Homework 2
Quantum Information Processing

Exercise 1 *Orthonormal basis and measurement principle*

Let $\{|x\rangle, |y\rangle\}$ an orthonormal basis of $\mathbb{C}^2$. This means that $\langle x|x\rangle = \langle y|y\rangle = 1$ and $\langle x|y\rangle = \langle y|x\rangle = 0$. Let $|\alpha\rangle = \cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle$, $|\alpha_\perp\rangle = -\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle$, $|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i\,|y\rangle)$, $|L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i\,|y\rangle)$.

1) Check that $\{|\alpha\rangle, |\alpha_\perp\rangle\}$ and $\{|R\rangle, |L\rangle\}$ are two orthonormal basis.

2) We measure the polarization with three different measurement apparatus. The first apparatus is modeled by the basis $\{|x\rangle, |y\rangle\}$; the second one is modeled by the basis $\{|R\rangle, |L\rangle\}$; and the third one by the basis $\{|\alpha\rangle, |\alpha_\perp\rangle\}$. Let

$$|\psi\rangle = \cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle$$

be the polarized state of a photon just before the measurement. For each of the three experiments, give the (two) possible outcoming states just after the measurement and their corresponding probabilities of outcome. *You are asked to give the probabilities in terms of real quantities (i.e., compute the complex modulus) and completely expand the squares of trigonometric expressions.*

Exercise 2 *Interferometer revisited*

Let $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, $R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ be the two matrices representing a semi-transparent mirror and a perfectly reflecting mirror.

1) Compute $S|H\rangle$, $S|V\rangle$ and $R|H\rangle$, $R|V\rangle$ and give the result in Dirac notation.

2) Compute the state $SRS\,|H\rangle$ as well as the probabilities $|\langle H|\,SRS\,|H\rangle|^2$ and $|\langle V|\,SRS\,|H\rangle|^2$ and verify they sum to one..

   Make a picture of the experimental set-up.

3) We introduce a "dephaser" described by $D = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix}$ where φ_1 and φ_2 are two different phases (angles). Verify this matrix is unitary. Compute $D|H\rangle$ and $D|V\rangle$ and give the result in Dirac notation.

   We consider the operation $SRDS$. Make a picture of the experimental situation. Compute $SRDS\,|H\rangle$, and the probabilities $|\langle H|\,SRDS\,|H\rangle|^2$ and $|\langle V|\,SRDS\,|H\rangle|^2$. *You are* *asked to compute the complex modulus so that you result involves only real quantities, and also give the expressions as a function of the difference $\varphi_1 - \varphi_2$. Check that you find expressions that are periodic functions of $\varphi_1 - \varphi_2$.*

Verify also that the matrix $SRDS$ is unitary and relate this fact to the other fact that the two probabilities should sum to one.