Exercise 1 *Orthonormal basis and measurement principle*

1) It involves the following checking:

$$
\begin{aligned}
\langle \alpha | \alpha \rangle &= \big(\cos\alpha \, \langle x | + \sin\alpha \, \langle y | \big)\big(\cos\alpha \, | x \rangle + \sin\alpha \, | y \rangle \big) \\
&= \cos^2\alpha + \sin^2\alpha = 1 \\
\langle \alpha_\perp | \alpha_\perp \rangle &= \big(-\sin\alpha \, \langle x | + \cos\alpha \, \langle y | \big)\big(-\sin\alpha \, | x \rangle + \cos\alpha \, | y \rangle \big) \\
&= \cos^2\alpha + \sin^2\alpha = 1 \\
\langle \alpha_\perp | \alpha \rangle &= \big(-\sin\alpha \, \langle x | + \cos\alpha \, \langle y | \big)\big(\cos\alpha \, | x \rangle + \sin\alpha \, | y \rangle \big) \\
&= -\sin\alpha \cos\alpha + \cos\alpha \sin\alpha = 0
\end{aligned}
$$

$$
\begin{aligned}
\langle R | R \rangle &= \frac{1}{2}\big(\langle x | - i\,\langle y | \big)\big(| x \rangle + i\,| y \rangle \big) = \frac{1}{2}(1^2 + (-i)i) = 1 \\
\langle L | L \rangle &= \frac{1}{2}\big(\langle x | + i\,\langle y | \big)\big(| x \rangle - i\,| y \rangle \big) = \frac{1}{2}(1^2 + i(-i)) = 1 \\
\langle R | L \rangle &= \frac{1}{2}\big(\langle x | - i\,\langle y | \big)\big(| x \rangle - i\,| y \rangle \big) = \frac{1}{2}(1^2 + (-i)(-i)) = 0
\end{aligned}
$$

2) For each experiment, the possible states just after the measurement would be the corresponding measurement basis with the following probabilities:

$$
\begin{aligned}
\mathrm{Prob}(|x\rangle) &= |\langle x | \psi \rangle|^2 = \cos^2\theta \\
\mathrm{Prob}(|y\rangle) &= |\langle y | \psi \rangle|^2 = |(\sin\theta)e^{i\varphi}|^2 = \sin^2\theta
\end{aligned}
$$

where we use $e^{i\varphi} = \cos\varphi + i\sin\varphi$ so that $|e^{i\varphi}|^2 = \cos^2\varphi + \sin^2\varphi = 1$. For the other probabilities we have:

$$
\begin{aligned}
\mathrm{Prob}(|R\rangle) &= |\langle R | \psi \rangle|^2 \\
&= \left| \frac{1}{\sqrt{2}} \big(\langle x | - i\,\langle y | \big)\big(\cos\theta\,| x \rangle + (\sin\theta)e^{i\varphi}\,| y \rangle \big) \right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \cos\theta - \frac{1}{\sqrt{2}} i \sin\theta\, e^{i\varphi} \right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \cos\theta + \frac{1}{\sqrt{2}} \sin\theta \sin\varphi - \frac{1}{\sqrt{2}} i \sin\theta \cos\varphi \right|^2 \\
&= \left(\frac{1}{\sqrt{2}} \cos\theta + \frac{1}{\sqrt{2}} \sin\theta \sin\varphi \right)^2 + \left(\frac{1}{\sqrt{2}} \sin\theta \cos\varphi \right)^2 \\
&= \frac{1}{2} + \cos\theta \sin\theta \sin\varphi
\end{aligned}
$$

$$\mathrm{Prob}(|L\rangle) = |\langle L|\psi\rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}} \left(\langle x| + i\,\langle y| \right) \left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle \right) \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} \cos\theta + \frac{1}{\sqrt{2}} i \sin\theta e^{i\varphi} \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} \cos\theta - \frac{1}{\sqrt{2}} \sin\theta \sin\varphi + \frac{1}{\sqrt{2}} i \sin\theta \cos\varphi \right|^2$$

$$= \left(\frac{1}{\sqrt{2}} \cos\theta - \frac{1}{\sqrt{2}} \sin\theta \sin\varphi \right)^2 + \left(\frac{1}{\sqrt{2}} \sin\theta \cos\varphi \right)^2$$

$$= \frac{1}{2} - \cos\theta \sin\theta \sin\varphi$$

$$\mathrm{Prob}(|\alpha\rangle) = |\langle \alpha|\psi\rangle|^2$$

$$= \left| \left(\cos\alpha\,\langle x| + \sin\alpha\,\langle y| \right) \left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle \right) \right|^2$$

$$= \left| \cos\alpha \cos\theta + \sin\alpha \sin\theta e^{i\varphi} \right|^2$$

$$= \left| \cos\alpha \cos\theta + \sin\alpha \sin\theta \cos\varphi + i \sin\alpha \sin\theta \sin\varphi \right|^2$$

$$= \left(\cos\alpha \cos\theta + \sin\alpha \sin\theta \cos\varphi \right)^2 + \left(\sin\alpha \sin\theta \sin\varphi \right)^2$$

$$= \cos^2\alpha \cos^2\theta + 2\cos\alpha \sin\alpha \cos\theta \sin\theta \cos\varphi + \sin^2\alpha \sin^2\theta$$

$$\mathrm{Prob}(|\alpha_\perp\rangle) = |\langle \alpha_\perp|\psi\rangle|^2$$

$$= \left| \left(\sin\alpha\,\langle x| + \cos\alpha\,\langle y| \right) \left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle \right) \right|^2$$

$$= \left| - \sin\alpha \cos\theta + \cos\alpha \sin\theta e^{i\varphi} \right|^2$$

$$= \left| - \sin\alpha \cos\theta + \cos\alpha \sin\theta \cos\varphi + i \cos\alpha \sin\theta \sin\varphi \right|^2$$

$$= \left(\sin\alpha \cos\theta - \cos\alpha \sin\theta \cos\varphi \right)^2 + \left(\cos\alpha \sin\theta \sin\varphi \right)^2$$

$$= \sin^2\alpha \cos^2\theta - 2\cos\alpha \sin\alpha \cos\theta \sin\theta \cos\varphi + \cos^2\alpha \sin^2\theta$$

One can verify that these probabilities are normalized to one,

$$\mathrm{Prob}(|x\rangle) + \mathrm{Prob}(|y\rangle) = \mathrm{Prob}(|R\rangle) + \mathrm{Prob}(|L\rangle) = \mathrm{Prob}(|\alpha\rangle) + \mathrm{Prob}(|\alpha_\perp\rangle) = 1.$$

Exercise 2 *Interferometer revisited*

1) Using Exercise 2.3, we have

$$S = |H\rangle\langle H| + i\,|H\rangle\langle V| + i\,|V\rangle\langle H| + |V\rangle\langle V|$$
$$R = i\,|H\rangle\langle V| + i\,|V\rangle\langle H|.$$

2) The computation in Dirac's notation is

$$RS = \frac{1}{\sqrt{2}} \left(-|H\rangle\langle H| + i\,|H\rangle\langle V| + i\,|V\rangle\langle H| - |V\rangle\langle V| \right),$$
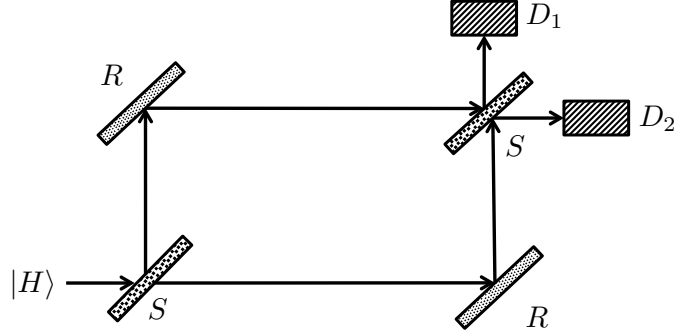$$SRS = -|H\rangle\langle H| - |V\rangle\langle V|.$$

The computation in usual matrix notation is

$$SRS = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$SRS\,|H\rangle = \big(-|H\rangle\langle H| - |V\rangle\langle V| \big)\,|H\rangle = -|H\rangle$$
$$|\langle H|\,SRS\,|H\rangle|^2 = |-\langle H|H\rangle|^2 = 1$$
$$|\langle V|\,SRS\,|H\rangle|^2 = |-\langle V|H\rangle|^2 = 0$$

The experimental set-up:

3) We have

$$SRDS = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -e^{i\varphi_1} - e^{i\varphi_2} & -ie^{i\varphi_1} + ie^{i\varphi_2} \\ ie^{i\varphi_1} - ie^{i\varphi_2} & -e^{i\varphi_1} - e^{i\varphi_2} \end{pmatrix}$$

and

$$SRDS\,|H\rangle = \frac{1}{2} \begin{pmatrix} -e^{i\varphi_1} - e^{i\varphi_2} \\ ie^{i\varphi_1} - ie^{i\varphi_2} \end{pmatrix},$$

which in Dirac notation is

$$SRDS\,|H\rangle = -\frac{e^{i\varphi_1} + e^{i\varphi_2}}{2}\,|H\rangle + \frac{ie^{i\varphi_1} - ie^{i\varphi_2}}{2}\,|V\rangle.$$
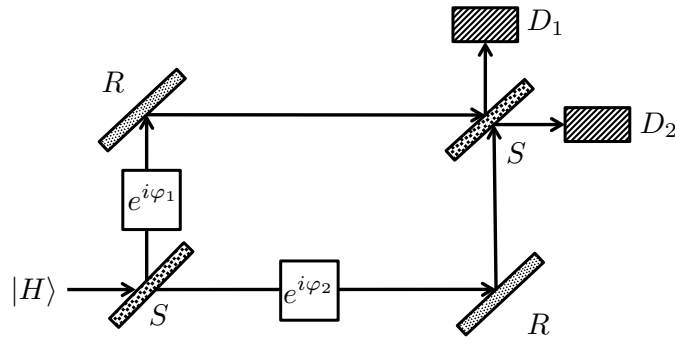
Then we have

$$\begin{aligned}
|\langle H|\,SRDS\,|H\rangle|^2 &= \left|\frac{e^{i\varphi_1} + e^{i\varphi_2}}{2}\right|^2 \\
&= \frac{1}{4}\,|\cos\varphi_1 + i\sin\varphi_1 + \cos\varphi_1 + i\sin\varphi_2|^2 \\
&= \frac{1}{4}\left((\cos\varphi_1 + \cos\varphi_2)^2 + (\sin\varphi_1 + \sin\varphi_2)^2\right) \\
&= \frac{1}{4}\left(2 + 2\cos\varphi_1\cos\varphi_2 + 2\sin\varphi_1\sin\varphi_2\right) \\
&= \frac{1}{2}(1 + \cos(\varphi_1 - \varphi_2)) \\
&= \cos^2\left(\frac{\varphi_1 - \varphi_2}{2}\right)
\end{aligned}$$

and

$$| \langle V | \, SRDS \, | H \rangle \, |^2 = \left| \frac{ie^{i\varphi_1} - ie^{i\varphi_2}}{2} \right|^2$$

$$= \frac{1}{4} \left| - \sin \varphi_1 + i \cos \varphi_1 + \sin \varphi_1 - i \cos \varphi_2 \right|^2$$

$$= \frac{1}{4} \left((\sin \varphi_1 - \sin \varphi_2)^2 + (\cos \varphi_1 - \cos \varphi_2)^2 \right)$$

$$= \frac{1}{4} \left(2 - 2 \cos \varphi_1 \cos \varphi_2 - 2 \sin \varphi_1 \sin \varphi_2 \right)$$

$$= \frac{1}{2} (1 - \cos(\varphi_1 - \varphi_2))$$

$$= \sin^2 \left(\frac{\varphi_1 - \varphi_2}{2} \right)$$

The experimental set-up:

A proof that $SRDS$ is unitary: Recall the notation $A^\dagger = A^{\top, *}$. We have checked that $SS^\dagger = S^\dagger S = I$ and $RR^\dagger = R^\dagger R = I$ in Homework 2. It is also easy to check $DD^\dagger = D^\dagger D = I$. The product of unitary matrices is unitary, indeed

$$(U_1 U_2)(U_1 U_2)^\dagger = U_1 U_2 U_2^\dagger U_1^\dagger = U_1 U_1^\dagger = I.$$

Remarks: The matrix elements are $\begin{pmatrix} \langle H | \, SRDS \, | H \rangle & \langle H | \, SRDS \, | V \rangle \\ \langle V | \, SRDS \, | H \rangle & \langle V | \, SRDS \, | V \rangle \end{pmatrix}$.

- We saw that for a unitary matrix the sum of the modulus squares of rows or columns equal 1. For example for the first column we have $| \langle H | \, SRDS \, | H \rangle \, |^2 + | \langle V | \, SRDS \, | H \rangle \, |^2 = 1$. This expresses the fact that the two probabilities of finding the photon in state $\langle H |$ or $\langle V |$ after the measurement is 1.

- Similarly if we would do an experiment with a photon coming in state $| V \rangle$ when it enters the interferometer, the probabilities of finding it in state $\langle H |$ or $\langle V |$ at the detectors should sum to 1. This means $| \langle H | \, SRDS \, | V \rangle \, |^2 + | \langle V | \, SRDS \, | V \rangle \, |^2 = 1$ which is the sum of the modulus squares of the second column.

- In fact for each of the sum of columns (or rows) that sums to 1, there is an experimental interpretation.