Nuclear Fusion and Plasma Physics - Exercises

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Solutions to Problem Set 4 - 17 October 2022

Exercise 1 - Small vs. large collision angles

a) For the collisions with small deflection angle, the momentum transfer cross-section is:

$$\begin{aligned}\sigma_p &= \sigma_{E_k}\frac{m_1+m_2}{2\,m_1} \\ &= \underbrace{8\pi\, b_{90}^2 \ln\Lambda \frac{m_1 m_2}{(m_1+m_2)^2}}_{\sigma_{Ek}} \frac{m_1+m_2}{2\,m_1} \\ &= 4\pi\, b_{90}^2 \ln\Lambda \frac{m_2}{m_1+m_2} \end{aligned} \quad (1)$$

The cross-section σ_p has to be compared with the cross-section for the collisions with a deflection angle $\theta \geq 90°$:

$$\sigma_{90} = \pi b_{90}^2$$

The ratio between them is:

$$\frac{\sigma_p}{\sigma_{90}} = 4\ln\Lambda \frac{m_2}{m_1+m_2}$$

and for $m_2 \gg m_1$ (ie electrons colliding with nucleus of deuterium or tritium):

$$\frac{\sigma_p}{\sigma_{90}} \approx 4\ln\Lambda$$

If $T_e = T_i = 10$ keV the coulomb logarithm is (formulary NRL, page 34)

$$\ln\Lambda = 24 - \ln\left(\frac{\sqrt{n_e[cm^{-3}]}}{T[eV]}\right) \approx 24 - \ln\left(\frac{\sqrt{10^{14}}}{10^4}\right) \approx 17$$

Therefore $\frac{\sigma_p}{\sigma_{90}} \approx 68 \gg 1$.

b) In general, for $m_2 \gg m_1$ and a typical temperature of a plasma used in thermonuclear fusion applications, we have $\frac{\sigma_p}{\sigma_{90}} \gg 1$. We can then conclude that it is possible to neglect the effects of the collisions with $\theta \geq 90°$.

Exercise 2 - Alpha particle thermalisation in a burning plasma

The α particles will collide with the three species (electrons, deuterium ions and tritium ions) losing their energy and therefore their velocity. The estimated relaxation time is necessarily a function of the α's velocity.

Using the given parameters, we can obtain the relevant thermal speeds:

$$v_{\text{th,e}} = \sqrt{\frac{T_e}{m_e}} = 4.19 \times 10^7\,\text{m/s}$$
$$v_{\text{th},D} = \sqrt{\frac{T_D}{2\,m_p}} = 6.92 \times 10^5\,\text{m/s}$$
$$v_{\text{th},T} = \sqrt{\frac{T_T}{3\,m_p}} = 5.65 \times 10^5\,\text{m/s}$$

The α's velocity is obtained from their energy:

$$v_\alpha = \sqrt{\frac{2\,E_\alpha}{4\,m_p}} = 6.92 \times 10^3 \sqrt{E_\alpha[eV]}\,\text{m/s}$$

At birth, $E_\alpha = 3.5$ MeV, $v_\alpha = 1.3 \times 10^7\,\text{m/s}$ and therefore:

$$v_{\text{th},D} \le v_{\text{th},T} \ll v_\alpha < v_{\text{th},e}$$

The last relation is valid for $E_\alpha > 10\,\text{keV}$. For $7\,\text{keV} \le E_\alpha \le 10\,\text{keV}$, $v_{\text{th},T} \le v_\alpha \le v_{\text{th},D}$ and for $E_\alpha \le 7\,\text{keV}$, we have $v_\alpha \le v_{\text{th},T}$. When $E_\alpha \approx 5\,\text{keV}$, the α's are decelerated to a velocity corresponding to $T_\alpha = 10\,\text{keV}$ (complete thermalization).

The three collision frequencies taking part in the energy transfer process are:

$$\nu_{E_k}^{\alpha/e} \simeq n_e \frac{4e^4}{2\pi\varepsilon_0^2} \frac{\ln\Lambda}{4\,m_p m_e} \frac{1}{v_{the}^3} \approx 20.4\ \text{s}^{-1}$$
$$\nu_{E_k}^{\alpha/D} \simeq \frac{n_e}{2} \frac{4e^4}{2\pi\varepsilon_0^2} \frac{\ln\Lambda}{4\,m_p\,2\,m_p} \frac{1}{v_\alpha^3} = \nu_{E_k}^{\alpha/e} \sqrt{\frac{m_p}{2\,m_e}} \left(\frac{T}{E_\alpha}\right)^{3/2}$$
$$\nu_{E_k}^{\alpha/T} = \frac{2}{3}\,\nu_E^{\alpha/D}$$

where we have used the relations:

$$\ln\Lambda = 24 - \ln\left(\frac{\sqrt{n_e[\text{cm}^{-3}]}}{T[\text{eV}]}\right) \approx 24 - \ln\left(\frac{\sqrt{10^{14}}}{10^4}\right) \approx 17.1$$

and

$$n_D = n_T = 0.5\,n_e$$

Strictly speaking, the velocity in the denominator is relative to the centre-of-mass reference frame. Usually, this relative velocity is dominated by the velocity of one particle (target particle or on-coming particle). That is the case for the three interactions studied and that is why the electron thermal speed (target particles) is present in the expression of $\nu_{E_k}^{\alpha/e}$ ($v_{\text{th},e} \gg v_\alpha$).

At the beginning of the relaxation process ($E_\alpha = 3.5$ MeV), the frequencies are equal to:

$$\nu_{E_k}^{\alpha/D} \approx 9.4 \times 10^{-2}\ \text{s}^{-1} \qquad \nu_{E_k}^{\alpha/T} \approx 6.3 \times 10^{-2}\ \text{s}^{-1}$$

therefore $\nu_{E_k}^{\alpha/D}, \nu_{E_k}^{\alpha/T} \ll \nu_{E_k}^{\alpha/e} \approx 20.4\ \text{s}^{-1}$: the α particles transmit their energy to the electrons.

In this case, the relaxation time is given by:

$$\tau = \left(\nu_{E_k}^{\alpha/D} + \nu_{E_k}^{\alpha/T} + \nu_{E_k}^{\alpha/e}\right)^{-1} \approx \left(\nu_{E_k}^{\alpha/e}\right)^{-1} = 49\ \text{ms}$$

We can conclude that:

- At the beginning the electrons are more effective in the α thermalisation process. Only when $E_\alpha \leq 10\,\text{keV}$, the deuterium ions are more effective than the electrons (the tritium ions are playing an important role in the thermalisation process for lower energies).
- Since the energy transfer due to the collisions between α particles and electrons during the transition $E_\alpha = 3.5\,\text{MeV} \to 10\,\text{keV}$ is more important than the transition $E_\alpha = 10\,\text{keV} \to 5\,\text{keV}$, the electrons are heated more than the ions.

Exercise 3 - Runaway electrons

a) We suppose to have uniform parameters (density, temperature, ...) in the considered volume. The relation between the electric field E and the current density j is $E = \eta\, j$. Having $I = j \cdot A$, we can write $I = \frac{A}{\eta} E$, where $A = \pi a^2$ is the section of the cylinder. The potential difference between the extremities of the cylinder is $\Delta V_{Plasma} = E \cdot L$, therefore:

$$\Delta V_{Plasma} = \frac{\eta I}{\pi a^2} L$$

Using the numerical values, we find:

$$E \cong \frac{5.1 \times 10^{-5} Z\, ln\Lambda}{(T_e[eV])^{\frac{3}{2}}} \frac{I}{\pi a^2} \cong 5.8 \times 10^{-4}\ \text{V/m}$$

where the Spitzer resistivity has been used.

Finally we have:

$$\Delta V_{Plasma} = E \cdot L = 19.3 \text{ mV}$$

If we consider a stainless steel cylinder with resistivity $\eta_{inox} \approx 7 \times 10^{-7}\ \Omega \cdot \text{m}$, we find:

$$E_{inox} = \frac{I\eta_{inox}}{\pi a^2} = 7{\times}10^{-7}\frac{15 \times 10^6}{\pi 2^2} \text{ V/m} \approx 0.835 \text{ V/m} \Rightarrow \Delta V_{inox} = 27.8 \text{ V} >> \Delta V_{Plasma}$$

b) What is the behaviour of the electrons in the "tail" of the distribution function (suprathermal electrons)?

For these electrons, we can no longer consider the value of the collision frequency averaged on the maxwellian distribution function. Moreover, the collisions between these electrons and the ones having a velocity of the order of the thermal velocity (bulk electrons) are not negligible. That's why we need a more complex expression for the collision frequency ($v_{se} \propto v^{-3}$).

The momentum equation for this electron population at the velocity v is:

$$m_e\frac{dv}{dt} = -eE - \nu_{se}(v)m_e v = -eE - F_c(v)$$

Consider the sign of the right hand side of this equation. If the friction force due to the collisions is bigger than the acceleration force due to the electric field ($F_c(v) > e|E|$), there is a deceleration until the thermal velocity is reached.

On the other hand, if $e|E| > F_c(v)$ there is an acceleration. When the velocity v increases, the friction force decreases ($F_c(v) \propto v^{-2}$) and the acceleration is amplified (run-away regime).

The expression for the minimum electric field necessary to be in the run-away regime is:

$$e|E| = F_c(v) = (2+Z)\frac{e^4}{2\pi\epsilon_0^2}n_e\frac{\ln\Lambda}{m_e^2 v^3}m_e v$$

$$|E| = (2+Z)\frac{e^3}{2\pi\epsilon_0^2}n_e\frac{\ln\Lambda}{m_e v^2}$$

The critical kinetic energy $E_{k,crit} = \frac{m_e v_{crit}^2}{2}$

$$E_{k,crit} = (2+Z)\frac{e^3}{4\pi\epsilon_0^2}n_e \ln\Lambda\frac{1}{|E|}$$

If we relate this kinetic energy to the electron temperature, given in eV, we get

$$\frac{E_{k,crit}}{eT_e} = (2+Z)\frac{e^2}{4\pi\epsilon_0^2}n_e \ln\Lambda\frac{1}{|E|T_e}$$

If we then define the critical field as:

$$E_{crit} \equiv (2+Z)\frac{e^2}{4\pi\epsilon_0^2}\ln\Lambda\frac{n_e}{T_e},$$

we get the expression

$$\frac{E_{k,crit}}{eT_e} = \frac{E_{crit}}{|E|}$$

This electric field E_{crit} is called the *Dreicer field*. This expression is particularly useful since it can be calculated from the plasma parameters (T_e, n_e,Z). Since the electric field is known from the parameters of the device (e.g. Tokamak), we immediately obtain the portion of electrons which can be in the runaway regime.

The E field in ITER is calculated as $V_{loop}/L = V_{loop}/(2\pi R) = 0.58\text{mV/m}$ The value of the critical kinetic energy then is:

$$E_{k,crit} = (2+1)\frac{(1.6\times10^{-19})^3}{4\pi\cdot(8.85\times10^{-12})^2}17\frac{10^{20}}{0.58\times10^{-3}} \simeq 3.7\times10^{-11}\text{J} \simeq 230\,\text{MeV}$$

Be careful with the notation: E_{crit} is the critical electric field and $E_{k,crit}$ is the critical kinetic energy.

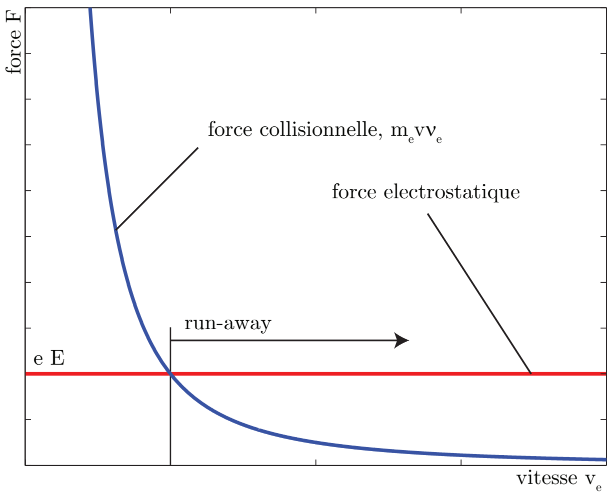

Figure 1: Force due to the electric field E and force due to the collisions for an electron with velocity v_e.