking values in $\mathbb{N}^*$ and such that $\mathbb{P}(\{X \geq k\}) = \frac{2}{k\,(k+1)}$ for $k \geq 1$. Compute $\mathbb{E}(X)$.

c) Let X be a $\mathcal{U}([0,1])$ random variable, Y be independent of X and such that $\mathbb{P}(\{Y = +1\}) = \mathbb{P}(\{Y = -1\}) = \frac{1}{2}$ and $Z = X \cdot Y$. Compute $\phi_Z(t) \in \mathbb{R}$ for $t \in \mathbb{R}$.

d) Let X_1, X_2 be two square-integrable random variables such that $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1 - X_2)$. Compute $\mathrm{Cov}(X_1, X_2)$.

e) Does there exist a non-negative random variable X such that $\mathbb{E}(X) = 10$ and $\mathbb{E}(2^X) = 1000$? Justify your answer.

Exercise 3. (13 points)
Let $n \geq 1$ and $X_1, \ldots, X_n$ be i.i.d. random variables with common cdf $F(t) = \exp(-\exp(-t))$ for $t \in \mathbb{R}$.

a) Verify that F is indeed a cdf.

b) Compute *both* the cdf and the pdf of

$$Y_n = \max\{X_1, \ldots, X_n\} - \ln(n)$$

c) Compute *both* the cdf and the pdf of

$$Z_n = \min\{\exp(-X_1), \ldots, \exp(-X_n)\}$$

Exercise 4. (14 points)
Let $\alpha > 0$ and $(X_n,\, n \geq 1)$ be a sequence of independent random variables such that

$$\mathbb{P}(\{X_n = +n^\alpha\}) = \mathbb{P}(\{X_n = -n^\alpha\}) = \frac{1}{2n} \quad \text{and} \quad \mathbb{P}(\{X_n = 0\}) = 1 - \frac{1}{n} \quad \text{for } n \geq 1$$

Let also $S_n = X_1 + \ldots + X_n$ for $n \geq 1$.

a) Compute $\mathbb{E}(S_n)$ and $\mathrm{Var}(S_n)$, then estimate both quantities as a function of n using the approximation (valid for a generic value of $\gamma \in \mathbb{R}$):

$$\sum_{j=1}^{n} j^\gamma \simeq \int_1^n dx\, x^\gamma$$

$\left(\text{as an example, such an approximation allows to estimate } \sum_{j=1}^{n} j \simeq \int_1^n dx\, x \simeq \frac{n^2}{2}.\right)$

b) For what values of $\beta > 0$ can you show that $\frac{S_n}{n^\beta} \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$?

c) For what values of $\beta > 0$ can you show that $\frac{S_n}{n^\beta} \underset{n\to\infty}{\to} 0$ almost surely ?