Final Exam

Note: Please pay attention to the presentation of your answers! **(3 points)**

Exercise 1. Quiz. (18 points)
For each assertion below, state whether it is correct or not (1 point) and provide a short justification for your answer (2 points).

a) Let A, B be two generic subsets of Ω. Then $\sigma(A, B) = \sigma(A, B\backslash A)$.

b) If the random variables X, Y, Z satisfy $\sigma(X) \perp\!\!\!\perp \sigma(Y)$ and $\sigma(X) \perp\!\!\!\perp \sigma(Z)$, then $\sigma(X) \perp\!\!\!\perp \sigma(Y, Z)$.

c) Let F be a generic cdf. Then $G(t) = \begin{cases} 1/(1 - \log(F(t))), & \text{if } F(t) > 0, \\ 0, & \text{if } F(t) = 0, \end{cases}$ is necessarily also a cdf.

d) The function $\phi(t) = \begin{cases} 1, & \text{if } |t| \leq 1, \\ 0, & \text{if } |t| > 1, \end{cases}$ is the characteristic function of a random variable X.

e) Let X, Y be two i.i.d. $\mathcal{N}(0,1)$ random variables and $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. Then $f(X + Y)$ and $f(X - Y)$ are independent.

f) Let $(X_n,\, n \geq 2)$ be a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables and $(M_n,\, n \in \mathbb{N})$ be the process defined recursively as follows:

$$M_0 = M_1 = 0, \quad M_{n+1} = \frac{M_n + M_{n-1}}{2} + X_{n+1}, \quad \text{for } n \geq 1.$$

Then $(M_n,\, n \geq 1)$ is a martingale (with respect to its natural filtration $\mathcal{F}_n = \sigma(M_0, \ldots, M_n), n \geq 0$).

Exercise 2. (15 points)

Hints for this exercise: For any $a, b \in \mathbb{C}$ and $n \geq 1$, $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k\, b^{n-k}$ and $e^z \simeq 1 + z$ when $z \in \mathbb{C}$ and $|z|$ is small.

Let $(B_n,\, n \geq 1)$ be a sequence of random variables such that

$$\mathbb{P}\left(\left\{B_n = \frac{k}{n}\right\}\right) = \binom{n}{k} p^k\, (1 - p)^{n-k} \quad \text{for } 0 \leq k \leq n$$

where $0 < p < 1$ is a fixed parameter.

a) Compute $\mathbb{E}(B_n)$ and $\mathrm{Var}(B_n)$ for $n \geq 1$. (*Note:* You might use "well known" formulas here.)

b) Compute the characteristic function $\phi_{B_n}(t)$ for $t \in \mathbb{R}$ and $n \geq 1$.

c) To what limiting random variable B does the sequence $(B_n,\, n \geq 1)$ converge in distribution? Justify your reasoning.

Exercise 3. (21 points + BONUS 3 points)

Let X, Y be two random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that

$$\mathbb{P}(\{X=+1, Y=+1\}) = p - \tfrac{q}{2} \qquad \mathbb{P}(\{X=+1, Y=-1\}) = \tfrac{q}{2}$$

$$\mathbb{P}(\{X=-1, Y=+1\}) = \tfrac{q}{2} \qquad \mathbb{P}(\{X=-1, Y=-1\}) = 1 - p - \tfrac{q}{2}$$

where $0 \le q \le 1$ and $\frac{q}{2} \le p \le 1 - \frac{q}{2}$ are fixed parameters.

a) Compute all values of p and q for which X and Y are independent.

b) Compute $\mathbb{P}(\{X = x\} \,|\, \{X + Y = z\})$ for all possible values of x and z (and all possible p, q).

c) Compute $\mathbb{E}(X|X+Y)$ and $C = \mathbb{E}((X - \mathbb{E}(X|X+Y))^2)$.

BONUS d) Does there exist a square-integrable random variable $U = f(X+Y)$ (with $f : \mathbb{R} \to \mathbb{R}$ Borel-measurable) such that $\mathbb{E}((X-U)^2) < C$? If yes, exhibit such a random variable U and compute $\mathbb{E}((X-U)^2)$; if not, justify why.

Consider now $((X_n, Y_n),\, n \ge 1)$ a sequence of independent random vectors defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that (X_n, Y_n) has the same distribution as (X, Y) above, for every $n \ge 1$.

Let also, for $n \ge 1$, $Z_n = X_n + Y_n$, $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$, $R_n = \sum_{j=1}^n X_j$ and $S_n = \sum_{j=1}^n Z_j$.

e) For $n \ge 1$, compute $\mathbb{E}(R_n \,|\, \mathcal{F}_n)$ and $\mathbb{E}(R_n \,|\, S_n)$.

Exercise 4. (18 points + BONUS 3 points)

Hint for this exercise: For $0 < a < 1$, $\sum_{j \ge 1} a^j = \frac{a}{1-a}$.

Let $(U_n,\, n \ge 1)$ and $(V_n,\, n \ge 1)$ be two independent sequences of i.i.d. random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}(\{U_1 = 1\}) = p = 1 - \mathbb{P}(\{U_1 = 0\})$ and $\mathbb{P}(\{V_1 = 1\}) = q = 1 - \mathbb{P}(\{V_1 = 0\})$, where $0 \le p, q \le 1$ are fixed parameters.

Let also $W_0 = 0$ and $W_n = \displaystyle\sum_{j=1}^n \frac{U_j + V_j}{3^j}$, for $n \ge 1$.

a) Show that $W = \lim_{n\to\infty} W_n$ exists a.s. and that $\lim_{n\to\infty} \mathbb{E}((W_n - W)^2) = 0$.

b) For a given $n \ge 1$, compute $\mathbb{E}(W|\mathcal{F}_n) - W_n$, where $\mathcal{F}_n = \sigma(U_1, \ldots, U_n, V_1, \ldots, V_n)$.

BONUS c) Are there values of p, q such that W is a uniform random variable on $[0, 1]$? If yes, compute these values; if not, justify why.

Let now $\mathcal{G}_0 = \{\emptyset, \Omega\}$ and $\mathcal{G}_n = \sigma(U_1, \ldots, U_n)$ for $n \ge 1$.

d) Compute $M_n = \mathbb{E}(W|\mathcal{G}_n)$ for $n \ge 0$.

e) Explain why there exists a random variable M_∞ such that $M_n \underset{n\to\infty}{\to} M_\infty$ almost surely, and compute M_∞.

f) Does it also hold that $\mathbb{E}(M_\infty|\mathcal{F}_n) = M_n$ for every $n \ge 0$? Justify your answer.