Solution 5
Traitement Quantique de l'Information

Exercise 1 *Dynamics of spin 1/2*

(a) The evolution matrix is given by (here notation $U(t,0)$ means evolution from 0 to t)

$$U(t,0) = \exp\Big(-i\frac{t\delta}{2}\sigma_z + i\frac{t\omega_1}{2}\sigma_x\Big) = \exp\Big(\frac{a}{2}(n_x\sigma_x + n_z\sigma_z\Big)$$

with $a = t(\delta^2+\omega_1^2)^{1/2}$ et $n_x = \frac{\omega_1}{(\delta^2+\omega_1^2)^{1/2}}$, $n_z = \frac{\delta}{(\delta^2+\omega_1^2)^{1/2}}$. So applying the "generalized Euler formula"

$$U(t,0) = \cos\left(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\right)I + i\sin\left(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\right)\Big(\frac{\omega_1}{(\delta^2+\omega_1^2)^{1/2}}\sigma_x + \frac{\delta}{(\delta^2+\omega_1^2)^{1/2}}\sigma_z\Big)$$

This gives the final matrix :

$$U(t,0) = \begin{bmatrix} \cos\big(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\big) + i\delta\frac{\sin\left(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\right)}{(\delta^2+\omega_1^2)^{1/2}} & i\omega_1\frac{\sin\left(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\right)}{(\delta^2+\omega_1^2)^{1/2}} \\ i\omega_1\frac{\sin\left(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\right)}{(\delta^2+\omega_1^2)^{1/2}} & \cos\big(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\big) - i\delta\frac{\sin\left(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\right)}{(\delta^2+\omega_1^2)^{1/2}} \end{bmatrix}$$

(b) In the limit $\omega_1 << \delta$ we obtain

$$U(t,0) = \begin{bmatrix} \cos\big(\frac{t\delta}{2}\big) + i\sin\big(\frac{t\delta}{2}\big) & 0 \\ 0 & \cos\big(\frac{t\delta}{2}\big) - i\sin\big(\frac{t\delta}{2}\big) \end{bmatrix}$$

If the initial state is $\frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle)$ then the final state is

$$\frac{1}{\sqrt{2}}(e^{it\delta/2}|\uparrow\rangle + e^{-it\delta/2}|\downarrow\rangle = \frac{e^{-it\delta/2}}{\sqrt{2}}(|\uparrow\rangle + e^{-it\delta}|\downarrow\rangle$$

<u>Trajectory</u> : On the Bloch sphere the trajectory is along the equator (since $\theta = \pi/2$ et $\varphi = -t\delta$). It is periodic with period $T = \frac{2\pi}{\delta}$.

<u>Remark</u> :In the rotating frame the state is (approximately) unaffected by the magnetic field (situation of extreme detuning).

(c) In the limit $\delta << \omega_1$ we obtain

$$U(t,0) = \begin{bmatrix} \cos\big(\frac{\omega_1 t}{2}\big) & i\sin\big(\frac{\omega_1 t}{2}\big) \\ i\sin\big(\frac{\omega_1 t}{2}\big) & \cos\big(\frac{\omega_1 t}{2}\big) \end{bmatrix}$$

If the initial state is $|\uparrow\rangle$ then the final state is

$$U(t,0)|\uparrow\rangle = \cos\left(\frac{\omega_1 t}{2}\right)|\uparrow\rangle + i\sin\left(\frac{\omega_1 t}{2}\right)|\downarrow\rangle$$

Trajectory : On the Bloch sphere the trajectory is rotating around the x -axis and is in the plane (yz) (because $\theta = \omega_1 t$ and $\varphi = \pi/2$). It is periodic with period $T = \frac{2\pi}{\omega_1}$: in effect this period corresponds to the parametrisation $\theta = \omega_1 t$.

Remark 1 : Within one period the evolution matrix changes sign but this sign gives a global phase which is not obervable at the level of probabilities of measurement outcomes.

Remark 2 :In the rotating frame this corresponds to spin flips in a situation of tuning.

Exercise 2 *Entanglement creation by a magnetic interaction*

We work entirely in Dirac notation but you can also check out the complement on the connection with component notation.

The final state is (using that $|\uparrow\rangle, |\downarrow\rangle$ are eigenvectors of σ_z with eigenvalues $+1$ et -1).

$$\begin{aligned}
e^{-\frac{it}{\hbar}\mathcal{H}}|\psi_0\rangle &= e^{-itJ\sigma_1^z\otimes\sigma_2^z}\cdot\frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\
&= \frac{1}{2}\left(e^{-itJ}|\uparrow\uparrow\rangle - e^{itJ}|\uparrow\downarrow\rangle + e^{itJ}|\downarrow\uparrow\rangle - e^{-itJ}|\downarrow\downarrow\rangle\right) \\
&= \frac{e^{-itJ}}{2}\left(|\uparrow\uparrow\rangle - e^{2itJ}|\uparrow\downarrow\rangle + e^{2itJ}|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right).
\end{aligned}$$

(a) for $t = \frac{\pi}{4J}$ on a $e^{2itJ} = e^{\frac{i\pi}{2}} = i$
$\Rightarrow |\psi_t\rangle = \frac{e^{-\frac{i\pi}{4}}}{2}\left(|\uparrow\uparrow\rangle - i|\uparrow\downarrow\rangle + i|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$.

(b) suppose the state can be written
$(\alpha|\uparrow\rangle + \beta|\downarrow\rangle)\otimes(\gamma|\uparrow\rangle + \delta|\downarrow\rangle) = \alpha\gamma|\uparrow\uparrow\rangle + \alpha\delta|\uparrow\downarrow\rangle + \beta\gamma|\downarrow\uparrow\rangle + \beta\delta|\downarrow\downarrow\rangle$,
then $\alpha\gamma = 1,\ \alpha\delta = -i,\ \beta\gamma = i,\ \beta\delta = -1$.
One can always set $\alpha = 1$ (global phase). Thus $\gamma = 1,\ \delta = -i,\ \beta = i$ et $\delta = i$ $\Rightarrow$ contradiction on δ. You can also take any value for α and show the contradiction appears.

(c) At $t = \frac{\pi}{2J}$ with $e^{\pm itJ} = e^{\pm i\frac{\pi}{2}} = \pm i$,

$$\begin{aligned}
|\psi_t\rangle &= \frac{1}{2}\left(-i|\uparrow\uparrow\rangle - i|\uparrow\downarrow\rangle + i|\downarrow\uparrow\rangle + i|\downarrow\downarrow\rangle\right) \\
&= \frac{-i}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)
\end{aligned}$$

is a product state. So another $\frac{\pi}{4J}$ time of evolution cancels the entanglement.

(d) At $t = \frac{\pi}{J}$ with $e^{\pm itJ} = e^{\pm i\pi} = -1$,

$$\begin{aligned} |\psi_t\rangle &= \frac{1}{2}\left(-|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\right) \\ &= \frac{-1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) \end{aligned}$$

is also a product state.

Exercise 3 *No cloning theorem*

(a) Taking the equation defining U for $i = 1$ and the Dirac conjugate of it for $i = 2$ we get

$$U(|\phi_1\rangle \otimes |O\rangle) = |\phi_1\rangle \otimes |\phi_1\rangle, \quad (\langle\phi_2| \otimes \langle O|)U^\dagger = \langle\phi_2| \otimes \langle\phi_2|$$

Using unitarity the inner product $U^\dagger U = I$ we find

$$(\langle\phi_2| \otimes \langle O|)(|\phi_1\rangle \otimes |O\rangle) = (\langle\phi_2| \otimes \langle\phi_2|)(|\phi_1\rangle \otimes |\phi_2\rangle)$$

Since $\langle O|O\rangle = 1$ we get $\langle\phi_2|\phi_1\rangle = \langle\phi_2|\phi_1\rangle^2$ which implies $\langle\phi_2|\phi_1\rangle = 0$ or $= 1$. The first is not possible because we assumed the two states are not orthogonal. The second case is also not possible because we assumed the two states are different. In conclusion such a unitary matrix cannot exist.

(b) Applying U on both sides of the equation given in the hint :

$$U(|\phi_1\rangle + |\phi_2\rangle) \otimes |O\rangle) = U(|\phi_1\rangle \otimes |O\rangle + |\phi_2\rangle \otimes |O\rangle)$$

Using U's definition on the left hand side, and on the right hand side linearity together with U's definition :

$$(|\phi_1\rangle + |\phi_2\rangle) \otimes (|\phi_1\rangle + |\phi_2\rangle) = |\phi_1\rangle \otimes |\phi_1\rangle + |\phi_2\rangle \otimes |\phi_2\rangle$$

Expanding the tensor product on the left hand side :

$$|\phi_1\rangle \otimes |\phi_2\rangle + |\phi_2\rangle \otimes |\phi_1\rangle = 0$$

This is not possible. Indeed if we take the inner product with $|\phi_1\rangle \otimes |\phi_2\rangle$ we find $\langle\phi_1|\phi_2\rangle\langle\phi_2|\phi_1\rangle = -1$ which is equivalent to $|\langle\phi_1|\phi_2\rangle|^2 = -1$.

Complement on the Hamiltonian in matrix and Dirac notation.

1. **Matrix notation.** In the canonical bases, we have $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Using the tensor product rule one obtains that

$$\begin{aligned}\sigma_1^z \otimes \sigma_2^z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

thus the Hamiltonian is

$$\mathcal{H} = \begin{pmatrix} \hbar J & 0 & 0 & 0 \\ 0 & -\hbar J & 0 & 0 \\ 0 & 0 & -\hbar J & 0 \\ 0 & 0 & 0 & \hbar J \end{pmatrix}.$$

2. **Dirac notation.** In the bra-ket formalism one has $\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$, thus

$$\begin{aligned}\sigma_1^z \otimes \sigma_2^z &= (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \otimes (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \\ &= |\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|.\end{aligned}$$

Therefore, we have

$$\mathcal{H} = \hbar J(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|).$$

3. **Connection between matrix and Dirac notations.** Notice that to verify this one can use

$$|\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which implies that

$$(|\uparrow\rangle\langle\uparrow|) \otimes (|\uparrow\rangle\langle\uparrow|) = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly one can show that

$$|\uparrow\downarrow\rangle\langle\uparrow\downarrow| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$|\downarrow\uparrow\rangle\langle\downarrow\uparrow| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$|\downarrow\downarrow\rangle \langle\downarrow\downarrow| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. **Eigenvalues and eigenvectors.** One can see that the eigen-values are $\hbar J$ corresponding to the eigenvectors $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$ and $-\hbar J$ corresponding to the eigenvectors $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$.