Nuclear Fusion and Plasma Physics - Exercises

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Solutions to Problem set 2 - 03 October 2022

Exercise 1 - Perfect Plasma Power Reactor

a) We must try to minimize the quantity

$$F = \frac{\rho_{blanket} V_{blanket}}{P_E} = \frac{\rho_{bl}}{P_E} 2\pi^2 R_0 \left[(b+a)^2 - a^2\right] = \frac{\rho_{bl}\, 2\pi^2}{P_E} R_0 (b^2 + 2ab)$$

which is a function of R_0 and a, under the constraints

$$P_E = \frac{1}{4}\eta_t (\Delta E_{fus} + \Delta E_{Li})\, n^2 \langle \sigma v \rangle_{D-T} (2\pi^2 R_0 a^2) = 1\,\text{GW} \quad (1)$$

$$L_W = \frac{\frac{1}{4}(\Delta E_\alpha + 0.3\,\Delta E_n)\, n^2 \langle \sigma v \rangle_{D-T} (2\pi^2 R_0 a^2)}{(4\pi^2 R_0 a)} < 4\,\text{MW/m}^2 \quad (2)$$

The second constraint can be written as

$$n^2 \langle \sigma v \rangle_{D-T} a = \frac{8 L_W^{max}}{(\Delta E_\alpha + 0.3\,\Delta E_n)}, \text{ where } L_W^{max} = 4\ \text{MW/m}^2 \quad (3)$$

Replacing in the equation for the first constraint gives:

$$P_E = 2\,\eta_t \frac{\Delta E_{fus} + \Delta E_{Li}}{\Delta E_\alpha + 0.3\,\Delta E_n} L_W^{max} (2\pi^2 R_0 a) \rightarrow R_0 = \frac{1}{4\pi^2 \eta_t} \frac{P_E}{L_W^{max}} \frac{\Delta E_\alpha + 0.3\,\Delta E_n}{\Delta E_{fus} + \Delta E_{Li}} \frac{1}{a} \quad (4)$$

Now we use this result in the expression for F, to express the quantity to be minimized as

$$F = \frac{\rho_{bl}}{2\eta_t} \frac{1}{L_W^{max}} \frac{\Delta E_\alpha + 0.3\,\Delta E_n}{\Delta E_{fus} + \Delta E_{Li}} \left(\frac{b^2}{a} + 2b \right) \quad (5)$$

The appearance of L_W^{max} in the denominator confirms our intuition that it is beneficial to operate at the maximum allowable wall loading of $4\,\text{MW/m}^2$. F is minimized for $a \rightarrow \infty$ and correspondingly $R_0 \rightarrow 0$. This is limited by the topological constraint $a < R_0 - b$: the plasma minor radius cannot be larger than the torus major radius minus the thickness of the blanket. So the minimum value of F is reached for $a = R_0 - b$. Replacing $R_0 = a + b$ in (**??**), we get a quadratic equation for a :

$$a^2 + ba - \frac{1}{4\pi^2 \eta_t} \frac{P_E}{L_W^{max}} \frac{\Delta E_\alpha + 0.3\,\Delta E_n}{\Delta E_{fus} + \Delta E_{Li}} = 0.$$

We are interested in the positive solution :

$$
\begin{aligned}
a &= \frac{-b+\sqrt{b^2-4\left(-\frac{1}{4\pi^2\eta_t}\frac{P_E}{L_W^{max}}\frac{\Delta E_\alpha+0.3\,\Delta E_n}{\Delta E_{fus}+\Delta E_{Li}}\right)}}{2} \\
&= \frac{-1.5\text{m}+\sqrt{(1.5\text{m})^2+\frac{1}{\pi^2\cdot 0.35}\frac{1\cdot 10^9\text{W}}{4\cdot 10^6\text{W/m}^2}\frac{(3.5+4.2)\text{MeV}}{22.4\text{MeV}}}}{2} = 1.85\,\text{m}.
\end{aligned}
$$

The major radius is then

$$R_0 = a + b = 3.35\,\text{m}$$

and the corresponding value of F is

$$F_{min} = \frac{3\cdot 10^3\,\text{kg/m}^3}{2\cdot 0.35}\frac{1}{4\text{MW/m}^2}\frac{(3.5+4.2)\text{MeV}}{22.4\,\text{MeV}}\left(\frac{(1.5\,\text{m})^2}{1.85\,\text{m}}+2\cdot 1.5\,\text{m}\right) \approx 1.56\times 10^3\,\text{kg/MW}.$$

This makes the mass utilization factor be ≈ 1.56 times that of a fission reactor.

Exercise 2 - Plasma β limit and diamagnetism

a) The current per unit length, generated by a single charged particle in the Larmor "circuit" is given by $i_j = -q_j\frac{\Omega_j}{2\pi}$. Now from $\nabla\times\vec{B} = \mu_0\vec{J} \to \delta B = \mu_0\delta i$ therefore $\delta B = -\frac{\mu_0 q_j^2 B}{2\pi m_j}\delta n$. This has the same sign for electrons and ions. Therefore the motion of the charged particles in a magnetic field tends to counteract the field and reduce its strength.

b) At a given point in space, the number of particles that contribute to the reduction of the B field is given by the number of particles whose Larmor orbit passes through that point. This is proportional to the particle density n and the surface swept by each Larmor orbit: $S_L = \pi\rho_L^2 \sim v_\perp^2$. Now $v_\perp^2$ is proportional to the plasma (thermal) kinetic energy and thus to the temperature T. We thus find the proportionality $\delta B \sim (nT)$.

   Here we have assumed that $v_\perp$ is equal for all particles at all points in space. In reality we would have to integrate over the distribution function $f(x,v)$ describing the distribution of velocity over the particle population: $\delta B(x) = -\frac{\mu_0}{B}\int\frac{1}{2}m_j v_\perp^2 f(x,v)d^3v = -\frac{\mu_0 nT}{B}$.

c) At some point, the diamagnetism will reduce the magnetic field to such an extent that the Larmor radius becomes larger than the machine size and the particles are no longer confined.

Exercise 3 - Violating quasi-neutrality

a) We first set up our simple 1D model of a plasma by considering a plasma located between $x = -d/2$ and $x = d/2$, where $d = 1\,\text{m}$. It is assumed homogeneous and infinite in all other directions (y and z).

The 1% violation of quasineutrality means that we will have a surplus of $0.01\,n$ ions. This will result in a charge density (charge per unit volume) $\rho = 0.01\,e\,n$. First we write Poisson's equation in one dimension:

$$\frac{\partial E}{\partial x} = \frac{\rho}{\epsilon_0} \tag{6}$$

The electric field in the plasma can be found by integrating this differential equation over the plasma, which is easy since we assumed the violation in quasineutrality - and hence the charge density - to be uniform.

$$\begin{aligned} E(x) &= \int \frac{\rho}{\epsilon_0}\,dx = \frac{\rho}{\epsilon_0}\int dx \\ &= \frac{\rho}{\epsilon_0}x + C \end{aligned}$$

Here C is an arbitrary constant of integration. Now note that charged particles at the exact center of the plasma will see no electric field, since there are equally many other charged particles on each side. Due to this symmetry we must choose the integration constant such that $E(0) = 0$, i.e. $C = 0$. The electrostatic force per unit volume is

$$F_e(x) = \rho E(x) = \frac{\rho^2}{\epsilon_0}x \tag{7}$$

We can now calculate its absolute value for our plasma parameters.

$$|F| = \frac{(0.01\,n\,e)^2 d}{\epsilon_0} = \frac{(0.01 \cdot 10^{20}\text{m}^{-3})^2 (1.6 \times 10^{-19}\text{C})^2 \cdot 1.0\text{m}}{8,85 \times 10^{-12}\frac{\text{C}^2}{\text{Nm}^2}} \sim 10^9\,\text{N/m}^3 \tag{8}$$

b) Compare this now to other forces acting on the plasma:

- Gravity exerts a force per unit volume $F_g = \rho_m g$ where ρ is the mass density, so

$$\begin{aligned} F_g &= (m_e n_e + m_i n_i)\,ng \approx m_i ng \\ &\approx 10^{-27}\text{kg} \cdot 10^{20}\text{m}^{-3} \cdot 9.8\,\text{m/s}^2 \qquad \approx 10^{-6}\,\text{N/m}^3 \end{aligned}$$

- Pressure is given by $p = nT$ where we have to take care to convert the T into Joules. For our plasma this is $p = nT = 10^{20}\,\text{m}^3 \cdot 1.6 \times 10^{-19}\,\text{J/eV} \cdot 10 \times 10^3\,\text{eV} = 10^5\text{N/m}^2$. The force exerted per cubic meter of plasma is $p/1\text{m} = 10^5\,\text{N/m}^3$

These forces are several orders of magnitude smaller than the electric force trying to maintain quasineutrality!