Solutions for the Moore-Penrose pseudoinverse
CS-526 Learning Theory

1) Define $\Sigma^\dagger$ as the $N \times M$ diagonal matrix with diagonal entries:

$$\forall i \in \{1, 2, \ldots, \min\{M, N\}\} : (\Sigma^\dagger)_{ii} \begin{cases} 1/\Sigma_{ii} & \text{if} \quad \Sigma_{ii} \neq 0 \; ; \\ 0 & \text{otherwise.} \end{cases}$$

Then both $\Sigma^\dagger\Sigma \in \mathbb{C}^{N\times N}$ and $\Sigma\Sigma^\dagger \in \mathbb{C}^{M\times M}$ are diagonal square matrices with diagonal entries:

$$\forall i \in [N] : (\Sigma^\dagger\Sigma)_{ii} = \begin{cases} 1 & \text{if} \quad i \leq \min\{M, N\} \text{ and } \Sigma_{ii} \neq 0 \; ; \\ 0 & \text{otherwise.} \end{cases}$$
$$\forall i \in [M] : (\Sigma\Sigma^\dagger)_{ii} = \begin{cases} 1 & \text{if} \quad i \leq \min\{M, N\} \text{ and } \Sigma_{ii} \neq 0 \; ; \\ 0 & \text{otherwise.} \end{cases}$$

It is then easy to check that $\Sigma^\dagger$ satisfies the first two conditions of the Moore-Penrose pseudoinverse: $\Sigma\Sigma^\dagger\Sigma = \Sigma$ and $\Sigma^\dagger\Sigma\Sigma^\dagger = \Sigma^\dagger$. Besides, $\Sigma^\dagger\Sigma$ and $\Sigma\Sigma^\dagger$ being real diagonal matrices, the last two conditions are clearly satisfied too.

2) We can check that the matrix $V\Sigma^\dagger U^*$ satisfies the four conditions of the Moore-Penrose pseudoinverse, i.e., $A^\dagger = V\Sigma^\dagger U^*$:

$$A[V\Sigma^\dagger U^*]A = U\Sigma(V^*V)\Sigma^\dagger(U^*U)\Sigma V^* = U\Sigma\Sigma^\dagger\Sigma V^* = U\Sigma V^* = A \; ;$$
$$[V\Sigma^\dagger U^*]A[V\Sigma^\dagger U^*] = V\Sigma^\dagger(U^*U)\Sigma(V^*V)\Sigma^\dagger U^* = V\Sigma^\dagger\Sigma\Sigma^\dagger U^* = V\Sigma^\dagger U^* \; ;$$
$$(AV\Sigma^\dagger U^*)^* = (U\Sigma\Sigma^\dagger U^*)* = U(\Sigma\Sigma^\dagger)^*U^* = U\Sigma\Sigma^\dagger U^* = AV\Sigma^\dagger U^* \; ;$$
$$(V\Sigma^\dagger U^*A)^* = (V\Sigma^\dagger\Sigma V^*)* = V(\Sigma^\dagger\Sigma)^*V^* = V\Sigma^\dagger\Sigma V^* = V\Sigma^\dagger U^*A \; .$$

3) A is full column rank, therefore A^*A is a full rank $N \times N$ matrix and has a unique inverse $(A^*A)^{-1}$. The matrix $(A^*A)^{-1}A^*$ satisfies the four conditions:

$$A[(A^*A)^{-1}A^*]A = A \; ; \; [(A^*A)^{-1}A^*]A[(A^*A)^{-1}A^*] = (A^*A)^{-1}A^* \; ;$$
$$(A[(A^*A)^{-1}A^*])^* = A[(A^*A)^{-1}A^*] \; ; \; ([(A^*A)^{-1}A^*]A)^* = A^*A(A^*A)^{-1} = I_{N\times N} = ([(A^*A)^{-1}A^*]A \; .$$

Hence $A^\dagger = (A^*A)^{-1}A^*$.

4) A is full row rank, therefore AA^* is a full rank $M \times M$ matrix and has a unique inverse $(AA^*)^{-1}$. The matrix $A^*(AA^*)^{-1}$ satisfies the four conditions:

$$A[A^*(AA^*)^{-1}]A = A \; ; \; [A^*(AA^*)^{-1}]A[A^*(AA^*)^{-1}] = A^*(AA^*)^{-1} \; ;$$
$$(A[A^*(AA^*)^{-1}])^* = (AA^*)^{-1}AA^* = I_{M\times M} = AA^\dagger \; ; \; ([A^*(AA^*)^{-1}]A)^* = A^*(AA^*)^{-1}A \; .$$

Hence $A^\dagger = A^*(AA^*)^{-1}$.

5) We have $AA^{-1}A = A$, $A^{-1}AA^{-1} = A^{-1}$, $(AA^{-1})^* = I_{M\times M} = AA^{-1}$, $(A^{-1}A)^* = I_{N\times N} = A^{-1}A$. Hence $A^\dagger = A^{-1}$.

6) A is full column rank so $A^\dagger A = I_{M\times M}$ and B is full column rank so $BB^\dagger = I_{N\times N}$. Therefore:

$$\begin{aligned}(AB)(B^\dagger A^\dagger)(AB) &= A(BB^\dagger)(A^\dagger A)B = AI_{M\times M}I_{N\times N}B = AB \;; \\ (B^\dagger A^\dagger)(AB)(B^\dagger A^\dagger) &= B^\dagger(A^\dagger A)(BB^\dagger)A^\dagger = B^\dagger I_{N\times N}I_{M\times M}A^\dagger = B^\dagger A^\dagger \;; \\ (ABB^\dagger A^\dagger)^* &= (AI_{N\times N}A^\dagger)^* = (AA^\dagger)^* = AA^\dagger = (AB)(B^\dagger A^\dagger) \;; \\ (B^\dagger A^\dagger AB)^* &= (B^\dagger I_{M\times M}B)^* = (B^\dagger B)^* = B^\dagger B = (B^\dagger A^\dagger)(AB) \;.\end{aligned}$$

Hence $(AB)^\dagger = B^\dagger A^\dagger$.