EPFL

4th year physics 05.10.2022

Exercises week 2 Autumn semester 2022

Astrophysics III : Stellar and galactic dynamics Solutions

Problem 1:

The disk surface density can be expressed as a function of the total mass and radius of the disk:

$$\Sigma = \frac{f M_{\text{tot}}}{\pi R^2}.$$
 (1)

Hence:

$$R = \sqrt{\frac{f \cdot M_{\text{tot}}}{\pi \Sigma}}.$$
 (2)

With f = 1/45, $M_{\text{tot}} = 2 \cdot 10^{12} \,\mathrm{M}_{\odot}$, we thus have : $R = 16.8 \,\mathrm{kpc}$.

The mean density is:

$$<\rho> = \frac{f \cdot M_{\text{tot}}}{\pi R^2 \cdot 500} \sim 0.1 \text{ M}_{\odot} \,\text{pc}^{-3}$$
 (3)

with a thickness of $500 \,\mathrm{pc}$, $R = 16800 \,\mathrm{pc}$ and f = 1/45.

The period of a circular orbit is:

$$T = \frac{2\pi R}{\sqrt{GM(R)/R}},\tag{4}$$

So:

$$M = \frac{4\pi^2 R^3}{GT^2}. (5)$$

With $G = 6.67 \cdot 10^{-11} \,\mathrm{m^3/kg/s^2}$, $R = 8 \,\mathrm{kpc} = 2.46 \cdot 10^{20} \,\mathrm{m}$ and $T = 220 \,\mathrm{Myr} = 6.9 \cdot 10^{15} \,\mathrm{s}$, we find approximately $10^{11} \,\mathrm{M_{\odot}}$.

Problem 2:

For a galaxy cluster, the cluster typical radius is 1 Mpc and a typical galaxy size is 10 kpc :

$$\frac{\text{Volume (N galaxies)}}{\text{Volume (galaxy cluster)}} \simeq \frac{10^3 \cdot (10 \text{ kpc})^3}{(1000 \text{ kpc})^3} = 10^{-3}$$

For a galaxy:

$$\frac{\text{Volume (N stars)}}{\text{Volume (galaxy)}} \simeq \frac{10^{11} \cdot (10^6 \text{ km})^3}{(10^4 \text{ pc} \cdot 3.09 \cdot 10^{13} \text{ km/pc})^3} \simeq 4 \times 10^{-24}$$

We thus see that those dynamical systems are largely made of void.

Problem 3:

In a galaxy cluster, the typical speed of galaxies is $v=10^3~{\rm km/s\cdot (3.09\cdot 10^{16}~km/kpc)^{-1}}=3.09\cdot 10^{-14}~{\rm kpc/s},$ hence :

$$\frac{\text{Volume (tube)}}{\text{Volume (galaxy cluster)}} \simeq \frac{\pi (10 \text{ kpc})^2 \cdot v \cdot t}{\frac{4\pi}{3} (1000 \text{kpc})^3} \simeq 7.7 \cdot 10^{-4}$$

Within a galaxy, stars typically have v = 200 km/s, hence:

$$\frac{\text{Volume (tube)}}{\text{Volume (galaxy)}} \simeq \frac{\pi (10^6 \text{ km})^2 \cdot v \cdot t}{\frac{4\pi}{3} (10^4 \text{ pc} \cdot 3 \cdot 10^{13} \text{ km/pc})^3} \simeq 1.6 \cdot 10^{-21}$$

Given the tiny portion of the volume crossed by the components, we conclude that the probability of a collision between two components of a given dynamical system is extremely low.

Problem 4:

We consider that the gravitational influence of an object is significant when the mutual potential energy is of the same order than the kinematic energy of the relative motion.

$$E_{\rm cin} \simeq E_{\rm grav} \qquad \Leftrightarrow \qquad \frac{1}{2} m v^2 \simeq \frac{{\rm G} m^2}{R_G} \qquad \Leftrightarrow \qquad R_G \simeq \frac{2 {\rm G} m}{v^2}$$

We observe that this result gives the value of $b_{90} = 2Gm/v^2$ seen in the course, so that $R_G = b_{90}$.

For a galaxy moving within a galaxy cluster :

$$R_G \simeq \frac{2 \cdot 6.67 \cdot 10^{-11} \text{m}^3/\text{kg/s}^2 \cdot 10^{11} \cdot 2 \cdot 10^{30} \text{ kg}}{(10^6 \text{ m/s})^2} \simeq 10^{19} \text{ m} \simeq 1 \text{ kpc}$$

For a star moving within a galaxy:

$$R_G \simeq \frac{2 \cdot 6.67 \cdot 10^{-11} \text{m}^3/\text{kg/s}^2 \cdot 2 \cdot 10^{30} \text{ kg}}{(2 \cdot 10^5 \text{ m/s})^2} \simeq 0.7 \cdot 10^{10} \text{ m} \simeq 0.05 \text{ A.U.}$$

Problem 5:

The solution to this is given by the fraction of galaxies which have their symmetry axis (normal to the disk) which covers the ten degrees of the spherical cap.

$$P(i < i_0) = \int_0^{i_0} \frac{2\pi \sin \theta}{2\pi} d\theta$$
$$= 1 - \cos i_0$$

For $i_0 = 10^{\circ}$, we then get

$$P(i < 10^{\circ}) = 1 - \cos 10^{\circ} = 0.015$$

For the fraction seen edge on, we have $P(i > 80^{\circ}) = 1 - P(i < 80^{\circ}) = 0.17$

Problem 6:

The relaxation time is given by

$$t_{relax} = \frac{0.1 N}{\ln N} t_{cross} = \frac{0.1 N}{\ln N} \frac{R}{v}$$

Therefore we have

1. For the open cluster:

$$t_{relax} = \frac{0.1 \cdot 300}{\ln 300} \, \frac{2 \cdot 3.09 \cdot 10^{13} \text{ km}}{0.5 \text{ km s}^{-1}} \simeq 6.5 \cdot 10^{14} \text{ s} \simeq 2 \cdot 10^7 \text{ yrs}$$

The youngest open clusters are not relaxed yet, but the oldest ones are fully relaxed.

2. For the globular cluster:

$$t_{relax} = \frac{0.1 \cdot 2 \cdot 10^5}{\ln(2 \cdot 10^5)} \frac{3 \cdot 3.09 \cdot 10^{13} \text{ km}}{6 \text{ km s}^{-1}} \simeq 2.5 \cdot 10^{16} \text{ s} \simeq 8 \cdot 10^8 \text{ yrs}$$

The globular clusters are fully relaxed.

3. For a dwarf spheroidal galaxy:

$$t_{relax} = \frac{0.1 \cdot 10^7}{\ln(10^7)} \frac{500 \cdot 3.09 \cdot 10^{13} \text{ km}}{10 \text{ km s}^{-1}} \simeq 1 \cdot 10^{20} \text{ s} \simeq 3 \cdot 10^{12} \text{ yrs}$$

Dwarf spheroidal galaxies are far from being relaxed.

$\underline{\text{Problem } 7}:$

The relaxation time is given by a ratio between the velocity (which is a factor of the total mass of the system) and the average change in velocity per unit time. If the number of particles increases, the average velocity of a particle increases as $\propto \sqrt{N}$, while the average change in velocity per unit time only increases by $\propto \log N$. Therefore, if other factors are held constant, adding more members (and therefore mass) will result in an increase in the relaxation time of a system.