Potential Theory I

Outlines

Newtonian Mechanics:

- refreshing memory

Potential Theory : general results

- Gravitational field force, gravitational potential
- Gauss Law
- Poisson Equation
- Total potential energy

Spherical systems:

- Newton's Theorems
- Circular speed, circular velocity, circular frequency, escape speed, potential energy

Refreshing memory...

Newtonian mechanics

Newtonian mechanics : a very short remainder

point mass :

mass m

position $\vec{x}$

velocity $\vec{v} = \frac{d\vec{x}}{dt}$

momentum $\vec{p} = m\vec{v} = m\frac{d\vec{x}}{dt}$

Newtonian mechanics : a very short remainder

point mass :

mass m

position $\vec{x}$

velocity $\vec{v} = \frac{d\vec{x}}{dt}$

momentum $\vec{p} = m\vec{v} = m\frac{d\vec{x}}{dt}$

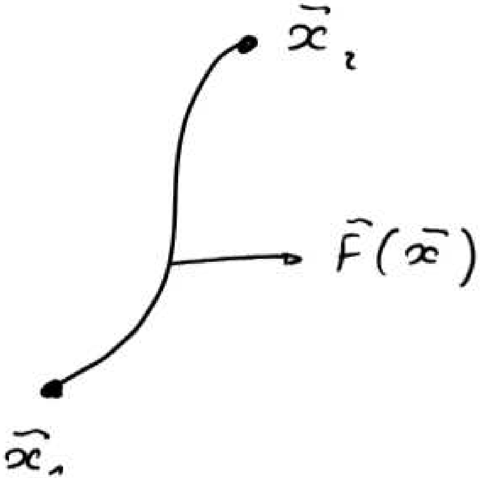
Newton second law

$$\boxed{\frac{d}{dt}(\vec{p}) = m\frac{d^2\vec{x}}{dt^2} = \vec{F}}$$

- $\vec{F}$: a force

$\vec{p}$ is constant in absence of a force

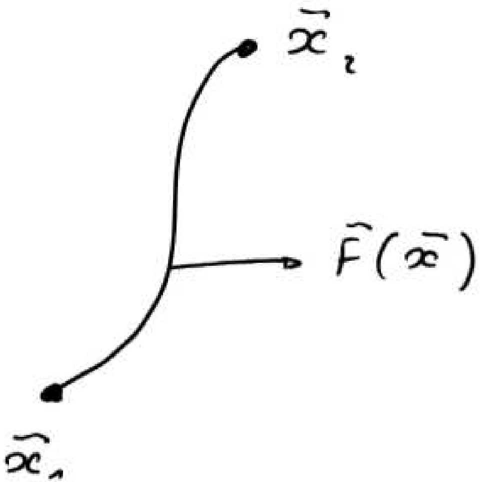
__Work__ : work done by a force in moving the particle from $\vec{x}_1$ to $\vec{x}_2$



$$W_{12} = -\int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

$$[W_{12}] = g\,\frac{cm^2}{s^2} = erg$$

Work : work done by a force in moving the particle from $\vec{x}_1$ to $\vec{x}_2$



$$W_{12} = -\int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

$$[W_{12}] = g\,\frac{cm^2}{s^2} = erg$$

Power of a force (energy rate)

$$\frac{erg}{s}$$

with $\frac{\partial \mathcal{F}(\vec{x})}{\partial x_i} = F_i(\vec{x})$

$$\frac{d}{dt} W_{12}(x(t)) = -\frac{d}{dt}\int_{\vec{x}_1}^{\vec{x}(t) = \vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x} = -\frac{d}{dt}\left(\mathcal{F}(\vec{x}(t)) - \mathcal{F}(\vec{x}_1) \right)$$

$$= -\vec{\nabla}_{\vec{x}}\, \mathcal{F}(\vec{x}) \cdot \frac{d}{dt}(\vec{x}(t)) = -\vec{F}(\vec{x})\,\vec{v}(\vec{x})$$

Kinetic energy

$\vec{x} = \vec{x}(t)$

$d\vec{x} = \frac{d\vec{x}}{dt}\,dt$

Newton 2nd law

$$W_{12} = -\int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x} = -m \int_{\vec{x}_1}^{\vec{x}_2} \frac{d\vec{v}}{dt} \cdot \frac{d\vec{x}}{dt} \cdot dt$$

$$= -m \int_{\vec{x}_1}^{\vec{x}_2} \frac{d\vec{v}}{dt} \cdot \vec{v} \cdot dt$$

integration by part gives

$$= -m \left[\vec{v}^2 \Big|_{\vec{x}_1}^{\vec{x}_2} - \int_{\vec{x}_1}^{\vec{x}_2} \vec{v} \frac{d\vec{v}}{dt} dt \right] = -m \vec{v}_2^{\,2} + m \vec{v}_1^{\,2} + \underbrace{m \int_{\vec{x}_1}^{\vec{x}_2} \vec{v} \frac{d\vec{v}}{dt} dt}_{-W_{12}}$$

Thus $\quad W_{12} = \frac{1}{2} m \vec{v}_1^{\,2} - \frac{1}{2} m \vec{v}_2^{\,2}$

$$\boxed{W_{12} = K_1 - K_2}$$

$K = \frac{1}{2} m \vec{v}^{\,2}$: Kinetic energy

Potential energy and conservative forces

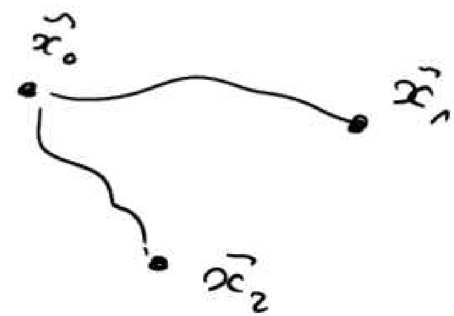
A force $\vec{F}(\vec{x})$ is called conservative if the work done by this force in moving the particle from $\vec{x}_1$ to $\vec{x}_2$ is independent of the path.

Then, for any given point $\vec{x}_0$ we can define the function (potential) $V_0(\vec{x})$

$$V_0(\vec{x}) := V_{0x} = -\int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{x}')\, d\vec{x}'$$

Then $W_{12} = W_{10} + W_{02}$

$$\boxed{W_{12} = V(\vec{x}_2) - V(\vec{x}_1)}$$



Useful convention: $\vec{x}_0 \to \infty$ (far away from all interacting bodies)

Gradient of the potential

with $\dfrac{\partial \mathcal{F}(\vec{x})}{\partial x_i} = F_i(\vec{x})$

$$\vec{\nabla}_{\vec{x}} \cdot V(\vec{x}) = -\vec{\nabla}_{\vec{x}} \cdot \left[\int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{x}\,')\, d\vec{x}\,' \right] = -\vec{\nabla}_{\vec{x}} \Big(\mathcal{F}(\vec{x}) - \underbrace{\mathcal{F}(\vec{x}_0)}_{\text{ct} \Rightarrow 0} \Big) = -\vec{F}(\vec{x})$$

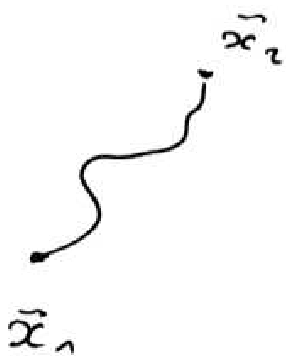
$$\boxed{\vec{\nabla}_{\vec{x}} \cdot V(\vec{x}) = -\vec{F}(\vec{x})}$$

We can represent a conservative force field by its potential

Total Energy E

$$E := K + V = \frac{1}{2} m \vec{v}^{\,2} + V(\vec{x})$$

Theorem The energy E of a system evolving under conservative forces $\vec{F}(\vec{x})$ (associated to a potential $V(\vec{x})$) is constant.



$$E_1 = E(\vec{x}_1) = \frac{1}{2} m v_1^2 + V(\vec{x}_1)$$

$$E_2 = E(\vec{x}_2) = \frac{1}{2} m v_2^2 + V(\vec{x}_2)$$

$$E_1 - E_2 = \underbrace{K_1 - K_2}_{W_{12}} + \underbrace{V(\vec{x}_1) - V(\vec{x}_2)}_{-W_{12}}$$

$$= 0 \qquad \#$$

Angular momentum and Tork

Angular momentum $\vec{L} = \vec{x} \times \vec{p}$

Tork $\vec{N} = \vec{x} \times \vec{F}$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{x}}{dt} \times \vec{p} + \vec{x} \times \frac{d\vec{p}}{dt}$$

$$= \underbrace{\vec{v} \times \vec{p}}_{=0} + \underbrace{\vec{x} \times \vec{F}}_{\vec{N}}$$

$$\boxed{\frac{d\vec{L}}{dt} = \vec{N}}$$

Potential Theory

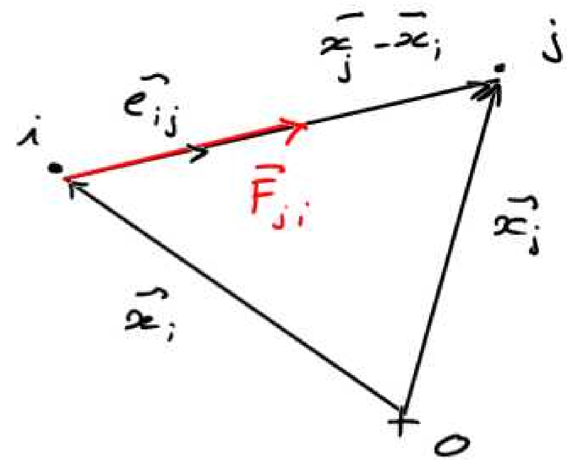
Potential theory : general results

Goal : compute the gravitational potential / forces due to a large number of stars (point masses)

$N \sim 10^{11}$ for a Milky Way like galaxy

As the relaxation time of such system is very large ($\gg$ the age of the Universe) we can describe the system with a smooth analytical potential / density .

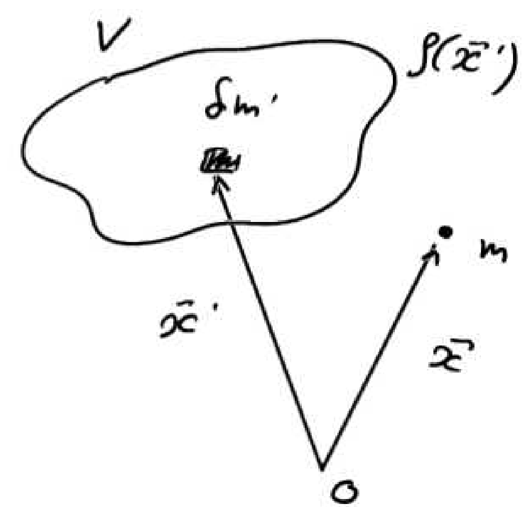
Newton Law



$$\vec{F}_{ji} = \frac{G m_i m_j}{|\vec{x}_j - \vec{x}_i|^2} \vec{e}_{ij} = \frac{G m_i m_j}{|\vec{x}_{ij}|^3} \vec{x}_{ij}$$

$$\vec{x}_{ij} = \vec{x}_j - \vec{x}_j$$

Force on a particle of mass m in $\vec{x}$ due to a distribution of mass $\rho(\vec{x})$



$$\delta \vec{F}(\vec{x}) = \frac{G m \, \delta m'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})$$

$$= \frac{G m \, \rho(\vec{x}') \, d^3\vec{x}'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})$$

So, the total force writes :

$$\boxed{\vec{F}(\vec{x}) = \int_V \frac{G\, m\, \rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'}$$

$$\equiv m \underbrace{G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'}_{\vec{g}(\vec{x}) \,:\, \text{gravitational field}}$$

$$[\vec{g}] = \frac{cm}{s^2} = \frac{erg}{g} \frac{1}{cm}$$

Gravitational Potential

It is easy to see that the function

$$\delta V(\vec{x}) = -\frac{G m \delta m}{|\vec{x}' - \vec{x}|} \qquad \text{is such that}$$

$$\vec{\nabla} \delta V(\vec{x}) = -\frac{G m \delta m}{|\vec{x}' - \vec{x}|^2} \frac{(\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|} = -\delta \vec{F}(\vec{x})$$

so, by defining

$$\boxed{V(\vec{x}) = -G \int_V \frac{m \rho(\vec{x}')}{|\vec{x}' - \vec{x}|} d^3\vec{x}'}$$

we ensure that $\quad \vec{\nabla} V(\vec{x}) = -\vec{F}(\vec{x})$

We define the specific potential $\phi(\bar{x}) = \frac{V(\bar{x})}{m}$

which writes

$$\boxed{\phi(\bar{x}) = -G \int_V \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|} \, d^3\bar{x}'}$$

$[\phi] = \frac{erg}{g}$

= specific energy

The gravitational field writes :

$$\boxed{\bar{g}(\bar{x}) = -\bar{\nabla}\phi(\bar{x})}$$

Notes

- The gravity is a conservative force

- $\phi(\vec{x})$: scalar field
  $\vec{g}(\vec{x})$: vector field
  } contain the same information

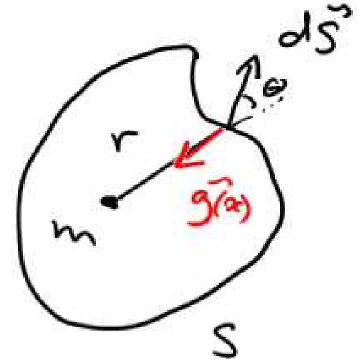
- we will always use "specific" quantities

$$V(\vec{x}) \to \phi(\vec{x})$$

$$K = \frac{1}{2} m \vec{v}^2 \to \frac{1}{2} \vec{v}^2$$

$$\frac{1}{2} v^2 + \phi(\vec{x}) = \text{specific energy (conserved quantity)}$$

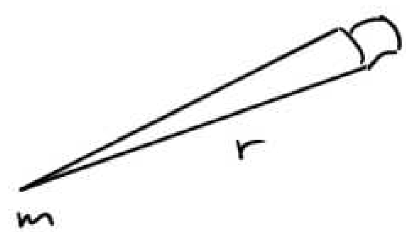
The Gauss's Law



Consider:
- a single point mass m
- a surface S around this point
- a point $\vec{x}$ on the surface at a distance r
- $\vec{g}(\vec{x})$ the gravitational field
- $d\vec{S}$, the normal at the surface
- θ the angle between $\vec{g}(\vec{x})$ and $d\vec{S}$

$$\vec{g}(\vec{x}) \cdot d\vec{S} = -\vec{g}(\vec{x}) \cdot |d\vec{S}| \cos\theta$$

But
$$|d\vec{S}| \cos\theta = r^2 \, d\Omega$$



$$|\vec{g}(x)| = \frac{Gm}{r^2}$$

$$\boxed{\vec{g}(\vec{x}) \cdot d\vec{S} = -G m \, d\Omega}$$

integrating over any surface

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = \begin{cases} -4\pi G m & \text{if } m \text{ inside } S \\ 0 & \text{instead} \end{cases}$$

For multiple masses m_i:

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \sum_{i \text{ in } S} m_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\boxed{\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \int_V \rho(\vec{x}) \, d\vec{x} = -4\pi G M}$$

Gauss's Law

Divergeance of the specific force

(A) $\vec{\nabla}_x \cdot \vec{g}(\vec{x})$

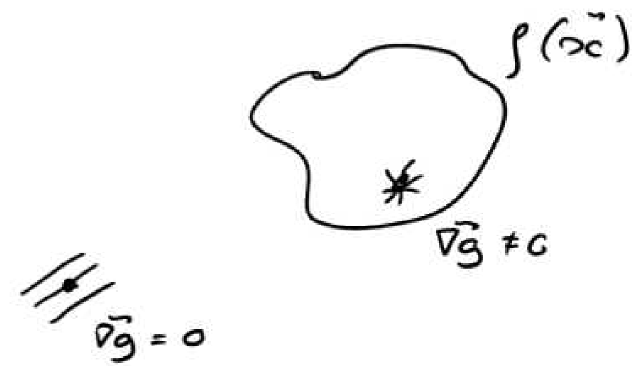
div. theorem

$$\int_V \vec{\nabla} \cdot \vec{g}(\vec{x}) \, d^3\vec{x} = \int_S \vec{g}(\vec{x}) \, d\vec{S}$$

Gauss's Law

$$= -4\pi G \int_V \rho(\vec{x}) \, d\vec{x}$$

$$\boxed{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \, \rho(\vec{x})}$$



Divergence of the specific force (B) $\vec{\nabla}_x \cdot \vec{g}(\vec{x})$

$$\vec{g}(\vec{x}) = G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \, d^3\vec{x}'$$

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left(\frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3\vec{x}'$$

- $$\vec{\nabla}_x \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) = \frac{d}{dx_1}\left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_2}\left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_3}\left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right)$$

$$= -\frac{3}{|\vec{x}' - \vec{x}|^3} + \frac{3(\vec{x}' - \vec{x}) \cdot (\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^5}$$

$$= 0 \quad \text{if} \quad \vec{x}' \neq \vec{x}$$

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left(\frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3\vec{x}'$$

$\rho(\vec{x}) = \text{cte}$

$$= G \rho(\vec{x}) \int_{|\vec{x}' - \vec{x}| \leq h} \vec{\nabla}_x \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) d^3\vec{x}'$$

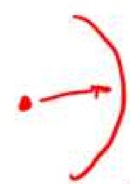
variable exchange

$\vec{\nabla}_x f(\vec{x} - \vec{x}') = - \vec{\nabla}_{x'} f(x - x')$

$$= - G \rho(\vec{x}) \int_{|\vec{x}' - \vec{x}| \leq h} \vec{\nabla}_{x'} \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) d^3\vec{x}'$$

divergence theorem

$$= - G \rho(\vec{x}) \underbrace{\int_{|\vec{x}' - \vec{x}| = h} \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} d^2\vec{S}'}_{4\pi h^2 \cdot \left.\frac{1}{r^2}\right|_{h=r} = 4\pi}$$



$r = |\vec{x}' - \vec{x}| = h$

$\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} = \frac{1}{r^2}$

$$\underline{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = - 4\pi G \rho(\vec{x})}$$

The Poisson Equation

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})$$

with : $\vec{\nabla}_x \phi(\vec{x}) = -\vec{g}(\vec{x})$ $\vec{\nabla}_x \cdot (\vec{\nabla}_x) = \vec{\nabla}_x^2$

$$\boxed{\vec{\nabla}_x^2 \phi(\vec{x}) = 4\pi G \rho(\vec{x})}$$

Poisson Equation

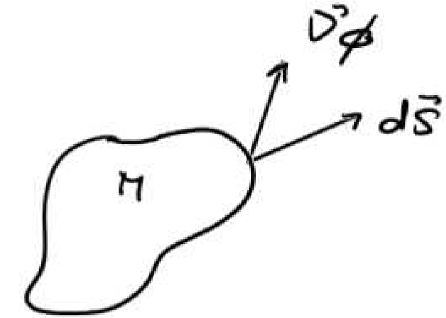
Note : To ensure a unique solution, boundary conditions are necessary (2nd order diff. eqn.)

ex : $\phi(\infty) = 0$

$\vec{\nabla}\phi(\infty) = \vec{g}(x) = 0$

Gauss theorem

integrate the Poisson equation over a volume V that contains a mass M



$$\int_V \vec{\nabla}^2 \phi(\vec{x})\, d^3\vec{x} = \underbrace{\int_V 4\pi G\, \rho(\vec{x})\, d^3\vec{x}}$$

div. theorem

$$\boxed{\int_S d^2\vec{S} \cdot \vec{\nabla}\phi = 4\pi G M}$$

Gauss theorem

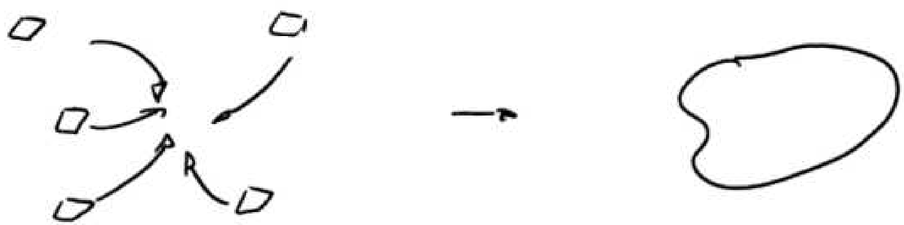
Equivalently :

$$\boxed{\int_S d^2\vec{S} \cdot \vec{g}(\vec{x}) = -4\pi G M}$$

Gauss's Law

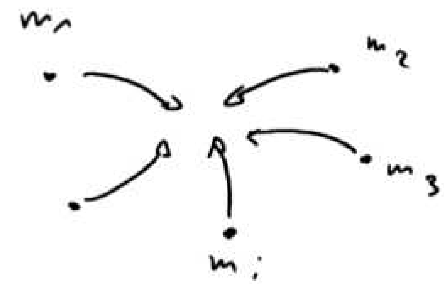
Total potential energy (A)

Total work needed to assemble a density distribution $\rho(\vec{x})$



$\rho(\vec{x})$, $\phi(\vec{x})$

Assume a set of discrete points



- The work to bring the 1^{st} point from ∞ to $\vec{x}_1$ is 0
- The work to bring the 2^{nd} point from ∞ to $\vec{x}_2$ is $-\dfrac{G m_1 m_2}{r_{12}}$
- The work to bring the 3^{dn} point from ∞ to $\vec{x}_3$ is $-\dfrac{G m_1 m_3}{r_{13}} - \dfrac{G m_2 m_3}{r_{23}}$

The total work is thus

$$W = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}} - \dots - \sum_{j=1}^{N-1} \frac{Gm_{jN}}{r_{jN}}$$

$$= -\sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{Gm_im_j}{r_{ij}} = -\frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{Gm_im_j}{r_{ij}}$$

With $\phi_i = -\sum_{\substack{j=1 \\ j\neq i}} \frac{Gm_j}{r_{ij}}$

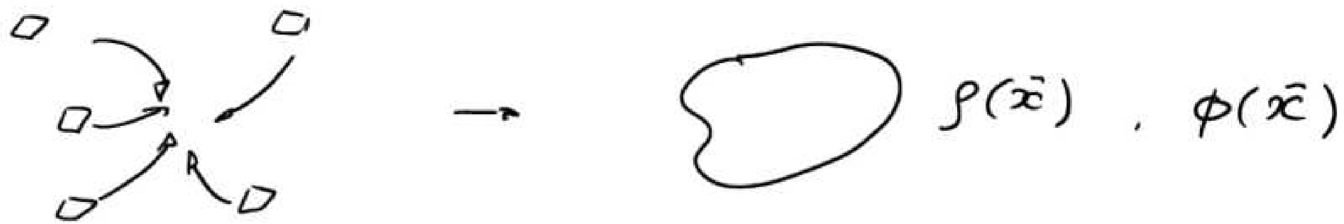
$$W = \frac{1}{2} \sum_{i=1}^{N} m_i \phi_i \equiv \frac{1}{2} \sum_{i=1}^{N} V_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\boxed{W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) \, d^3\vec{x}}$$

Total potential energy B

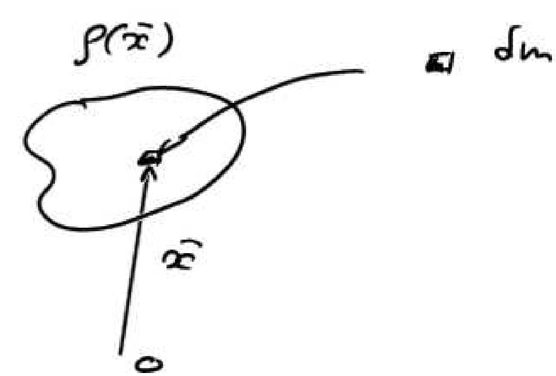
Total work needed to assemble a density distribution $\rho(\vec{x})$



① Work done to assemble a piece of mass $\delta m = \delta\rho \, d\vec{x}^3$ from ∞ to $\vec{x}$ assuming an existing mass distribution $\rho(\vec{x})$, $\phi(\vec{x})$

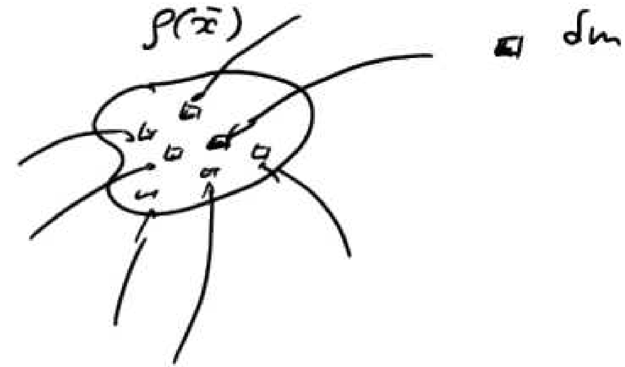
$$\delta W_{\vec{x}} = V(\vec{x}) - \underbrace{V(\infty)}_{=0}$$

$$= \delta m \, \phi(\vec{x}) = \delta\rho(\vec{x}) \, d^3\vec{x} \; \phi(\vec{x})$$



To increase everywhere the mass distribution by $\delta\rho$

$$\rho(\bar{x}) \to \rho(\bar{x}) + \delta\rho(\bar{x})$$



$$\delta W = \int \delta\rho(\bar{x})\, d^3\bar{x}\ \phi(\bar{x})$$

Poisson: $\delta\rho(\bar{x}) = \frac{1}{4\pi G} \vec{\nabla}^2 \delta\phi(x)$

$$\delta W = \frac{1}{4\pi G} \int \vec{\nabla}^2 \delta\phi(x)\ \phi(\bar{x})\ d^3\bar{x}$$

divergence theorem

$$\int_V d^3x\, g\, \vec{\nabla}\cdot\vec{F} = \int_S g\, \vec{F}\cdot d^2\vec{s} - \int_V d^3x\, \vec{F}\cdot\vec{\nabla} g$$

$$= \underbrace{\frac{1}{4\pi G} \int_{S \text{ at } \infty} \phi(\bar{x})\, \vec{\nabla}\delta\phi(\bar{x})}_{=0} - \frac{1}{4\pi G} \int \vec{\nabla}\phi(\bar{x}) \cdot \vec{\nabla}(\delta\phi(\bar{x}))\ d^3\bar{x}$$

as $\phi(\infty) = 0$

$\vec{\nabla}\delta\phi(\infty) = \delta g(\infty) = 0$

$$\delta W = -\frac{1}{4\pi G}\int \vec{\nabla}\phi(\vec{x})\cdot\vec{\nabla}(\delta\phi(\vec{x}))\, d^3\vec{x}$$

with

$$\frac{1}{2}\,\delta|\vec{\nabla}\phi(\vec{x})|^2 = \delta\vec{\nabla}\phi(\vec{x})\cdot\vec{\nabla}\phi(\vec{x}) = \vec{\nabla}(\delta\phi(\vec{x}))\cdot\vec{\nabla}\phi(\vec{x})$$

$$\delta W = -\frac{1}{8\pi G}\int \delta|\vec{\nabla}\phi|^2\, d^3x \equiv -\frac{1}{8\pi G}\,\delta\int |\vec{\nabla}\phi|^2\, d^3x$$

② <u>Contribution of all δW to W</u>

$$\boxed{W = -\frac{1}{8\pi G}\int |\vec{\nabla}\phi|^2\, d^3x}$$

$$\boxed{W = -\frac{1}{8\pi G}\int |\vec{\nabla}\phi|^2 \, d^3x}$$

<u>Other expression</u> using the divergeance theorem

$$\int_V d^3\vec{x}\; \vec{F}\cdot\vec{\nabla} g = \int_S g \cdot \vec{F}\cdot d^2\vec{S} - \int d^3\vec{x}\; g\, \vec{\nabla}\cdot\vec{F}$$

$$\int |\vec{\nabla}\phi|^2 \, d^3x = \int_V d^3\vec{x}\; \vec{\nabla}\phi\cdot\vec{\nabla}\phi = \int_S \underbrace{\phi\,\vec{\nabla}\phi}_{=0\ \vec{x}\to\infty}\, d^2\vec{S} - \int_V d^3\vec{x}\; \phi\; \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\phi)}_{=\vec{\nabla}^2\phi\, =\, 4\pi G\rho}$$

Poisson

$$W = -\frac{1}{8\pi G}\cdot\int -4\pi G\,\phi\cdot\rho \; d^3\vec{x}$$

$$\boxed{W = \frac{1}{2}\int \rho(\vec{x})\,\phi(\vec{x})\, d^3\vec{x}}$$

Other useful expression

$$\boxed{W = -\int \rho(\vec{x})\,\vec{x}\cdot\vec{\nabla}\phi(\vec{x})\, d^3\vec{x}}$$

Spherical Systems

$$\rho(\vec{x}) = \rho(r)$$

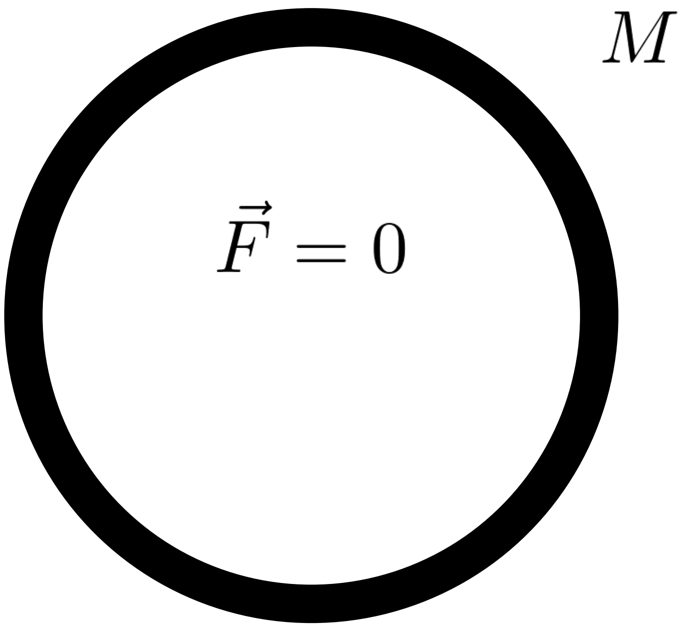
$$r = \sqrt{x^2 + y^2 + z^2}$$

Newton's Theorems

Newton (1642-1727)

First theorem:

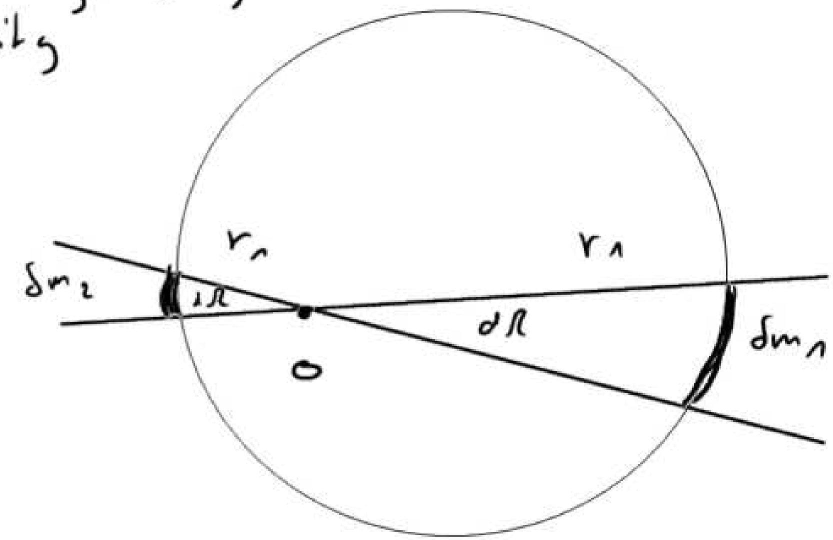
A body that is inside a spherical shell of matter experiences no net gravitational force from the shell.



First Newton theorem

A body that is inside a spherical shell of matter experiences no net gravitational force from that shell

a shell of constant density $\rho(\vec{x}) = \rho$



$$\begin{cases} \delta m_1 = \rho(r_1) \cdot r_1^2 \, d\Omega \, dr \\ \delta m_2 = \rho(r_2) \cdot r_2^2 \, d\Omega \, dr \end{cases}$$

thus : $$\frac{\delta m_1}{\delta m_2} = \frac{r_1^2}{r_2^2}$$

and $$\frac{\delta m_1}{r_1^2} = \frac{\delta m_2}{r_2^2}$$

<u>consequently</u> : $\delta \vec{F_1} = - \delta \vec{F_2}$

by integrating over the entire shell ($d\Omega$)

all forces cancel out !

#

Corollary The gravitational potential $\phi(\vec{x})$ is constant inside the sphere.

As $\vec{\nabla}_x \phi(\vec{x}) = \vec{g} = 0 \qquad \phi(\vec{x}) = ct$ #

What is the value of $\phi(\vec{x})$?

$$\phi(\vec{x}) = -\int_V \frac{G\,\rho(\vec{x}')}{|\vec{x}' - \vec{x}|}\, d^3\vec{x}'$$

Spherical coordinates $\quad d^3\vec{x} = r^2\, dr\, d\Omega = 4\pi\, r^2 dr$

At the center $\vec{x} = 0$

$$\phi(0) = -4\pi G \int_0^\infty \frac{\rho(r')}{r}\, r^2 dr = -4\pi G \int_0^\infty \rho(r')\, r\, dr$$

Density of a shell of mass M, radius R : $\rho(r) = \frac{M}{4\pi r^2} \delta(R-r)$

$$\left(\text{as} \quad 4\pi \int_0^\infty \frac{M}{4\pi r^2} \delta(R-r)\, r^2\, dr = M \right)$$

$$\phi(r) = - GM \int_0^\infty \frac{\delta(R-r)}{r^2}\, r\, dr = - \frac{GM}{R}$$

As the potential is constant for $r < R$

$$\boxed{\phi(\bar{x}) = - \frac{GM}{R} \qquad \bar{x} \in \text{sphere}}$$

Newton's Theorems

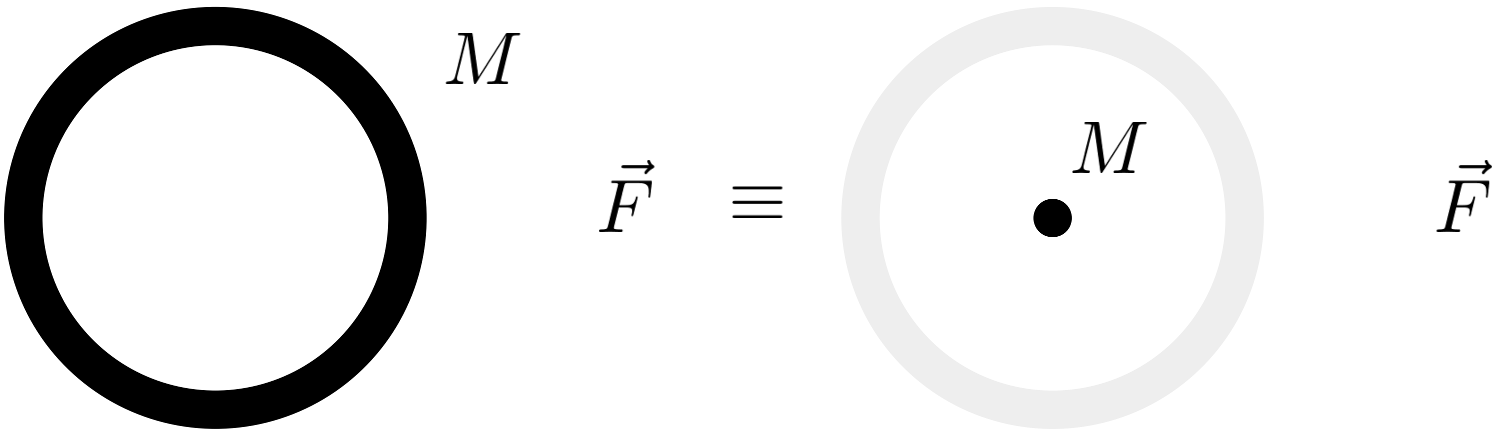
Newton (1642-1727)

First theorem:

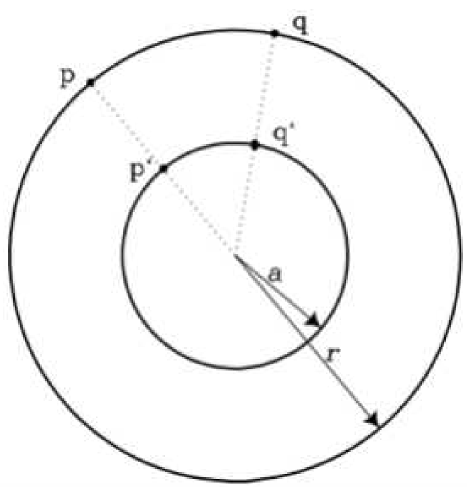
A body that is inside a spherical shell of matter experiences no net gravitational force from the shell.

Second theorem:

The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter where concentrated into a point at its centre.



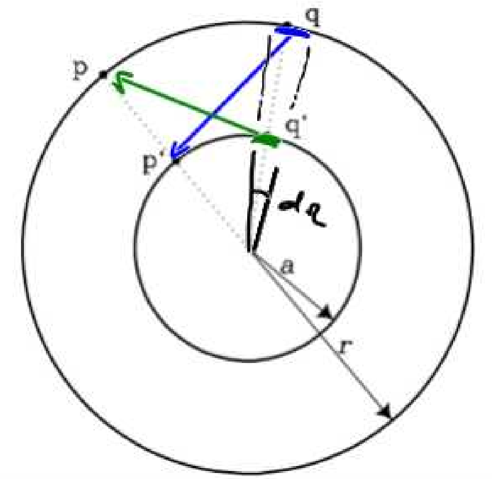
Second Newton theorem



The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center

Consider two shells

1. inner, with radius a and mass M
2. outer, with radius r and mass M



<u>Compute</u>

1. $\phi_p = \phi_i(r)$

2. $\phi_{p'} = \phi_o(a) = -\dfrac{GM}{r}$

mass inside the solid angle

$$\delta m = \frac{M\,\delta\Omega}{4\pi}$$

1. contribution of shell$_i$, in q' with solid angle $\delta\Omega$

- $$\delta\phi_i(p) = -\frac{G\,\delta m_{q'}}{|p-q'|} = -\frac{GM}{|p-q'|}\frac{\delta\Omega}{4\pi}$$

2. contribution of shell$_o$, in q with solid angle $\delta\Omega$

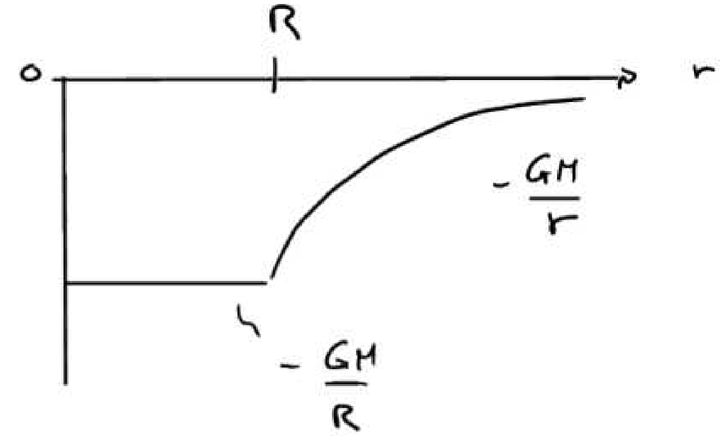
- $$\delta\phi_o(p') = -\frac{G\,\delta m_q}{|p'-q|} = -\frac{GM}{|p'-q|}\frac{\delta\Omega}{4\pi} = \delta\phi_i(p)$$

Summing over all q' $\equiv$ Summing over all q

$$\phi_i(p) = \phi_o(p') = -\frac{GM}{r} \qquad \#$$

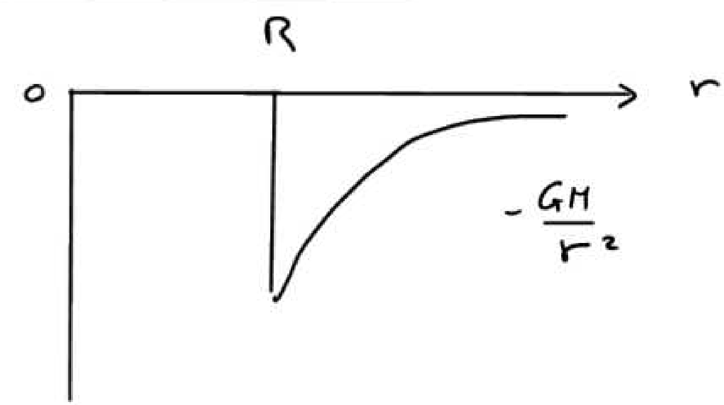
Total potential of a shell of mass M, radius R

$$\phi(r) = \begin{cases} -\dfrac{GM}{R} & r < R \\ -\dfrac{GM}{r} & r \geq R \end{cases}$$



Total gravitational field of a shell of mass M, radius R

$$\vec{g}(r) = \begin{cases} 0 & r < R \\ -\dfrac{GM}{r^2}\,\vec{e}_r & r \geq R \end{cases}$$



Spherical Systems
general distribution of mass

$$\rho(\vec{x}) = \rho(r)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

Gravitational field of a spherical model $\rho(r)$

Sum of shells

$$g(r) = \int_0^\infty \delta g_{r'}(r)$$

$\delta g_{r'}(r)$ = force due to the shell of radius r'

$$= \underbrace{\int_0^r \delta g_{r'}(r)}_{\text{inner shells}} + \underbrace{\int_r^\infty \delta g_{r'}(r)}_{\text{outer shells}}$$

= 0 as we are inside

mass of a shell

$$\delta M(r') = 4\pi r'^2 dr' \rho(r') \qquad \delta g_{r'}(r) = -\frac{G\,\delta M(r')}{r^2} = -4\pi \rho(r') \frac{r'^2}{r^2} dr'$$

$$g(r) = -\frac{G}{r^2} \underbrace{4\pi \int_0^r \rho(r') r'^2 dr'}_{M(r)} = -\frac{G M(r)}{r^2}$$

Potential of a spherical system $\rho(r)$

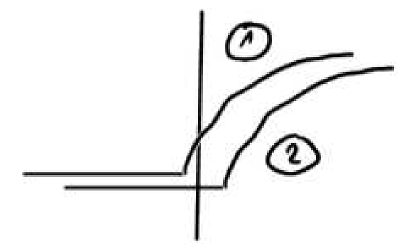
Sum of shells

$$\phi(r) = \int_0^\infty \delta\phi_{r'}(r)$$

$\delta\phi_{r'}(r)$ = potential on r due to a shell in r'

$$= \int_0^r \delta\phi_{r'}(r) + \int_r^\infty \delta\phi_{r'}(r)$$

$$= -\int_0^r \frac{G\,\delta M(r')}{r} - \int_r^\infty \frac{G\,\delta M(r')}{r'}$$



1 shells of radius $r' < r$

2 shells of radius $r' > r$
constant potential

with $\delta M(r') = 4\pi r'^2 dr' \rho(r')$

$$\phi(r) = -\frac{G}{r}\,\underbrace{4\pi\int_0^r \rho(r')\, r'^2\, dr'}_{M(r)} - 4\pi G\int_r^\infty \rho(r')\, r'\, dr'$$

$$\boxed{\phi(r) = \underbrace{-\frac{G M(r)}{r}}_{\substack{\text{contribution}\\ \text{of the mass}\\ \text{inside } r}} \underbrace{- 4\pi G \int_r^\infty \rho(r')\, r'\, dr'}_{\substack{\text{contribution}\\ \text{of the mass}\\ \text{outside } r}}}$$

Summary : for any spherical mass distribution $\rho(r)$

$$\boxed{g(r) = -\frac{GM(r)}{r^2}}$$

$$M(r) = 4\pi \int_0^\infty \rho(r')\, r'^2\, dr'$$

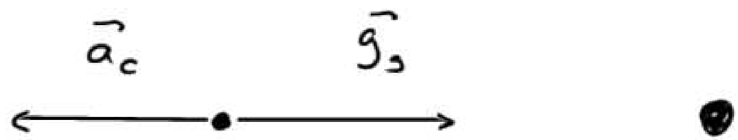
$$\boxed{\phi(r) = -\frac{GM(r)}{r} - 4\pi G \int_r^\infty \rho(r')\, r'\, dr'}$$

<u>Note</u> $g(r) = -\frac{\partial \phi}{\partial r}$ as expected from

$$\vec{g}(\vec{x}) = \vec{\nabla} \phi(\vec{x})$$

Spherical systems : circular speed, circular velocity

Speed of a test particle in a cirular orbit in the potential $\phi(r)$ at a radius r :



$\vec{a}_c$: centrifugal acceleration $\qquad \dfrac{v_c^2}{r}$

$\vec{g}_s$: gravity acceleration (spec force) $\qquad -\dfrac{GM(r)}{r^2} = -\dfrac{\partial\phi}{\partial r}$

$$\boxed{v_c^2 = \frac{GM(r)}{r}}$$

$$\boxed{v_c^2 = r\,\frac{\partial\phi}{\partial r}}$$

$$\boxed{GM(r) = r^2\,\frac{\partial\phi}{\partial r}}$$

$[v_c^2]$: $\dfrac{erg}{g}$

as ϕ

= specific energy

Velocity composition

<u>Note</u>: V_0^2 scale with the mass ($M(r)$) : it is thus
the "important" quantity (spec. energy)

Multi-components system : ex: bulge + stellar halo + DM halo

$$\begin{cases} \rho_B(r), & M_B(r), & \phi_B(r) \to V_{c,B}(r) \\ \rho_H(r), & M_H(r), & \phi_H(r) \to V_{c,H}(r) \\ \rho_{DM}(r), & M_{DM}(r), & \phi_{DM}(r) \to V_{c,DM}(r) \end{cases}$$

$$V_{c,tot}^2 = \frac{G M_{tot}(r)}{r} = \frac{G}{r} \sum_i M(r)$$

$$\boxed{V_{c,tot}^2 = \sum_i V_{c,i}^2}$$

$V_c^2 \sim$ energy : extensive quantity

Period of the circular orbit

$$\boxed{T(r) = \frac{2\pi r}{V_c(r)} = 2\pi\sqrt{\frac{r^3}{GM(r)}} = 2\pi\sqrt{\frac{r}{\frac{\partial\phi}{\partial r}}}}$$

Circular frequency (angular frequency)

$$\boxed{\Omega(r) = \frac{2\pi}{T(r)} = \sqrt{\frac{GM(r)}{r^3}} = \sqrt{\frac{1}{r}\frac{\partial\phi}{\partial r}}}$$

Escape speed V_e

if $\frac{1}{2}V_e^2 > \phi(r) = E > 0$
the partcle may escape the system

$$\boxed{V_e(r) = \sqrt{2|\phi(r)|}}$$

Potential energy

from $$W = - \int \rho(x)\, \vec{x} \cdot \vec{\nabla}\phi(\vec{x})\, d^3\vec{x}$$

$$\boxed{W = -4\pi G \int_0^\infty \rho(r)\, M(r)\, r\, dr}$$

Gravitational radius

radius at which $\frac{GM^2}{r} = W$

(estimation of the system size)

$$\boxed{r_g = \frac{GM^2}{|W|}}$$

Spherical systems : useful relations

	$\rho(r)$	$\Phi(r)$	$M(r)$	$\frac{\mathrm{d}\Phi}{\mathrm{d}r}$
$\rho(r)$	$\rho(r)$	$\frac{1}{4\,\pi\,G}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$	$\frac{1}{4\pi r^2}\frac{\mathrm{d}M(r)}{\mathrm{d}r}$	$\frac{1}{4\,\pi\,G}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$
$\Phi(r)$	$-\frac{G\,M(r)}{r}-4\pi G\int_r^\infty \mathrm{d}r'\,r'\,\rho(r')$	$\Phi(r)$	$-G\int_r^\infty \mathrm{d}r'\,\frac{M(r')}{r'^2}$	$-\int_r^\infty \mathrm{d}r'\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$
$M(r)$	$4\pi\int_0^r \mathrm{d}r'\,r'^2\,\rho(r')$	$\frac{r^2}{G}\frac{\mathrm{d}\Phi}{\mathrm{d}r}$	$M(r)$	$\frac{r^2}{G}\frac{\mathrm{d}\Phi}{\mathrm{d}r}$
$\frac{\mathrm{d}\Phi}{\mathrm{d}r}$	$\frac{4\pi G}{r^2}\int_0^r \mathrm{d}r'\,r'^2\,\rho(r')$	$\frac{\mathrm{d}\Phi}{\mathrm{d}r}$	$\frac{G M(r)}{r^2}$	$\frac{\mathrm{d}\Phi}{\mathrm{d}r}$

Poisson in spherical coordinates

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)=4\,\pi\,G\,\rho(r)$$

Potential in spherical coordinates

$$\Phi(r)=-\frac{G\,M(r)}{r}-4\pi G\int_r^\infty \rho(r')r'\mathrm{d}r'$$

Mass inside a radius r

$$M(r)=4\pi\int_0^r dr'\,r'^2\,\rho(r')$$

Gradient of the potential in spherical coordinates

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r}=\frac{G\,M(r)}{r^2}$$

The End