Final Exam: Solutions

Note: Please pay attention to the presentation of your answers! **(3 points)**

Exercise 1. Quiz. (18 points)
For each assertion below, state whether it is correct or not (1 point) and provide a short justification for your answer (2 points).

a) Let A, B be two generic subsets of Ω. Then $\sigma(A, B) = \sigma(A, B\backslash A)$.

Answer: Incorrect. The set $A \cap B$ does not belong to the second.

b) If the random variables X, Y, Z satisfy $\sigma(X) \perp\!\!\!\perp \sigma(Y)$ and $\sigma(X) \perp\!\!\!\perp \sigma(Z)$, then $\sigma(X) \perp\!\!\!\perp \sigma(Y, Z)$.

Answer: Incorrect. Ctr-ex: $Y \perp\!\!\!\perp Z$, each taking values $\{0, 1\}$ wp $1/2$, $X = Y + Z \pmod 2$.

c) Let F be a generic cdf. Then $G(t) = \begin{cases} 1/(1 - \log(F(t))), & \text{if } F(t) > 0, \\ 0, & \text{if } F(t) = 0, \end{cases}$ is necessarily also a cdf.

Answer: Correct. G is non-decreasing, right-continuous, $\lim_{t\to-\infty} G(t) = 0$ and $\lim_{t\to+\infty} G(t) = 1$.

d) The function $\phi(t) = \begin{cases} 1, & \text{if } |t| \leq 1, \\ 0, & \text{if } |t| > 1, \end{cases}$ is the characteristic function of a random variable X.

Answer: Incorrect. ϕ is not continuous.

e) Let X, Y be two i.i.d. $\mathcal{N}(0, 1)$ random variables and $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. Then $f(X + Y)$ and $f(X - Y)$ are independent.

Answer: Correct, as $X + Y$ and $X - Y$ are independent.

f) Let $(X_n,\, n \geq 2)$ be a sequence of i.i.d. $\mathcal{N}(0, 1)$ random variables and $(M_n,\, n \in \mathbb{N})$ be the process defined recursively as follows:

$$M_0 = M_1 = 0, \quad M_{n+1} = \frac{M_n + M_{n-1}}{2} + X_{n+1}, \quad \text{for } n \geq 1.$$

Then $(M_n,\, n \geq 1)$ is a martingale (with respect to its natural filtration $\mathcal{F}_n = \sigma(M_0, \ldots, M_n), n \geq 0$).

Answer: Incorrect: $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \frac{M_n + M_{n-1}}{2} \neq M_n$.

Exercise 2. (15 points)

Hint for this exercise: For any $a, b \in \mathbb{C}$ and $n \geq 1$, $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k \, b^{n-k}$ and $e^z \simeq 1 + z$ when $z \in \mathbb{C}$ and $|z|$ is small.

Let $(B_n,\, n \geq 1)$ be a sequence of random variables such that

$$\mathbb{P}\left(\left\{B_n = \frac{k}{n}\right\}\right) = \binom{n}{k} p^k \, (1-p)^{n-k} \quad \text{for } 0 \leq k \leq n$$

where $0 < p < 1$ is a fixed parameter.

a) Compute $\mathbb{E}(B_n)$ and $\mathrm{Var}(B_n)$ for $n \geq 1$. (*Note:* You might use “well known” formulas here.)

Answer: (5 points) nB_n is a Binomial(n,p) random variable, so

$$\mathbb{E}(B_n) = \frac{np}{n} = p \quad \text{and} \quad \mathrm{Var}(B_n) = \frac{np\,(1-p)}{n^2} = \frac{p\,(1-p)}{n}$$

b) Compute the characteristic function $\phi_{B_n}(t)$ for $t \in \mathbb{R}$ and $n \geq 1$.

Answer: (5 points) Using the same argument (or via an explicit computation using the hint):

$$\phi_{B_n}(t) = \phi_{nB_n}(t/n) = (p\, e^{it/n} + (1-p))^n$$

c) To what limiting random variable B does the sequence $(B_n,\, n \geq 1)$ converge in distribution? Justify your reasoning.

Answer: (5 points) Using b) together with the criterion that convergence in distribution holds if and only if the respective characteristic functions converge, we find:

$$\phi_{B_n}(t) = (p\, e^{it/n} + (1-p))^n \simeq \left(p\left(1 + \frac{it}{n}\right) + 1 - p\right)^n = \left(1 + \frac{itp}{n}\right)^n \underset{n\to\infty}{\to} e^{itp}$$

which is the characteristic function of the constant random variable $B = p$.

Exercise 3. (21 points + BONUS 3 points)

Let X, Y be two random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that

$$\mathbb{P}(\{X=+1, Y=+1\}) = p - \tfrac{q}{2} \qquad \mathbb{P}(\{X=+1, Y=-1\}) = \tfrac{q}{2}$$

$$\mathbb{P}(\{X=-1, Y=+1\}) = \tfrac{q}{2} \qquad \mathbb{P}(\{X=-1, Y=-1\}) = 1 - p - \tfrac{q}{2}$$

where $0 \leq q \leq 1$ and $\frac{q}{2} \leq p \leq 1 - \frac{q}{2}$ are fixed parameters.

a) Compute all values of p and q for which X and Y are independent.

Answer: (5 points) $\mathbb{P}(\{X=+1\}) = p$ and $\mathbb{P}(\{Y=+1\}) = p$, so to obtain independence, we need:

$$p^2 = p - \frac{q}{2} \quad \text{i.e.} \quad q \in [0, 1/2] \quad \text{and} \quad p = \frac{1 \pm \sqrt{1-2q}}{2}$$

(and one checks that the above p indeed satisfies $q/2 \leq p \leq 1 - q/2$).

b) Compute $\mathbb{P}(\{X = x\} \,|\, \{X + Y = z\})$ for all possible values of x and z (and all possible p, q).

Answer: (5 points) For $X+Y=\pm 2$, we necessarily have $X=\pm 1$, so

$$\mathbb{P}(\{X=+1\}\,|\,\{X+Y=+2\})=1 \quad \text{and} \quad \mathbb{P}(\{X=-1\}\,|\,\{X+Y=-2\})=1$$

For $X+Y=0$, we have

$$\begin{aligned}\mathbb{P}(\{X=+1\}\,|\,\{X+Y=0\}) &= \frac{\mathbb{P}(\{X=+1, X+Y=0,\})}{\mathbb{P}(\{X+Y=0\})} \\ &= \frac{\mathbb{P}(\{X=+1, Y=-1\})}{\mathbb{P}(\{X=+1, Y=-1\})+\mathbb{P}(\{X=-1, Y=+1\})} = \frac{q/2}{q/2+q/2} = \frac{1}{2}\end{aligned}$$

and similarly $\mathbb{P}(\{X=-1\}\,|\,\{X+Y=0\})=\frac{1}{2}$.

c) Compute $\mathbb{E}(X|X+Y)$ and $C=\mathbb{E}((X-\mathbb{E}(X|X+Y))^2)$.

Answer: (5 points) We find that

$$\mathbb{E}(X|\{X+Y=j\}) = \begin{cases} +1 & \text{if } j=+2 \\ 0 & \text{if } j=0 \\ -1 & \text{if } j=-2 \end{cases} = \frac{j}{2}$$

so $\mathbb{E}(X|X+Y)=\frac{X+Y}{2}$ and

$$C=\mathbb{E}((X-\mathbb{E}(X|X+Y))^2) = \mathbb{E}\left(\left(\frac{X-Y}{2}\right)^2\right) = \frac{q}{2}+\frac{q}{2}=q$$

BONUS d) Does there exist a square-integrable random variable $U=f(X+Y)$ (with $f:\mathbb{R}\to\mathbb{R}$ Borel-measurable) such that $\mathbb{E}((X-U)^2)<C$? If yes, exhibit such a random variable U and compute $\mathbb{E}((X-U)^2)$; if not, justify why.

Answer: (3 points, 1 for the answer, 2 for the justification)

No: the conditional expectation $\mathbb{E}(X|X+Y)$ is by definition the random variable which minimizes $\mathbb{E}((X-U)^2)$ among all $\sigma(X+Y)$-measurable and square-integrable random variables U.

Consider now $((X_n,Y_n),\, n\geq 1)$ a sequence of independent random vectors defined on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that (X_n,Y_n) has the same distribution as (X,Y) above, for every $n\geq 1$.

Let also, for $n\geq 1$, $Z_n=X_n+Y_n$, $\mathcal{F}_n=\sigma(Z_1,\ldots,Z_n)$, $R_n=\sum_{j=1}^n X_j$ and $S_n=\sum_{j=1}^n Z_j$.

e) For $n\geq 1$, compute $\mathbb{E}(R_n\,|\,\mathcal{F}_n)$ and $\mathbb{E}(R_n\,|\,S_n)$.

Answer: (6 points) By independence and part c), we obtain

$$\mathbb{E}(R_n\,|\,\mathcal{F}_n)=\sum_{j=1}^n \mathbb{E}(X_j\,|\,\mathcal{F}_n)=\sum_{j=1}^n \mathbb{E}(X_j\,|\,Z_j)=\sum_{j=1}^n \frac{Z_j}{2}=\frac{S_n}{2}$$

and therefore, by the towering property of conditional expectation:

$$\mathbb{E}(R_n\,|\,S_n)=\mathbb{E}(\mathbb{E}(R_n\,|\,\mathcal{F}_n)|S_n)=\mathbb{E}\left(\frac{S_n}{2}\,\middle|\,S_n\right)=\frac{S_n}{2}$$

Exercise 4. (18 points + BONUS 3 points)

Hint for this exercise: For $0 < a < 1$, $\sum_{j\geq 1} a^j = \frac{a}{1-a}$.

Let $(U_n,\, n \geq 1)$ and $(V_n,\, n \geq 1)$ be two independent sequences of i.i.d. random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}(\{U_1 = 1\}) = p = 1 - \mathbb{P}(\{U_1 = 0\})$ and $\mathbb{P}(\{V_1 = 1\}) = q = 1 - \mathbb{P}(\{V_1 = 0\})$, where $0 \leq p, q \leq 1$ are fixed parameters.

Let also $W_0 = 0$ and $W_n = \displaystyle\sum_{j=1}^{n} \frac{U_j + V_j}{3^j}$, for $n \geq 1$.

a) Show that $W = \lim_{n\to\infty} W_n$ exists a.s. and that $\lim_{n\to\infty} \mathbb{E}((W_n - W)^2) = 0$.

Answer: (5 points) For every $\omega \in \Omega$, $W_n(\omega)$ is a Cauchy sequence, as for $n \geq m \geq 1$

$$0 \leq W_n(\omega) - W_m(\omega) = \sum_{j=m+1}^{n} \frac{U_j(\omega) + V_j(\omega)}{3^j} \leq 2 \sum_{j\geq m+1} \frac{1}{3^j} = \frac{1}{3^m} \underset{m\to\infty}{\to} 0$$

so the sequence $W_n(\omega)$ converges for every $\omega \in \Omega$. Moreover, by independence,

$$\mathbb{E}((W_n - W)^2) = \mathrm{Var}(W_n - W) + \mathbb{E}((W_n - W)^2) = \sum_{j\geq n+1} \mathrm{Var}\left(\frac{U_j + V_j}{3^j}\right) + \left(\sum_{j\geq n+1} \frac{\mathbb{E}(U_j + V_j)}{3^j}\right)^2$$

$$\leq \sum_{j\geq n+1} \frac{4}{3^{2j}} + \left(\sum_{j\geq n+1} \frac{2}{3^j}\right)^2$$

which converges to 0 as $n \to \infty$ for the same reasons as above.

b) For a given $n \geq 1$, compute $\mathbb{E}(W|\mathcal{F}_n) - W_n$, where $\mathcal{F}_n = \sigma(U_1, \ldots, U_n, V_1, \ldots, V_n)$.

Answer: (3 points) Using again independence, we find:

$$\mathbb{E}(W|\mathcal{F}_n) - W_n = W_n + \mathbb{E}\left(\sum_{j\geq n+1} \frac{U_j + V_j}{3^j}\right) - W_n = (p+q) \sum_{j\geq n+1} \frac{1}{3^j} = \frac{p+q}{2}\,\frac{1}{3^n}$$

BONUS c) Are there values of p, q such that W is a uniform random variable on $[0,1]$? If yes, compute these values; if not, justify why.

Answer: (3 points, 1 for the answer, 2 for the justification)

No: To obtain a uniform W, we would need $U_1 + V_1$ to be uniformly distributed on $\{0,1,2\}$. But this would mean

$$1/3 = pq = p(1-q) + q(1-p) = (1-p)(1-q)$$

and there are no values of p and q in $[0,1]$ satisfying these 3 equalities at once.

Let now $\mathcal{G}_0 = \{\emptyset, \Omega\}$ and $\mathcal{G}_n = \sigma(U_1, \ldots, U_n)$ for $n \geq 1$.

d) Compute $M_n = \mathbb{E}(W|\mathcal{G}_n)$ for $n \geq 0$.

Answer: (4 points) As the V's are independent of $\mathcal{G}_n$, we obtain

$$\begin{aligned} M_n = \mathbb{E}(W|\mathcal{G}_n) &= \sum_{j=1}^{n} \frac{U_j}{3^j} + \mathbb{E}\left(\sum_{j \geq n+1} \frac{U_j}{3^j}\right) + \mathbb{E}\left(\sum_{j \geq 1} \frac{V_j}{3^j}\right) \\ &= \sum_{j=1}^{n} \frac{U_j}{3^j} + \frac{p}{3^n}\,\frac{1}{2} + \frac{q}{2} \end{aligned}$$

e) Explain why there exists a random variable M_∞ such that $M_n \underset{n \to \infty}{\to} M_\infty$ almost surely, and compute M_∞.

Answer: (3 points) M is a non-negative martingale, so by the martingale convergence theorem (version 2), the a.s. limit M_∞ exists. Using the above formula, we find moreover that

$$M_\infty = \sum_{j \geq 1} \frac{U_j}{3^j} + \frac{q}{2}$$

f) Does it also hold that $\mathbb{E}(M_\infty|\mathcal{F}_n) = M_n$ for every $n \geq 0$? Justify your answer.

Answer: (3 points, 1 for the answer, 2 for the justification)

Yes, it does, as $\mathbb{E}(M_\infty|\mathcal{F}_n) = \mathbb{E}(M_\infty|\mathcal{G}_n)$ and

$$\sup_{n \geq 1} \mathbb{E}(M_n^2) = \sup_{n \geq 1} \mathbb{E}(\mathbb{E}(W|\mathcal{G}_n)^2) \overset{(*)}{\leq} \sup_{n \geq 1} \mathbb{E}(\mathbb{E}(W^2|\mathcal{G}_n)) = \sup_{n \geq 1} \mathbb{E}(W^2) = \mathbb{E}(W^2) < +\infty$$

where $(*)$ follows from Jensen's inequality for conditional expectation. So the first version of the martingale convergence theorem applies.