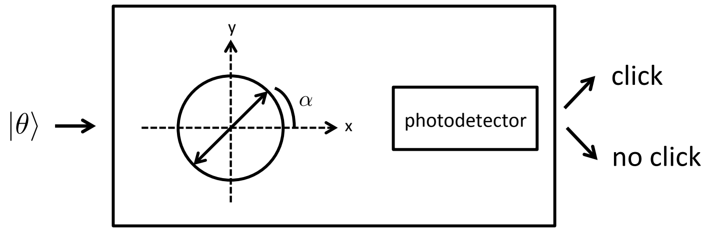
Homework 3
Traitement Quantique de l'Information

**Exercise 1** *Polarization observable and measurement principle*

Consider the "measurement apparatus" (in the below figure) constituted of "an analyzer and a detector". The incoming (initial) state of the photon is linearly polarized:

$$|\theta\rangle = \cos\theta\,|x\rangle + \sin\theta\,|y\rangle\,.$$




When the photodetector clicks we record $+1$ and when it does not click we record $-1$. Thus the "polarization observable" is represented by the $2\times 2$ matrix

$$P_\alpha = (+1)\,|\alpha\rangle\langle\alpha| + (-1)\,|\alpha_\perp\rangle\langle\alpha_\perp|$$

where $|\alpha\rangle = \cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle$ and $|\alpha_\perp\rangle = -\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle$ are the two vectors of the measurement basis. Note that the two orthogonal projectors of the measurement basis are $\Pi_\alpha = |\alpha\rangle\langle\alpha|$ and $\Pi_{\alpha_\perp} = |\alpha_\perp\rangle\langle\alpha_\perp|$.

1) Show that $\Pi_\alpha^2 = \Pi_\alpha$, $\Pi_{\alpha_\perp}^2 = \Pi_{\alpha_\perp}$ and $\Pi_\alpha\Pi_{\alpha_\perp} = \Pi_{\alpha_\perp}\Pi_\alpha = 0$.

2) Check the following formulas:

$$|\langle\theta|\alpha\rangle|^2 = \langle\theta|\,\Pi_\alpha\,|\theta\rangle\,,$$
$$|\langle\theta|\alpha_\perp\rangle|^2 = \langle\theta|\,\Pi_{\alpha_\perp}\,|\theta\rangle$$

3) Let $p = \pm 1$ the random variable corresponding to the event click / no-click of the detector. Express $\text{Prob}(p = \pm 1)$ with simple trigonometric functions and check that the two probabilities sum to one.

4) Deduce from 3) $\mathbb{E}(p)$ and $\text{Var}(p)$ and check that you find the same expressions by directly computing $\langle\theta|\,P_\alpha\,|\theta\rangle$ and $\langle\theta|\,P_\alpha^2\,|\theta\rangle - \langle\theta|\,P_\alpha\,|\theta\rangle^2$ in Dirac notation.

**Exercise 2** *Product versus entangled states*

Prove whether the following states are product or entangled states ? (check also they are correctly normalized)

1. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$
2. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$
3. $\frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{6}|10\rangle - \frac{1}{\sqrt{3}}|11\rangle$
4. $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, $|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, .
5. $\frac{1}{\sqrt{1+\epsilon^2}}(|00\rangle + \epsilon|11\rangle)$, for $0 \leq \epsilon \leq 1$
6. $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$
7. $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
8. $\frac{1}{2\sqrt{2}}(|000\rangle + |100\rangle + |010\rangle + |001\rangle + |110\rangle + |101\rangle + |011\rangle + |111\rangle)$

**Exercise 3** *Unitary transformations*

Verify that the following transformations are unitary (check also the identities between matrix tables and Dirac notation):

1. Simple time evolution of the type $|\psi_t\rangle = e^{i\omega t}|\psi_0\rangle$. This is for example the time evolution of a free photon of frequency $\nu = \omega/2\pi$ or energy $E = h\nu = \hbar\omega$.
2. The Hadamard gate.

   $$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right) \tag{1}$$

   Check how the basis $|0\rangle$, $|1\rangle$ is transformed. Remark: in interferometers models for example a semi-transparent mirror.
3. The $X$ or $NOT$ gate

   $$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

   Check how the basis $|0\rangle$, $|1\rangle$ is transformed. Remark: in interfreometers it models for example a reflecting mirror.
4. $U_1 \otimes U_2$ if $U_1$ and $U_2$ are unitary. Remark: if $U_i$, $i = 1, 2$ act each on a one-qubit Hilbert space $\mathbb{C}^2$ then the tensor product acts on the two-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

5. The control-NOT gate. This gate flips the control bit (the second) if the target bit (the first) is 1.

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10| = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

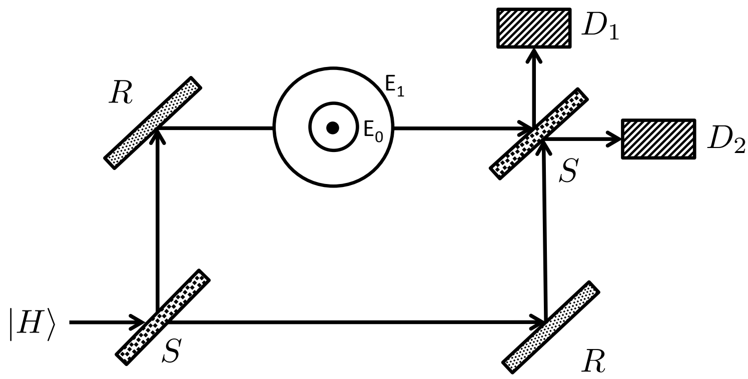
Verify the following identity:

$$|\beta_{ij}\rangle = CNOT\left((H|i\rangle) \otimes |j\rangle\right), \quad i,j = 0,1$$

Deduce that $\{|\beta\rangle, i,j = 0,1\}$ is an orthonormal basis. This identity shows that $CNOT$ entangles the two qubits.

**Exercise 4** *Interferometer with an atom on the upper path*

Consider the following set-up where an atom may absorb the photon on the upper arm of the interferometer.




The Hilbert space of the photon is here $\mathbb{C}^3$ with basis states

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |V\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |\text{abs}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

the semi-transparent and reflecting mirrors are modeled by the unitary matrices

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the "absorption-reemission" process[1] is modeled by the unitary matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

[1] On the picture $E_0$ and $E_1$ are two energy levels of the atom corresponding to ground state and excited state; but you can ignore this aspect in this problem.

Note that in Dirac notation

$$A = |H\rangle \langle \text{abs}| + |V\rangle \langle V| + |\text{abs}\rangle \langle H|$$

This models three possible transitions: $A|H\rangle = |\text{abs}\rangle$ (absorption); $A|\text{abs}\rangle = |H\rangle$ (emission); and $A|V\rangle = |V\rangle$ (nothing happens).

1) Write down all matrices in Dirac notation and then compute the unitary operator $U = SARS$ representing the total evolution process of this interferometer.

2) Given that the initial state is $|H\rangle$, what is the state after the second semi-transparent mirror? What are the probabilities of the following three events: click in $D_1$; or click in $D_2$; or no clicks in $D_1$ nor $D_2$? Verify the probabilities sum to to 1.

3) Suppose the photon-atom interaction is not absorption-reemission but some other process modeled by a matrix. Which of the two following matrices would be legitimate in QM?

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and why?