Final exam

Exercise 1. Quiz. (20 points)
Answer each yes/no question below (2 pts) and provide a short justification for your answer (2 pts).

a) Does there exist a sequence of independent random variables $(X_n,\, n \geq 1)$ such that the sequence of σ-fields $(\mathcal{F}_n,\, n \geq 1)$ defined as

$$\mathcal{F}_n = \sigma(X_1 + \ldots + X_n), \quad n \geq 1$$

is a filtration, and moreover, $\mathcal{F}_n \neq \mathcal{F}_{n+1}$ for every $n \geq 1$? If yes, exhibit such a sequence of random variables; if no, explain why such a sequence cannot exist.

b) Let X, Z be two independent random variables such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = +2\}) = \frac{1}{2}$ and $\mathbb{P}(\{Z = -1\}) = \mathbb{P}(\{Z = 0\}) = \frac{1}{2}$. Let also $Y = X \cdot Z$. Does it hold that $t \mapsto F_Y(t)$ has four jumps (i.e., four values of t where it is not continuous) ? Justify your answer.

c) Let X_1, X_2 be two random variables such that $X_1 \sim \mathcal{N}(0, \sigma_1^2)$, $X_2 \sim \mathcal{N}(0, \sigma_2^2)$ and

$$\mathbb{E}(\exp(it_1X_1 + it_2X_2)) = \exp\left(-\frac{(\sigma_1\, t_1 + \sigma_2\, t_2)^2}{2}\right), \quad \forall t_1, t_2 \in \mathbb{R}$$

Does it hold that X_1 and X_2 are independent ? Justify your answer.

d) Let X_1, X_2 be two independent random variables taking values in $\{-1, +1\}$ and such that $\mathbb{P}(\{X_1 = +1\}) = \mathbb{P}(\{X_1 = -1\}) = \frac{1}{2}$. Does it hold that

$$\mathbb{E}(\,|X_1 + X_2| \mid X_2) = |X_2| \quad ?$$

Justify your answer.

e) Do there exist real numbers $p, q \in [0, 1]$ and random variables X_1, X_2, X_3 taking values in $\{-1, +1\}$ satisfying the following properties ?

- $\mathbb{P}(\{X_1 = x_1, X_2 = x_2, X_3 = x_3\}) = \begin{cases} p & \text{if } x_1 = x_2 = x_3 \\ q & \text{otherwise} \end{cases}$
- $\mathbb{E}(X_j\, X_k) = -\frac{1}{2}$ for every $j, k \in \{1, 2, 3\}$ with $j < k$.

If yes, exhibit such random variables X_1, X_2, X_3, along with the corresponding values of p and q; if no, explain why they cannot exist.

Exercise 2. (15 points)

Note: For this exercise, the following facts might help:

- For $s, t \in \mathbb{R}$, $\cosh(s) = \frac{\exp(s)+\exp(-s)}{2}$ and $\cos(t) = \frac{\exp(it)+\exp(-it)}{2}$
- For $x \in \mathbb{R}$ and $n \geq 1$, $(1 + \frac{x}{n})^n \leq \exp(x)$; also, $\lim_{n\to\infty}(1 + \frac{x}{n})^n = \exp(x)$

For a given value of $n \geq 1$, let $(X_1^{(n)}, \ldots, X_n^{(n)})$ be i.i.d. random variables such that

$$\mathbb{P}(\{X_1^{(n)} = +1\}) = \mathbb{P}(\{X_1^{(n)} = -1\}) = \frac{1}{2n} \quad \text{and} \quad \mathbb{P}(\{X_1^{(n)} = 0\}) = 1 - \frac{1}{n}$$

Let also $S_n = X_1^{(n)} + \ldots + X_n^{(n)}$.

a) Compute $\mathbb{E}(S_n)$ and $\mathrm{Var}(S_n)$ for $n \geq 1$.

b) Using Chebyshev's inequality with $\varphi(x) = \exp(sx)$, $s > 0$, show that for every $c > 0$

$$\mathbb{P}(\{|S_n| \geq c\,\log(n) \quad \text{infinitely often}\}) = 0$$

Hint: There is no need here to optimize over s your upper bound on the probability: for each value of $c > 0$, you just need to find an appropriate value of $s > 0$ that allows to conclude.

c) Show that the sequence of random variables $(S_n,\, n \geq 1)$ converges in distribution.

Hint: You may use here the fact (not seen in class) that if a sequence of characteristic functions converges to a limit which is a continuous function, then the corresponding sequence of random variables converges in distribution (no need to compute the limiting distribution).

d) Can you tell whether the limiting distribution in part c) is discrete or continuous ? Justify your answer.

Hint: Again, there is no need to compute explicitly the limiting distribution here.

Exercise 3. (17 points) Let $N \geq 1$ be a fixed integer and $Z = (Z_1, \ldots, Z_N)$ be a random vector composed of i.i.d. $\mathcal{N}(0,1)$ random variables. Let also

$$\|Z\| = \sqrt{\sum_{n=1}^{N} Z_n^2}, \quad Y_n = \frac{Z_n}{\|Z\|}, \quad \text{for } 1 \leq n \leq N$$

Hint: For this exercise, you may use the rather remarkable fact that *conditioned on* $\|Z\|$, the vector $(Z_1, \ldots, Z_N)$ is uniformly distributed on the N-dimensional sphere of radius $\|Z\|$. This implies in particular that for every n, the vector $(Z_1, \ldots, -Z_n, \ldots, Z_N)$ is also uniformly distributed on the N-dimensional sphere of radius $\|Z\|$.

a) Compute $\mathbb{E}(Y_n)$, $\mathrm{Var}(Y_n)$ and $\mathbb{E}(Y_n\, Y_m)$ for $1 \leq n, m \leq N$ with $n \neq m$.

Let now $M_0 = 0$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $M_n = \sum_{j=1}^{n} Y_j$, $\mathcal{F}_n = \sigma(\|Z\|, Y_1, \ldots, Y_n)$ for $1 \leq n \leq N$.

b) Compute $\mathbb{E}(M_n)$ and $\mathrm{Var}(M_n)$ for $1 \leq n \leq N$.

c) Show that $\mathcal{F}_n = \sigma(\|Z\|, Z_1, \ldots, Z_n)$ for every $1 \leq n \leq N$.

d) Show that $(M_n,\, 0 \leq n \leq N)$ is martingale with respect to $(\mathcal{F}_n,\, 0 \leq n \leq N)$.

e) Using a theorem seen in the course (not forgetting to check its assumptions), show that

$$\mathbb{P}(\{|M_n - M_0| \geq nt\}) \leq 2\, \exp(-nt^2/2), \quad \text{for every } 1 \leq n \leq N \text{ and } t > 0$$

Exercise 4. (18 points + BONUS 5 points) Let us consider the process $(M_n,\, n \geq 0)$ defined recursively as:

$$M_0 = M_1 = 1, \quad M_{n+1} = \begin{cases} M_n + M_{n-1} & \text{with probability } \frac{1}{2} \\ \\ M_n - M_{n-1} & \text{with probability } \frac{1}{2} \end{cases} \qquad \text{for } n \geq 1$$

a) Show that $(M_n,\, n \geq 0)$ is a martingale with respect to its natural filtration $(\mathcal{F}_n,\, n \geq 0)$.

b) Does it also hold that $\mathbb{E}(M_{n+1} \,|\, M_n) = M_n$ for every $n \geq 0$? Justify your answer.

c) Compute the distribution of M_4.

Hint: It is highly advised to draw carefully the binary tree leading to the possible values of M_4.

d) Compute recursively the sequence of numbers $f(n) = \mathbb{E}(M_n^2)$, for $n \geq 0$. What is this sequence ?

e) Are the conditions of the first martingale convergence theorem satisfied ?

f) Does there exist a random variable M_∞ such that $M_n \underset{n\to\infty}{\to} M_\infty$ almost surely ? Justify your answer.

Hint: Consider the values that M_n can take when $n = 2 \pmod 3$ and $n \neq 2 \pmod 3$.

BONUS g) For a fixed value of $n \geq 3$, what is the value of $\max_{\omega\in\Omega} M_n(\omega)$? And what is the value of $\min_{\omega\in\Omega} M_n(\omega)$? Justify.