Potential Theory

end of the 2nd part

Stellar orbits

1st part

Outlines

Ideal but useful models

- the infinite wire, the infinite slab
- infinite slab with oscillatory surface density, tightly wound spiral

Orbits

- some generalities

Lagrangian and Hamiltonian mechanics

- Euler-Lagrange equations
- Hamilton's equations

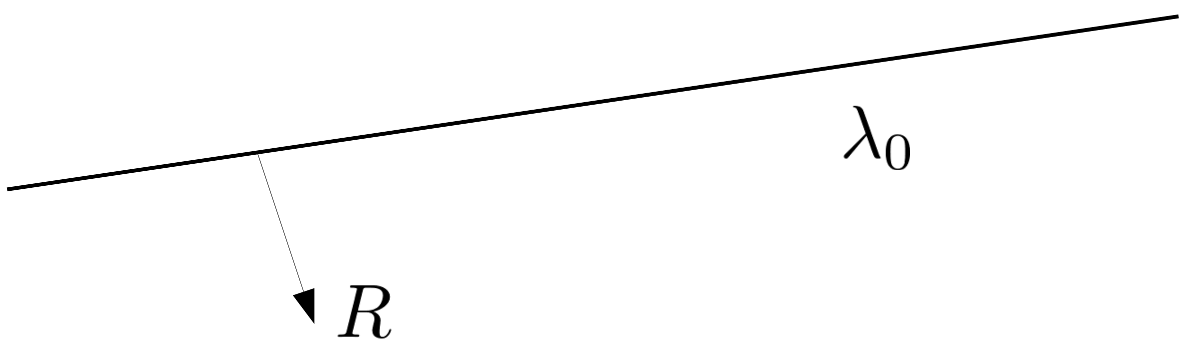
Orbits in spherical potentials

- angular momentum conservation
- equations of motion
- radial orbits
- non radial orbits

Potential Theory

Ideal but useful models

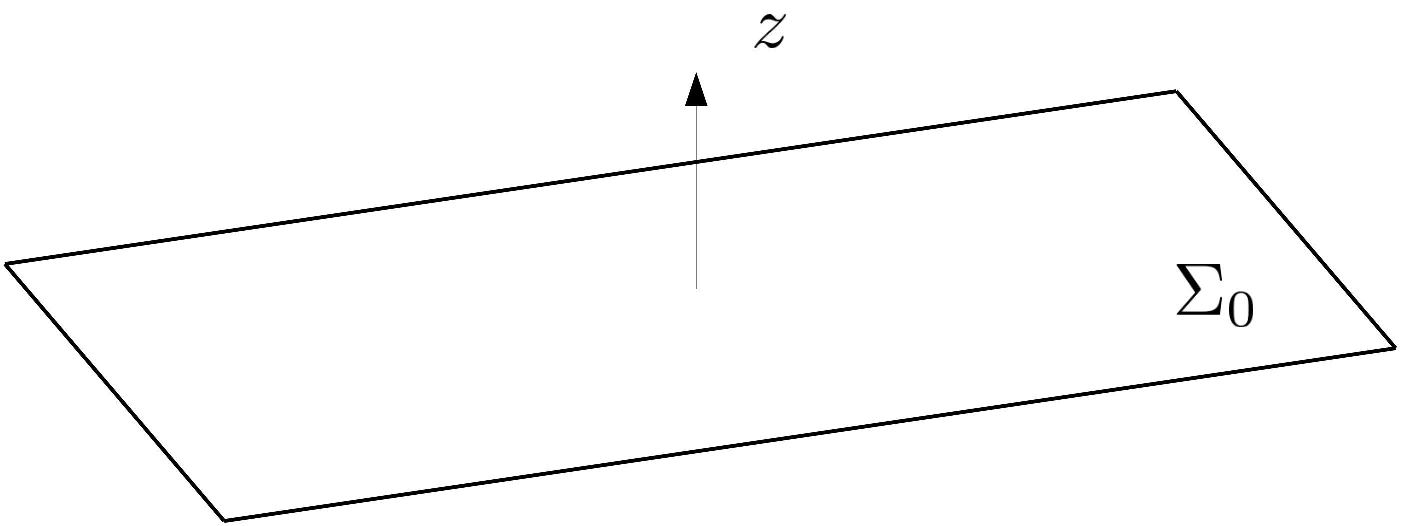
Potential of an infinite wire of constant linear density



$$\boxed{\Phi(R) = 2\,G\,\lambda_0\,\ln(R) + C}$$

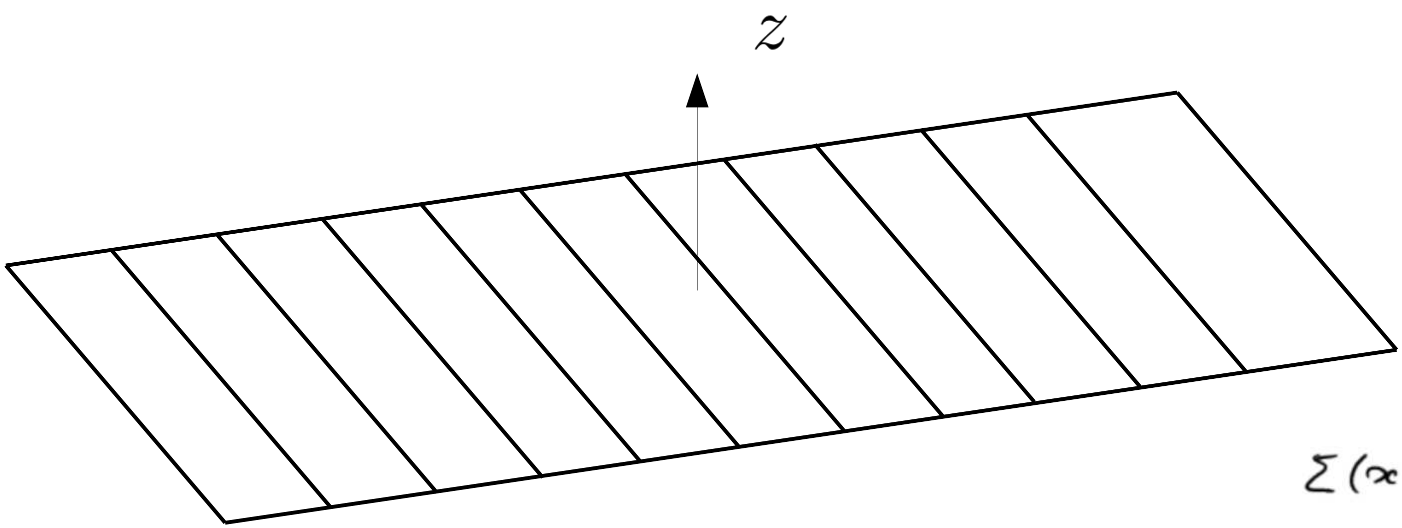
EXERCICE

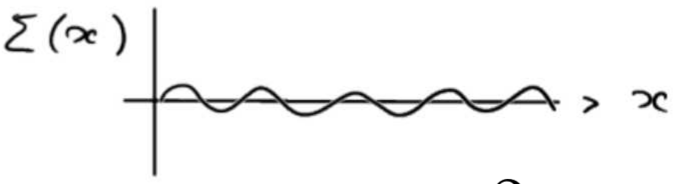
Potential of an infinite slab of constant surface density



$$\boxed{\Phi(z) = 2\pi G\,\Sigma_0\,|z| + C}$$

Potential of an infinite slab with an oscillatory surface density





$$k = |\vec{k}| = \frac{2\pi}{\lambda}$$

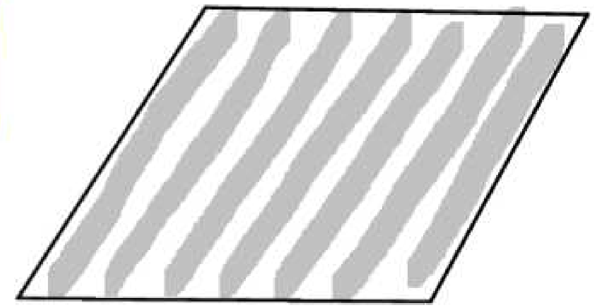
$$\Sigma(x,y) = \Sigma_1 \,\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)$$

! will be negative !

$$\boxed{\Phi(x,y,z) = -\frac{2\pi G\,\Sigma_1}{|\vec{k}|}\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right) e^{-|\vec{k}\,z|}}$$

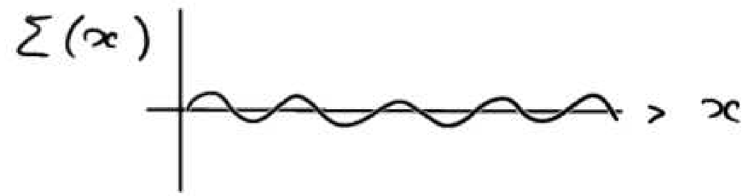
Potential of an infinite slab with an oscillatory surface density

$$\Sigma(x,y) = \mathrm{Re}\left(\Sigma_0 \, e^{i(\vec{k}\vec{x})}\right)$$



Without loss of generality we can restrict to:

$$\Sigma(x) = \Sigma_0 \, e^{ikx}$$



Poisson equation $\qquad \vec{\nabla}^2 \phi(x,z) = 4\pi G \, \Sigma(x) \, \delta(z)$

Assume a corresponding potential of the type

$$\phi(x,z) = \phi_0 \, e^{ikx - |kz|}$$

Method : Integrate the Poisson equation over z

$$\vec{\nabla}^2 \phi = 4\pi G \rho$$

$$\int_{-\xi}^{\xi} dz \, \vec{\nabla}^2 \phi = \int_{-\xi}^{\xi} dz \, 4\pi G \rho$$

and take the limit $\xi \to 0$

$$\underbrace{\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \, \vec{\nabla}^2 \phi}_{1} = \underbrace{\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \, 4\pi G \rho}_{2}$$

(1) $$\bar{\nabla}^2 = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$\frac{\partial^2 \phi}{\partial x^2}(x, y, 0)$ is continuous across $z = 0$

$$\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \, \frac{\partial^2 \phi}{\partial x^2}(x, y, z) = 0$$

$$\lim_{\xi \to 0^+} \equiv \lim_{\xi \to 0^-}$$

$$\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \, \frac{\partial^2 \phi}{\partial y^2}(x, y, z) = 0$$

$$\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \, \frac{\partial^2 \phi}{\partial z^2}(x, y, z) = \lim_{\xi \to 0} \left. \frac{\partial \phi}{\partial z} \right|_{-\xi}^{\xi}$$

$$= \lim_{\xi \to 0} \left. \phi_0 \, |k| \, \mathrm{sgn}(z) \, e^{ikx - |kz|} \right|_{-\xi}^{\xi} = -2|k| \, \phi_0 \, e^{ikx}$$

$$② \quad \lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \; 4\pi G \rho = \lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \; 4\pi G \, \Sigma_0 \, e^{ikx} \, \delta(z)$$

$$= 4\pi G \, \Sigma_0 \, e^{ikx}$$

Combining ① and ②

$$-2|k| \, \phi_0 \, e^{ikx} = 4\pi G \, \Sigma_0 \, e^{ikx}$$

$$\phi_0 = - \frac{2\pi G \Sigma_0}{|k|}$$

$$\boxed{\phi(x, y, z) = - \frac{2\pi G \Sigma_0}{|k|} \, e^{ikx - |kz|}}$$

Thus for $\Sigma(x,y) = \Sigma_0 \, e^{i\vec{k}\vec{x}}$

$$\boxed{\phi(x,y,z) = -\frac{2\pi G \Sigma_0}{|\vec{k}|} \, e^{i\vec{k}\vec{x} - |\vec{k}z|}}$$

Note if the surface density evolves as a plane wave

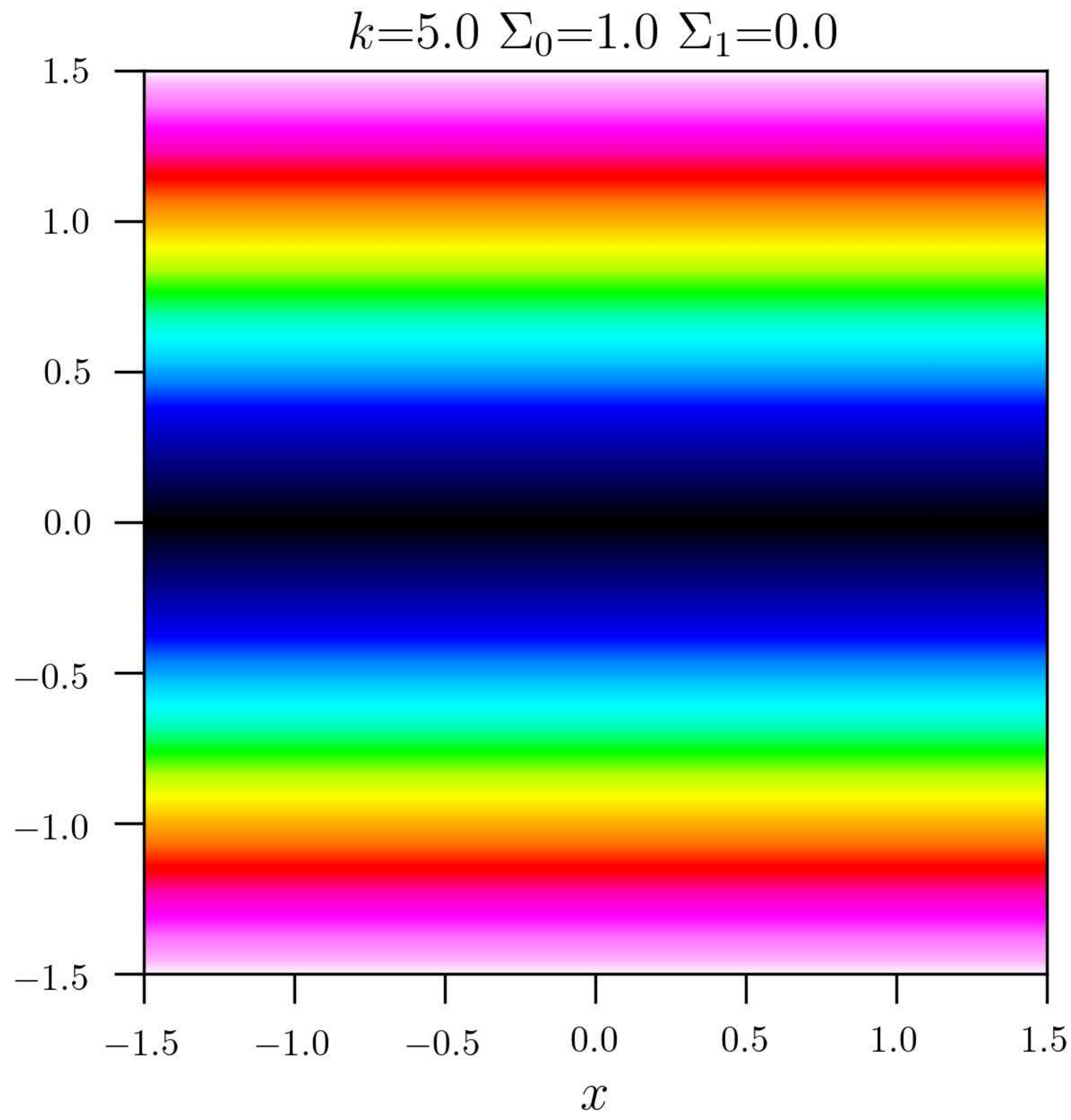
$$\Sigma(x,y,t) = \Sigma_0 \, e^{i(\vec{k}\vec{x} - \omega t)}$$

$$\boxed{\phi(x,y,z,t) = -\frac{2\pi G \Sigma_0}{|\vec{k}|} \, e^{i(\vec{k}\vec{x} - \omega t) - |\vec{k}z|}}$$

Potential of an Infinite slab

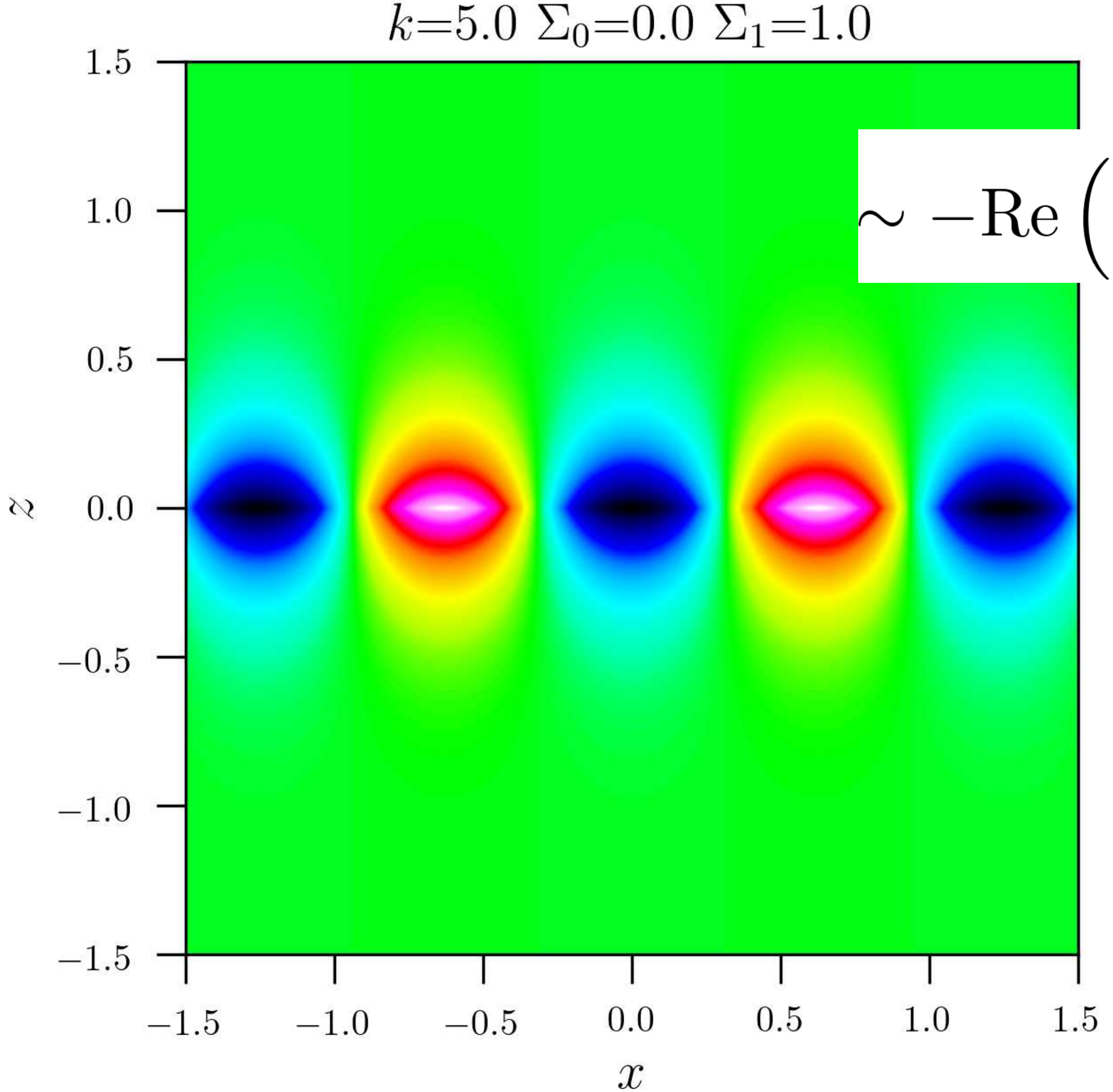
$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$

$\sim \Sigma_0 \, |z|$



Potential of an Infinite slab

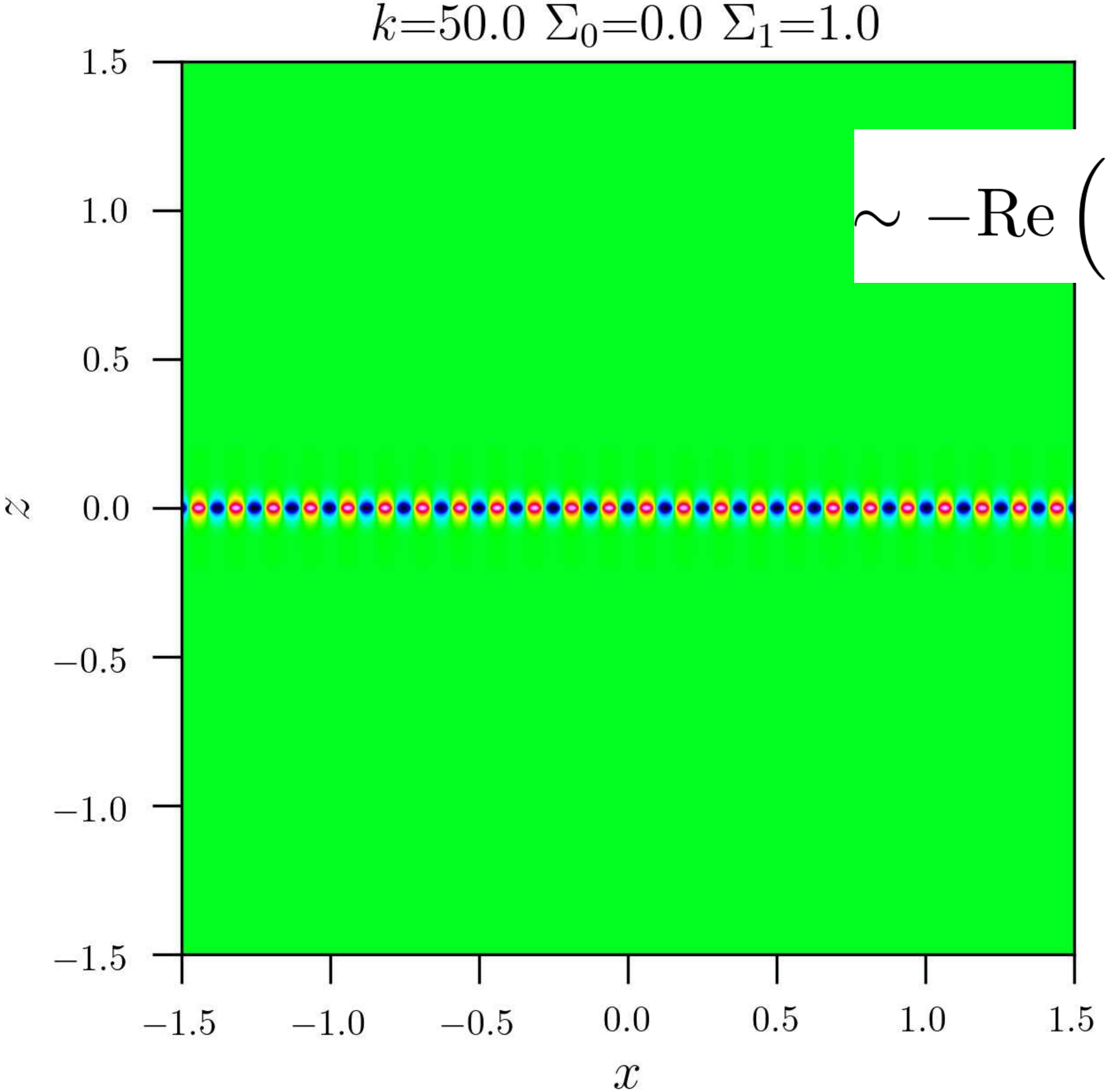
$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$



$$\sim -\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)\frac{e^{-k\,|z|}}{k}$$

Potential of an Infinite slab

$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$

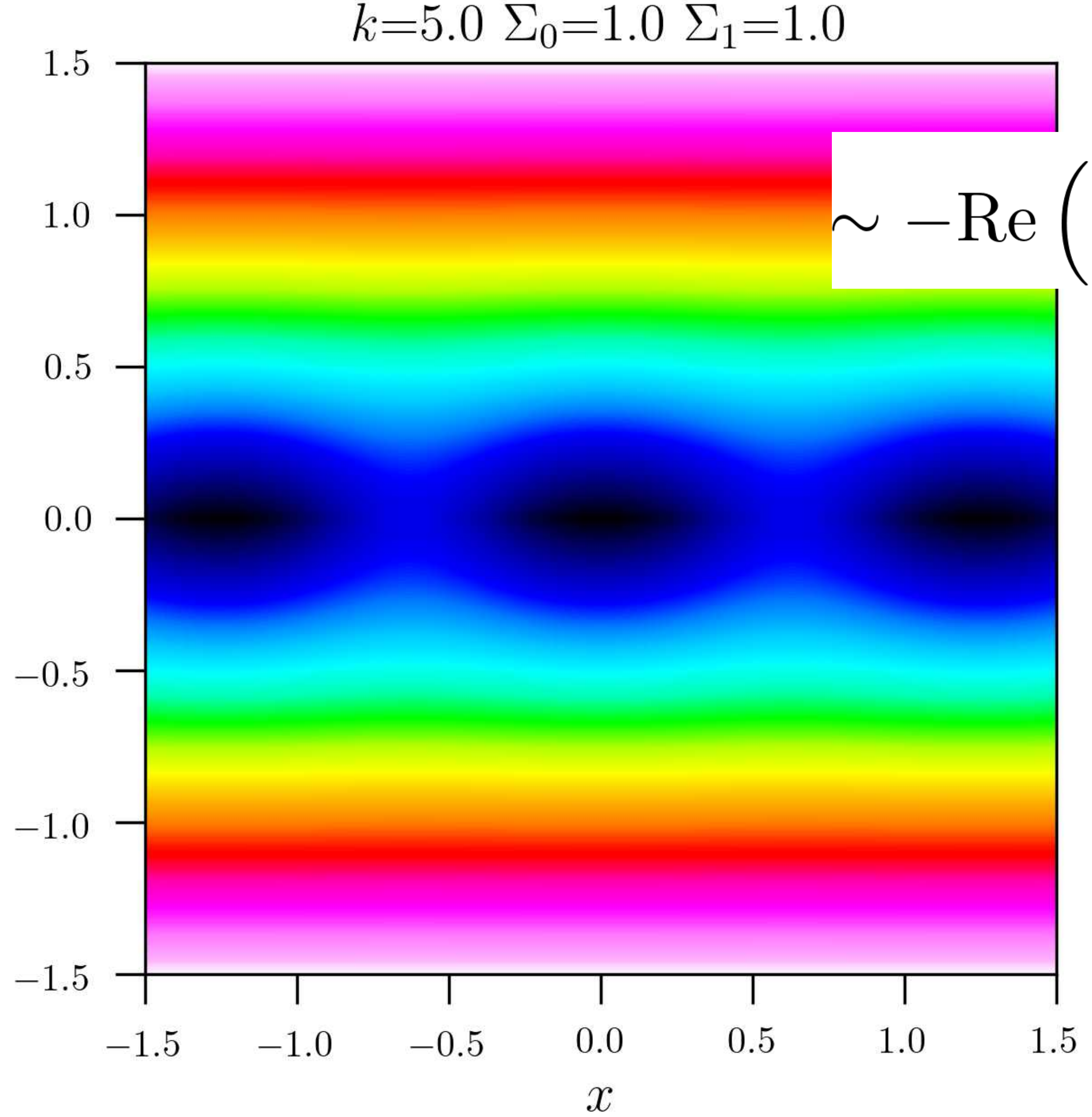


$$\sim -\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)\frac{e^{-k\,|z|}}{k}$$

Potential of an Infinite slab

$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$

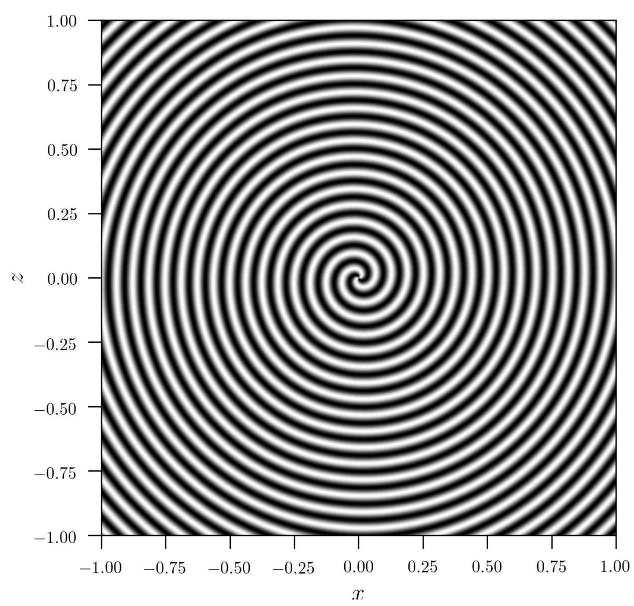
$$\sim \Sigma_0 \left|z\right|$$



$$\sim -\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)\frac{e^{-k\,|z|}}{k}$$

Potential of an infinite slab with a tightly wound spiral pattern

m=2



$$\Sigma(R,\phi) = H(R)\,\mathrm{Re}\left(e^{i[m\,\phi + f(R)]}\right)$$

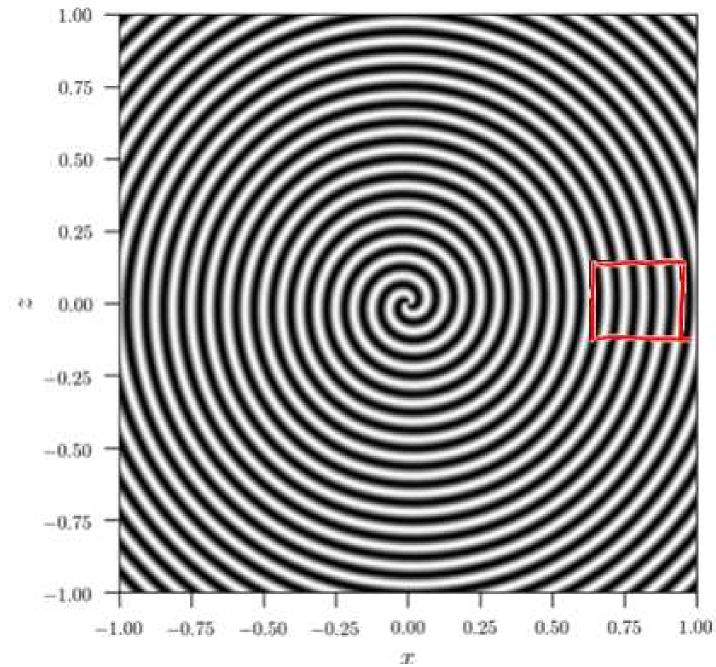
if $$\left|\frac{\partial f}{\partial R}\cdot R\right| \ll 1$$

WKB approximation
(Wentzel,Kramers,Brillouin)

$$\boxed{\Phi(R,\phi) = -\frac{2\pi G\,\Sigma_0}{\left|\frac{\partial f}{\partial R}\right|}\,H(R)\,\mathrm{Re}\left(e^{if(R)}\right)\,e^{-\left|\frac{\partial f}{\partial R}\cdot z\right|}}$$

Potential of an infinite slab with a tightly wound spiral pattern

$m = 2$



$$\boxed{\Sigma(R,\phi) = R_e\left(\underbrace{H(R)}_{\text{slow variation}} \; \underbrace{e^{i(m\theta + g(R))}}_{\text{rapid variation}} \right)}$$

Note $\boxed{m\theta + g(R) = \text{cte}}$

describe a spiral $g(R)$ = shape function

Idea :   WKB approximation

far from the center, Σ is nearly $\sim e^{i(kx)}$

Indeed   Developping $g(R)$ around R_0 gives

$$g(R) \cong g(R_0) + \left.\frac{\partial g}{\partial R}\right|_{R_0} (R - R_0)$$

For $\theta = 0$

$$\Sigma(R,\theta) = \underbrace{\mathcal{M}(R_0)\, e^{i f(R_0)}}_{\text{no radial dependency}} \; \underbrace{e^{i \frac{\partial f}{\partial R}\big|_{R_0} (R-R_0)}}_{e^{ikx}}$$

$$\left\{ \begin{array}{l} k = \frac{\partial f}{\partial R}\big|_{R_0} \\ x = R - R_0 \end{array} \right.$$

We directely have the solution from the infinite slab

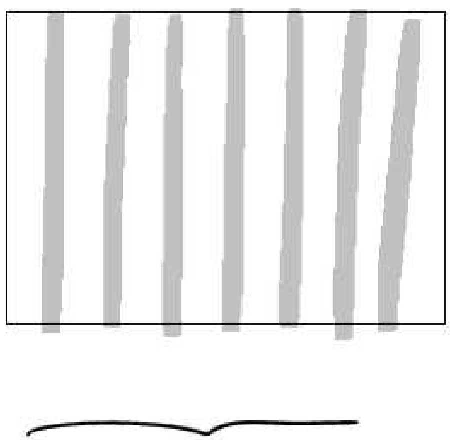
$$\phi(R,\theta) = - \frac{2\pi G}{\left|\frac{\partial f}{\partial R}\right|} \mathcal{M}(R_0)\, e^{i f(R_0)}\, e^{i \frac{\partial f}{\partial R}\big|_{R_0}(R-R_0)}\, e^{-\left|\frac{\partial f}{\partial R} z\right|}$$

Choosing $R_0 = R$

$$\boxed{\phi(R,\theta) = - \frac{2\pi G}{\left|\frac{\partial f}{\partial R}\right|} \mathcal{M}(R)\, e^{i f(R)}\, e^{-\left|\frac{\partial f}{\partial R} z\right|}}$$

Validity of the approximation

- we want a large number of "oscillations" over a small radius compared to R



$\sim R$

$$\boxed{\left|\frac{\partial \delta}{\partial R}\right| \cdot R \gg 1}$$

Stellar orbits

1st part

Generalities

Stellar orbits

Why studying stellar orbits ?

- understand the motion of stars in stellar systems and galaxies

  → understand the observed kinematics
  → constraints the mass model

We will assume :

- a smoothed gravitational field
- time independent potentials

Stellar orbits

Definitions

- trajectory — solution of the equation of motion

$$\ddot{\vec{x}} = -\vec{\nabla}\Phi(\vec{x})$$

defined on a finite interval:

$$\vec{x}(t), \vec{x}(t_0) = \vec{x_0}, t \in [t_0, t_1]$$

- orbit — a trajectory defined on an infinite time interval

$$\vec{x}(t), \vec{x}(t_0) = \vec{x_0}, t \in [-\infty, \infty[$$

- periodic orbit — a closed orbit

$$\forall\, t, \exists\, T, \vec{x}(t+T) = \vec{x}(t),\ \ \vec{v}(t+T) = \vec{v}(t)$$

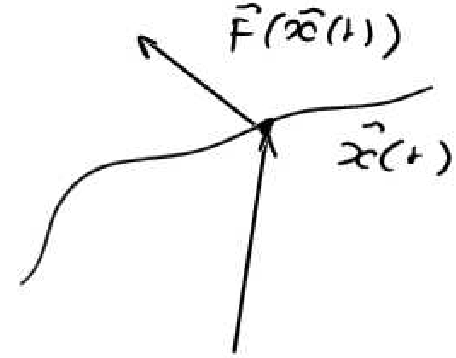
- stationary point — a point such that:

$$\ddot{\vec{x}} = \dot{\vec{x}} = 0$$

Lagrangian and Hamiltonian mechanics

Lagrangian Mechanics

Assume a mass point moving in a force field $\vec{F}(\vec{x})$

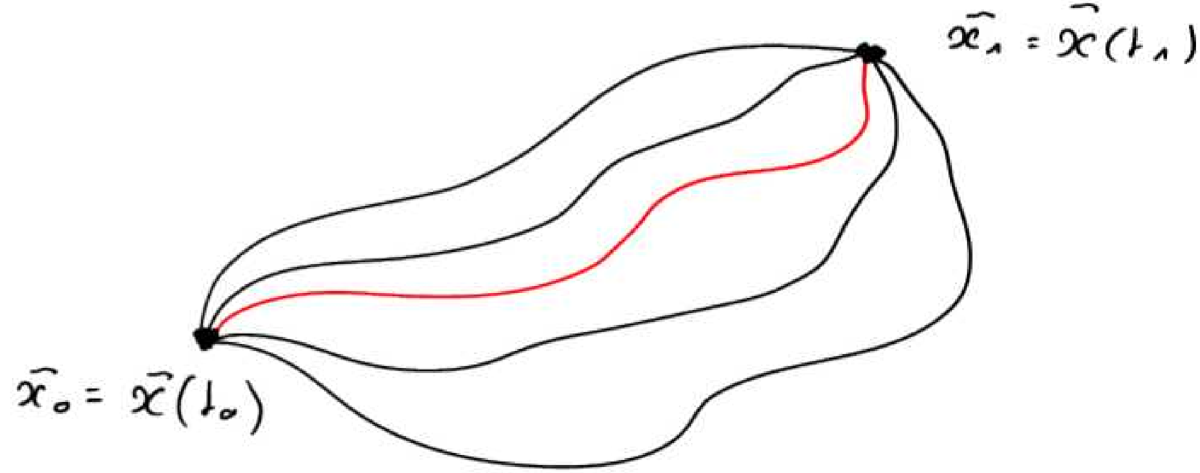


Definition Lagrangian, a scalar function of $\vec{x}$, $\dot{\vec{x}}$, t

$$\boxed{\mathcal{L}(\vec{x}, \dot{\vec{x}}, t) = K - V = \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}, t)}$$

Principle of least action or Hamiltonian principle

The motion of the particle from $\vec{x}_0$ to $\vec{x}_1$ is along a curve $\vec{x}(t)$ such that $\vec{x}(t_0) = \vec{x}_0$, $\vec{x}(t_1) = \vec{x}_1$ that is an extremal of the action I.



$$\boxed{I = \int_{t_0}^{t_1} \mathcal{L}(\vec{x}, \dot{\vec{x}}, t)\, dt = \int_{t_0}^{t_1} K(t) - V(t)\, dt}$$

Euler - Lagrange equation

The trajectory is an extremal of I if and only if

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{x}} = 0}$$

With cathesian coordinates, we get:

$$\boxed{m\ddot{\vec{x}} = -\vec{\nabla} V(\vec{x})}$$

which is nothing else than the second Newton law.

However: $\mathcal{L}$ can be a fuction of arbitrary coordinates $(\vec{q}, \dot{\vec{q}})$ "generalized" coordinates $\mathcal{L}(\vec{q}, \dot{\vec{q}})$.

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = 0}$$

Lagrange's equations

We can easily write equations of motions in any coord. system.

Hamiltonian mechanics

Note: Lagrangian mechanics generate 2^{nd} order differential equations

$$m\ddot{\vec{x}} = -\vec{\nabla}V(\vec{x})$$

> It is always possible to split a 2^{nd} order differential equation into two first order differential equations.

This is what is done in Hamiltonian mechanics

Definition

(1) For $\vec{q}, \dot{\vec{q}}$, a set of generalized coordinates, the generalized momentum are:

$$\boxed{\vec{p} := \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}}$$

Note: inverting $\vec{p} = \vec{p}(\vec{q}, \dot{\vec{q}})$, it is possible to write $\dot{\vec{q}} = \dot{\vec{q}}(\vec{p}, \vec{q})$

(2) Hamiltonian The scalar function

$$\boxed{H(\vec{q}, \vec{p}, t) := \vec{p} \cdot \dot{\vec{q}} - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)}$$

Note: $\dot{\vec{q}}$ is replaced by $\vec{q}, \vec{p}$ through the definition of $\vec{p}$

Hamilton equations

Compute the total derivative of $H(\vec{q}, \vec{p}, t) = \vec{p} \cdot \dot{\vec{q}} - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$

① right hand side (diff. with respect of $\vec{q}, \vec{p}$)

$$\frac{\partial H}{\partial \vec{q}} d\vec{q} + \frac{\partial H}{\partial \vec{p}} d\vec{p} + \frac{\partial H}{\partial t} dt$$

② left hand side (diff with respect of $\vec{p}\ \vec{q}$) with $\dot{\vec{q}} = \dot{\vec{q}}(\vec{p})$

$$\vec{p} \cdot d\dot{\vec{q}} + \dot{\vec{q}}\, d\vec{p} - \frac{\partial \mathcal{L}}{\partial \vec{q}} d\vec{q} - \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} d\dot{\vec{q}} - \frac{\partial \mathcal{L}}{\partial t} dt$$

$$= -\frac{\partial \mathcal{L}}{\partial \vec{q}} d\vec{q} + \dot{\vec{q}}\, d\vec{p} + \cancel{\vec{p}\, \frac{d\dot{\vec{q}}}{d\vec{p}} d\vec{p}} - \cancel{\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}}_{\vec{p}} \frac{\partial \dot{\vec{q}}}{\partial \vec{p}} d\vec{p}} - \frac{\partial \mathcal{L}}{\partial t} dt$$

$$= -\frac{\partial \mathcal{L}}{\partial \vec{q}} d\vec{q} + \dot{\vec{q}}\, d\vec{p} - \frac{\partial \mathcal{L}}{\partial t} dt$$

Equating ① and ②

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}} \qquad -\frac{\partial \mathcal{L}}{\partial \vec{q}} = \frac{\partial H}{\partial \vec{p}} \qquad \frac{\partial \mathcal{L}}{\partial t} = \frac{\partial H}{\partial t}$$

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}} \qquad -\frac{\partial \mathcal{L}}{\partial \vec{q}} = \frac{\partial H}{\partial \vec{p}} \;\circledast \qquad \frac{\partial \mathcal{L}}{\partial t} = \frac{\partial H}{\partial t}$$

Using Euler-Lagrange

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}}_{\vec{p}\ \text{def}\ \vec{p}}\Big) - \underbrace{\frac{\partial \mathcal{L}}{\partial \vec{q}}}_{\frac{\partial H}{\partial \vec{q}}\ \circledast} = 0 \implies \frac{d}{dt}\vec{p} = -\frac{\partial H}{\partial \vec{q}}$$

In conclusion, we have transformed a set of 2^{nd} order differential equations into 2x more 1^{st} order differential equations:

$$\boxed{\begin{aligned} \dot{\vec{q}} &= \frac{\partial H}{\partial \vec{p}} \\ \dot{\vec{p}} &= -\frac{\partial H}{\partial \vec{q}} \end{aligned}} \qquad \frac{\partial \mathcal{L}}{\partial t} = \frac{\partial H}{\partial t}$$

Hamilton's equations

Hamiltonian conservation

Lets compute the time derivative of $H(\vec{q},\vec{p},t)$

$$\frac{d}{dt} H(\vec{q},\vec{p},t) = \frac{\partial H}{\partial \vec{q}} \frac{d\vec{q}}{dt} + \frac{\partial H}{\partial \vec{p}} \frac{d\vec{p}}{dt} + \frac{\partial H}{\partial t}$$

$$- \dot{\vec{p}} \cdot \dot{\vec{q}} + \dot{\vec{q}}\,\dot{\vec{p}} = 0$$

If $\mathcal{L}$ is time independant, i.e. $\mathcal{L} = \mathcal{L}(\vec{q},\dot{\vec{q}})$

($\equiv V(\vec{q})$ is time independant)

$\Longrightarrow$

By construction, $H(\vec{q},\vec{p})$ is conserved along a trajectory

Definitions

for a system with n-dimensions

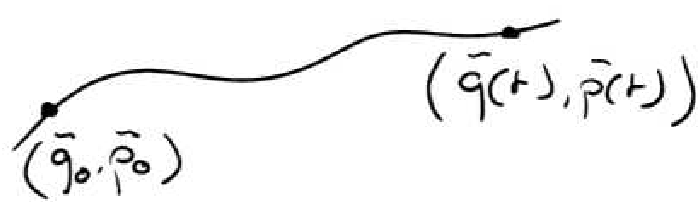
Configuration space $(q_1 \dots q_n)$ n-dimensions

Momentum space $(p_1 \dots p_n)$ n-dimensions

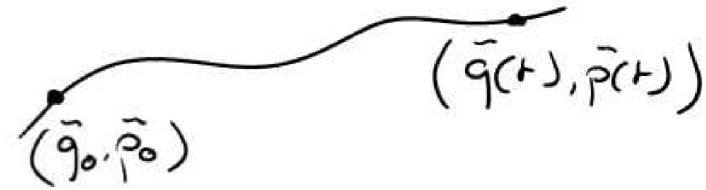
Phase space $(q_1 \dots q_n, p_1 \dots p_n)$ 2n-dimensions

$$\equiv (w_1 \dots w_{2n})$$

Note As Hamilton's equations are 1^{st} order differential equations, a trajectory is uniquely defined by a point in the phase space



Time evolution operator



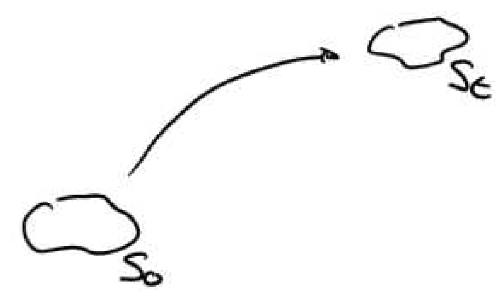
It is the possible to define a time evolution operator $\mathcal{H}_t$ that will bring $(\vec{q}_0, \vec{p}_0)$ to $(\vec{q}(t), \vec{p}(t))$

$$(\vec{q}(t), \vec{p}(t)) = \mathcal{H}_t(\vec{q}_0, \vec{p}_0)$$

$\mathcal{H}_t$ will map any 2D surface S_0 in the phase space to an other 2D surface S_t in the phase space.

Poincaré invariant theorem

$$\boxed{\iint_{S_0} d\vec{q} \cdot d\vec{p} = \iint_{S_t} d\vec{q}\, d\vec{p}}$$



Poisson brackets

two operaters A, B

$$[A, B] := \frac{\partial A}{\partial \vec{q}} \frac{\partial B}{\partial \vec{p}} - \frac{\partial A}{\partial \vec{p}} \frac{\partial B}{\partial \vec{q}} = \sum_{i}^{n} \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i}$$

Hamilton's equations

$$\boxed{\dot{w}_\alpha = [w_\alpha, H]} = \frac{\partial w_\alpha}{\partial \vec{q}} \frac{\partial H}{\partial \vec{p}} - \frac{\partial w_\alpha}{\partial \vec{p}} \frac{\partial H}{\partial \vec{q}}$$

$$\equiv \left\{ \begin{array}{l} \dot{q}_\alpha = \dfrac{\partial H}{\partial p_\alpha} \\[2em] \dot{p}_\alpha = - \dfrac{\partial H}{\partial q_\alpha} \end{array} \right.$$

Orbits in Spherical Systems

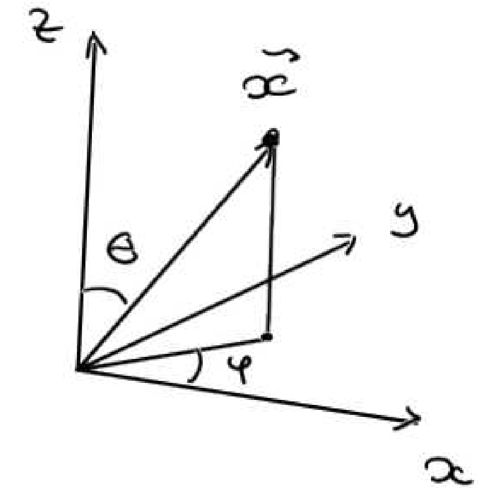
Orbits in spherical potentials $\quad \phi(\vec{x}) = \phi(r)$

Spherical coordinates

$$\begin{cases} x = r \cos\varphi \sin\theta \\ y = r \sin\varphi \sin\theta \\ z = r \cos\theta \end{cases}$$

$$\vec{x} = r\,\vec{e}_r = \vec{r}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$



Equation of motion $\quad$ (Newton law)

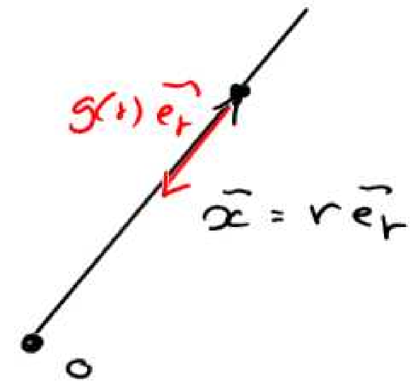
$$\boxed{\frac{d^2}{dt^2}(\vec{x}) = \vec{g}(\vec{x}) \equiv g(r)\,\vec{e}_r}$$

$$g(\vec{x}) = -\vec{\nabla}\phi(\vec{x}) = -\frac{\partial}{\partial r}\phi(r)\,\vec{e}_r - \frac{1}{r}\cancel{\frac{\partial}{\partial \theta}\phi(r)}\,\vec{e}_\theta - \frac{1}{r\sin\theta}\cancel{\frac{\partial}{\partial \varphi}\phi(r)}\,\vec{e}_\varphi$$

$$= g(r)\,\vec{e}_r$$

with $\quad g(r) = -\frac{\partial}{\partial r}\phi(r)$

Angular momentum conservation



$$\vec{L} = \vec{x} \times \frac{d\vec{x}}{dt} \qquad \text{(specific angular momentum)}$$

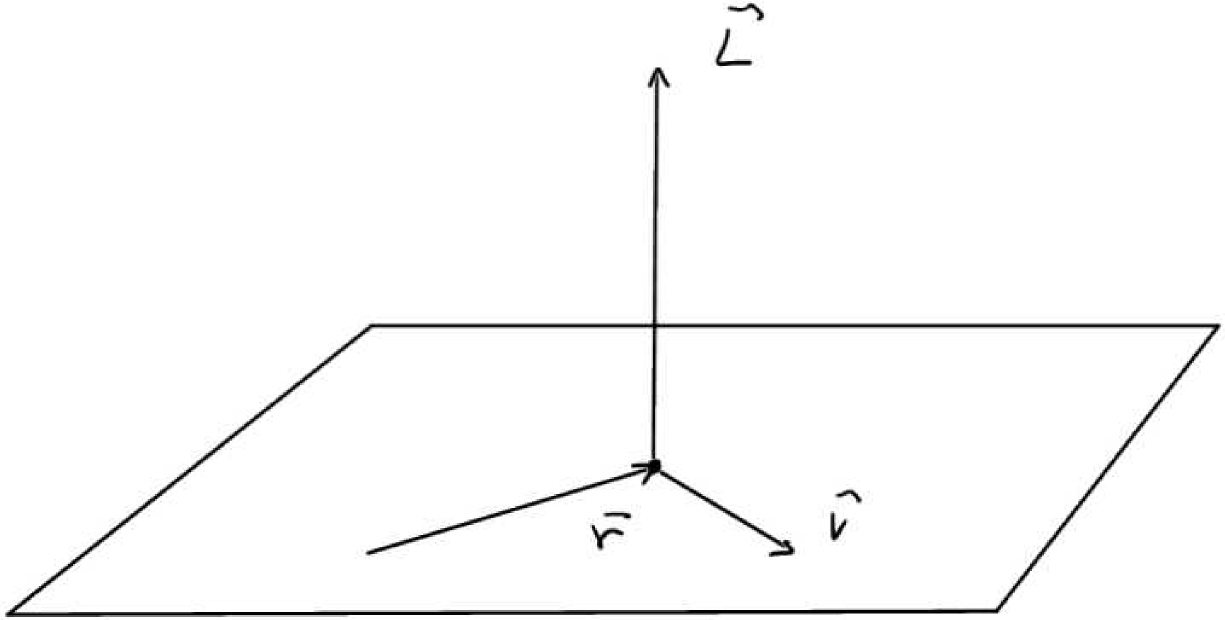
$$\frac{d}{dt}(\vec{L}) = \frac{d\vec{x}}{dt} \times \frac{d\vec{x}}{dt} + \vec{x} \times \frac{d^2\vec{x}}{dt^2}$$

$$= 0 + \underbrace{r\vec{e_r} \times g(r)\,\vec{e_r}}_{=0 \;\; (= \vec{N}\text{, the tork})} = 0$$

In a spherical system, the angular momentum of a particle is conserved ! $\vec{L} = \text{cte}$

(A spherical potential induces no tork $\vec{N} = \vec{x} \times \vec{F} = 0$)

Corollary As $\vec{L}$ is conserved the orbit of a particle is limited to a plane (the orbital plane)

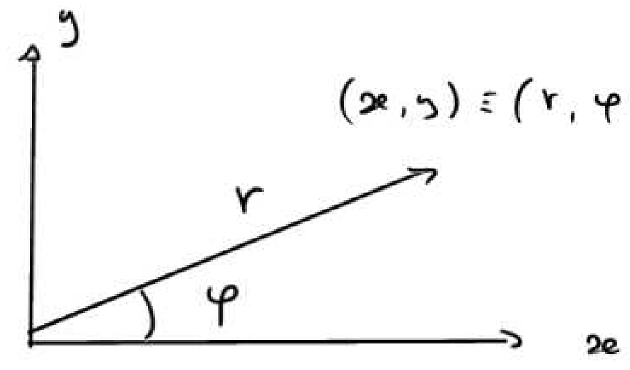


2D problem

Equations of motion in the orbital plane

Polar coordinates (in the orbital plane)

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \begin{cases} \dot{x} = \dot{r}\cos\varphi - r\sin\varphi\,\dot{\varphi} \\ \dot{y} = \dot{r}\sin\varphi + r\cos\varphi\,\dot{\varphi} \end{cases}$$



Lagrangian (specific) in polar coordinates

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \phi(\sqrt{x^2+y^2}) = \frac{1}{2}(\dot{r}^2 + (r\dot{\varphi})^2) - \phi(r)$$

Lagrange equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\vec{q}}}\right) - \frac{\partial\mathcal{L}}{\partial\vec{q}} = 0}$$

$$\begin{cases} \ddot{r} - r\dot{\varphi}^2 + \dfrac{\partial\phi}{\partial r} = 0 \\ \dfrac{d}{dt}(r^2\dot{\varphi}) = 0 \end{cases}$$

Angular momentum

$$\boxed{r^2 \dot{\varphi} = |\vec{L}| =: L}$$

Indeed

in spherical coordinates

$$\vec{x} = r \vec{e}_r$$

$$\vec{v} = \dot{r} \vec{e}_r + r \dot{\varphi} \vec{e}_\varphi$$

$$\vec{L} = \vec{x} \times \vec{v} = r \vec{e}_r \times \left(\dot{r} \vec{e}_r + r \dot{\varphi} \vec{e}_\varphi \right)$$

$$= r^2 \dot{\varphi} \vec{e}_z$$

Hamiltonian / Energy

$$\boxed{H(\vec{q},\vec{p},t) := \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}(\vec{q},\dot{\vec{q}},t)}$$

$$\vec{q} = \begin{cases} r \\ \varphi \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{r} \\ \dot{\varphi} \end{cases} \qquad \vec{p} = \begin{cases} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r} = p_r \\ \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = r^2\dot{\varphi} = p_\varphi \end{cases}$$

$$H(r,\varphi,\dot{r},r^2\dot{\varphi}) = \dot{r}^2 + r^2\dot{\varphi}^2 - \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r)$$

$$= \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r) = E$$

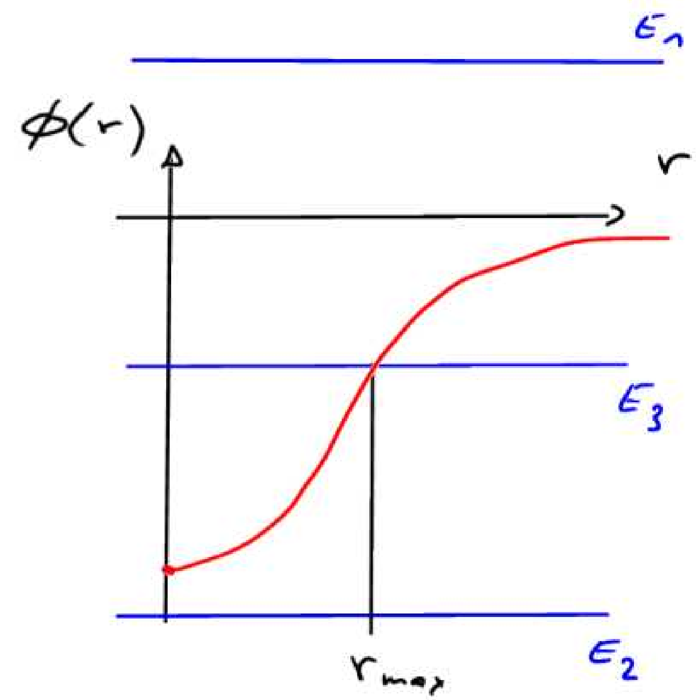
or

$$H(r,\varphi,p_r,p_\varphi) = \frac{1}{2}p_r^2 + \frac{1}{2}\frac{p_\varphi^2}{r^2} + \phi(r) = E$$

E (Energy) is conserved
as $\mathcal{L}$ is time independant

Radial orbits $\quad \dot{\varphi} = 0 \quad \Rightarrow \quad L = 0$

$$\begin{cases} \text{Equation of motion} : & \ddot{r} = -\dfrac{\partial \phi}{\partial r} \\ \text{Energy} : & E = \frac{1}{2}\dot{r}^2 + \phi(r) \end{cases}$$



3 cases

① $E > \phi(\infty) \quad \Rightarrow \quad \forall t,\ \dot{r}^2 > 0$

orbit not bounded

② $E < \phi(0) \quad \Rightarrow \quad$ impossible

③ $\phi(0) < E < \phi(\infty)$

$\exists\, r$ tq $\dot{r} = 0$ i.e $E = \phi(r)$

$r = r_{max}$

Non radial orbits $\quad r \neq 0 \quad \dot\varphi \neq 0 \quad L \neq 0$

$$\text{EOM}\quad \begin{cases} \ddot r - r\dot\varphi^2 + \dfrac{\partial \phi}{\partial r} = 0 & (1) \\ \dfrac{d}{dt}\left(r^2\dot\varphi\right) = 0 \end{cases}$$

replace t by φ

$$\frac{d}{dt} = \frac{d}{d\varphi}\dot\varphi = \frac{L}{r^2}\frac{d}{d\varphi}$$

(1) becomes

$$\frac{L^2}{r^2}\frac{d}{d\varphi}\left(\frac{1}{r^2}\frac{dr}{d\varphi}\right) - \frac{L^2}{r^3} = -\frac{\partial \phi}{\partial r}$$

use $u = \frac{1}{r}$

$$\boxed{\frac{d^2u}{d\varphi^2} + u = \frac{1}{L^2u^2}\frac{\partial \phi}{\partial r}\left(\frac{1}{u}\right)}$$

No analytical general solution

Radial energy equation

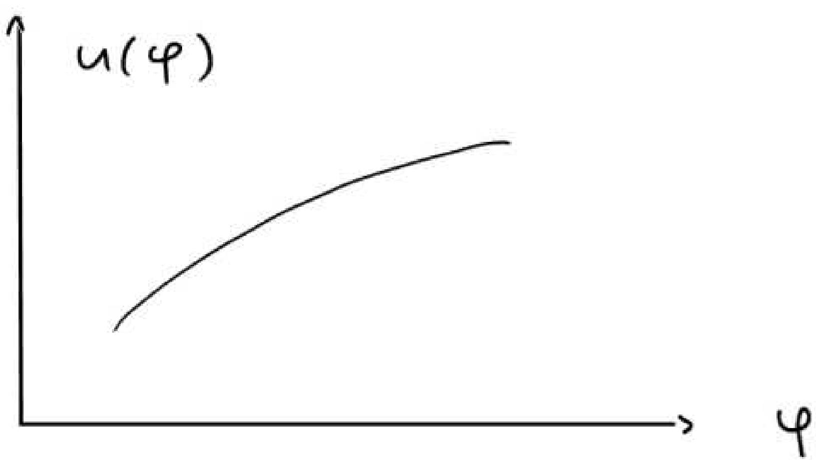
From the energy $\boxed{E = \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r)}$

1) multiply by $\frac{2}{L^2}$

2) use $u = \frac{1}{r}$ and $\frac{d}{dt} = \frac{L}{r^2}\frac{d}{d\varphi}$

we get

$$\boxed{\left(\frac{du}{d\varphi}\right)^2 + u^2 + \frac{2\phi(\frac{1}{u})}{L^2} = \frac{2E}{L^2}}$$



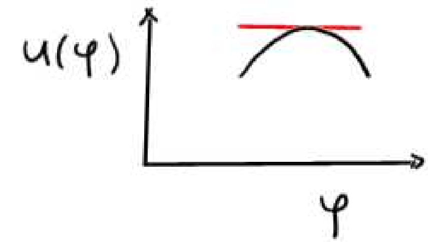
Orbit properties

Minimal radius

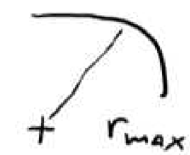
As $L \neq 0$, the orbit cannot cross the center

there must be a minimal radius

$$\forall \ \varphi \ \text{ such that } \ \frac{du}{d\varphi} = 0$$



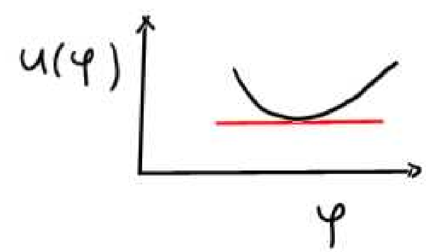
Maximal radius



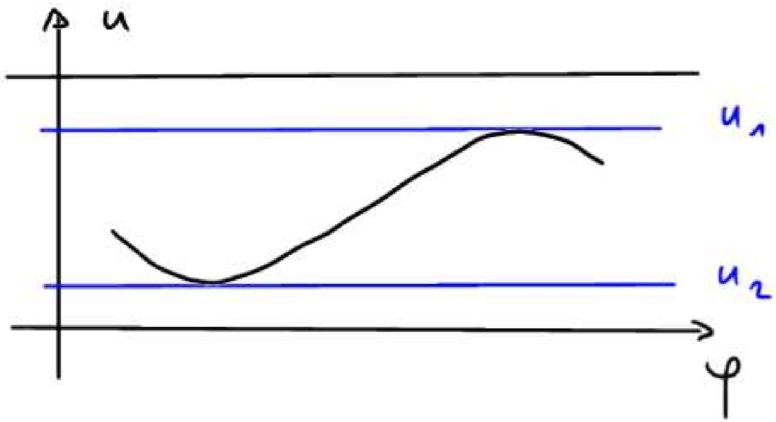
If the orbit is bounded there must be a

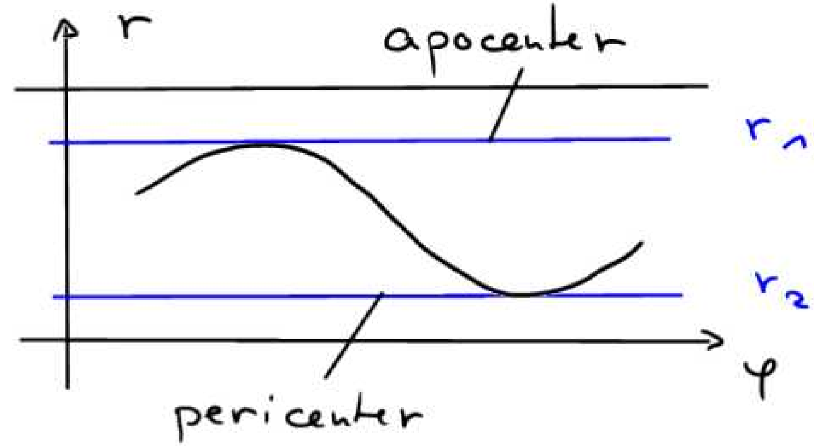
maximal radius

$$\forall \ \varphi \ \text{ such that } \ \frac{du}{d\varphi} = 0$$



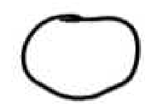
$$\boxed{\text{For } \ \frac{du}{d\varphi} = 0 \qquad u^2 = \frac{2\left[E - \phi(1/u)\right]}{L^2}}$$



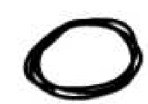


Notes

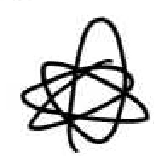
- if $u_1 = u_2$: periodic orbit



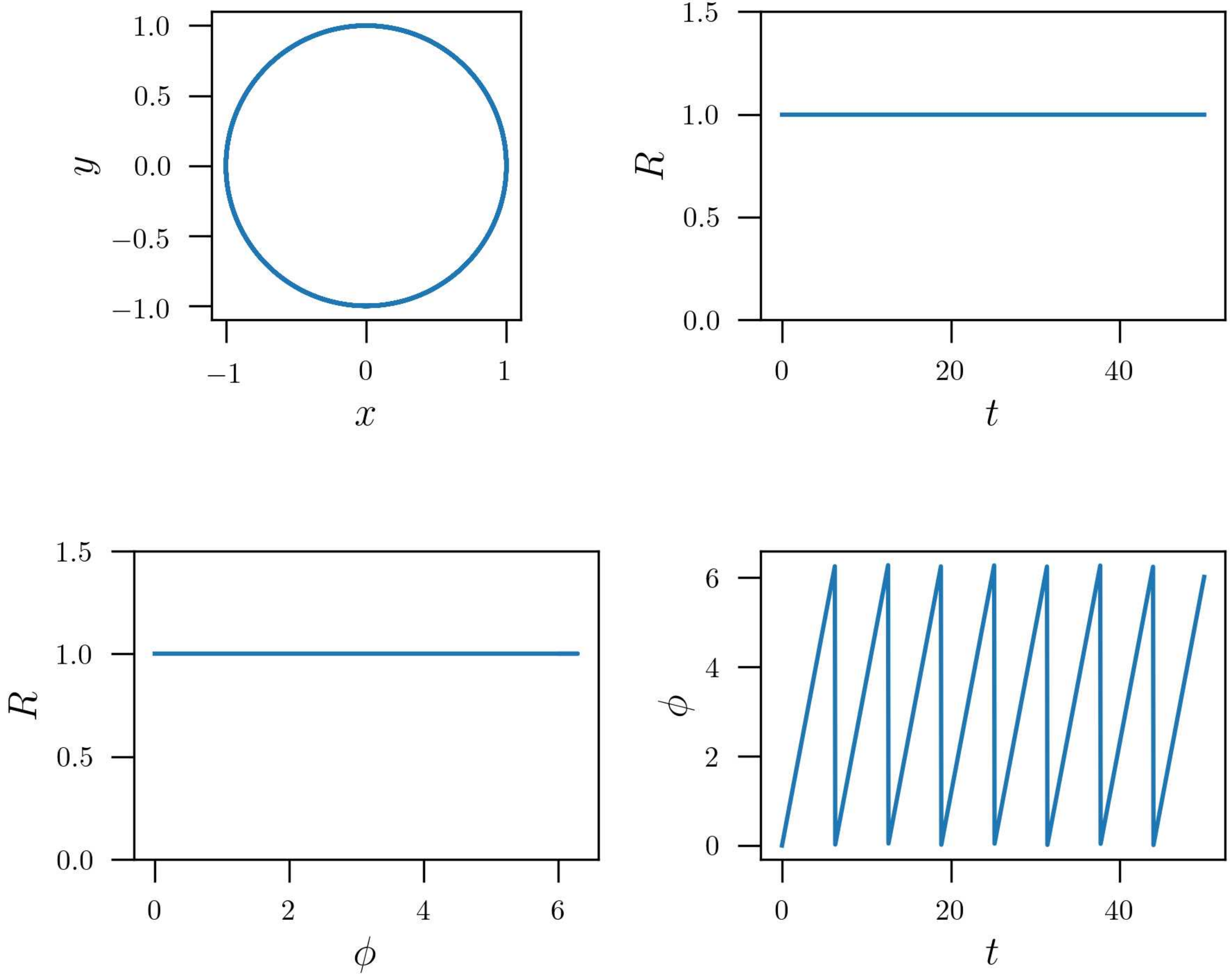
- if $u_1 \cong u_2$: orbit with a small eccentricity



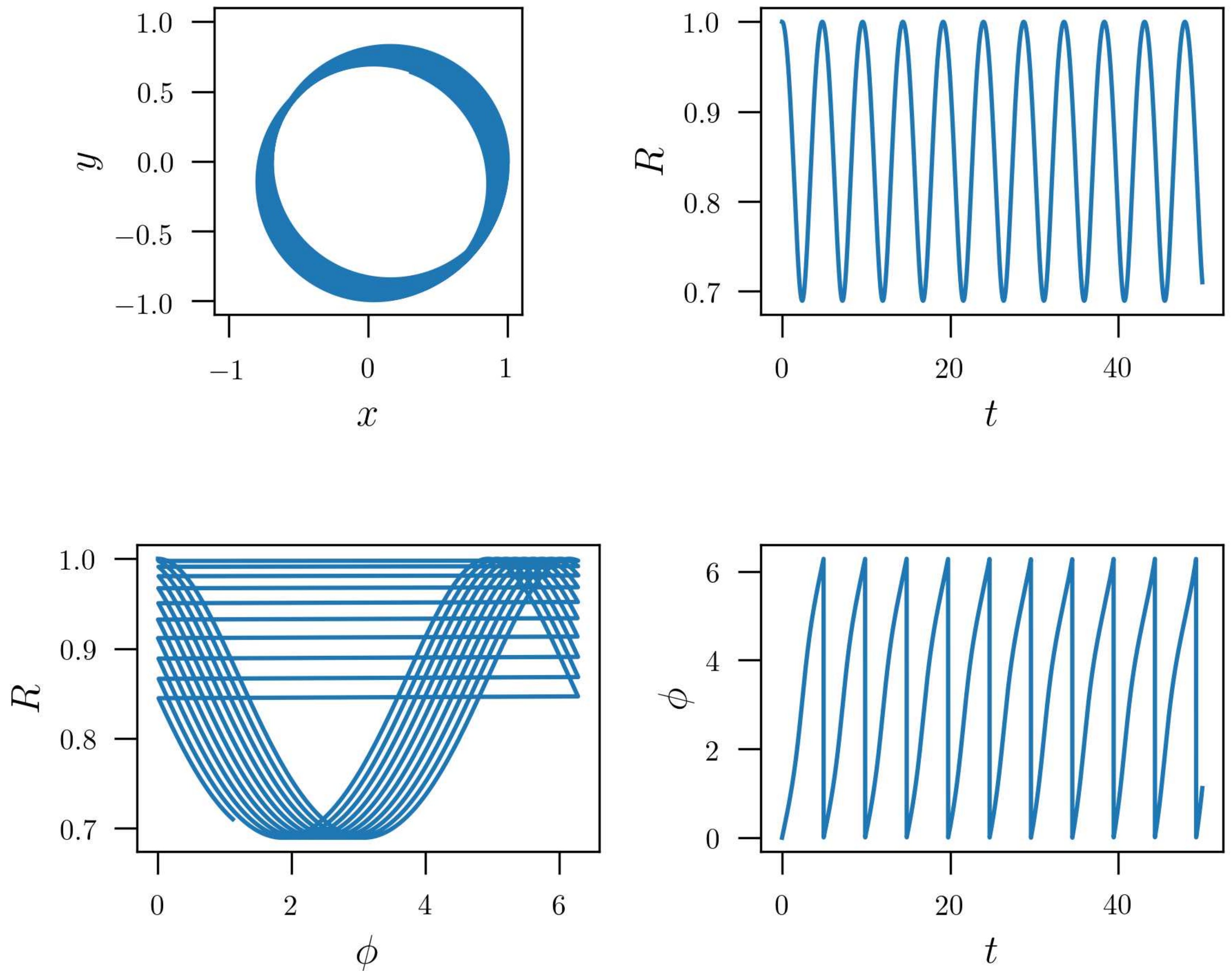
- if $u_1 \gg u_2$: orbit eccentricity is nearly 1



Plummer

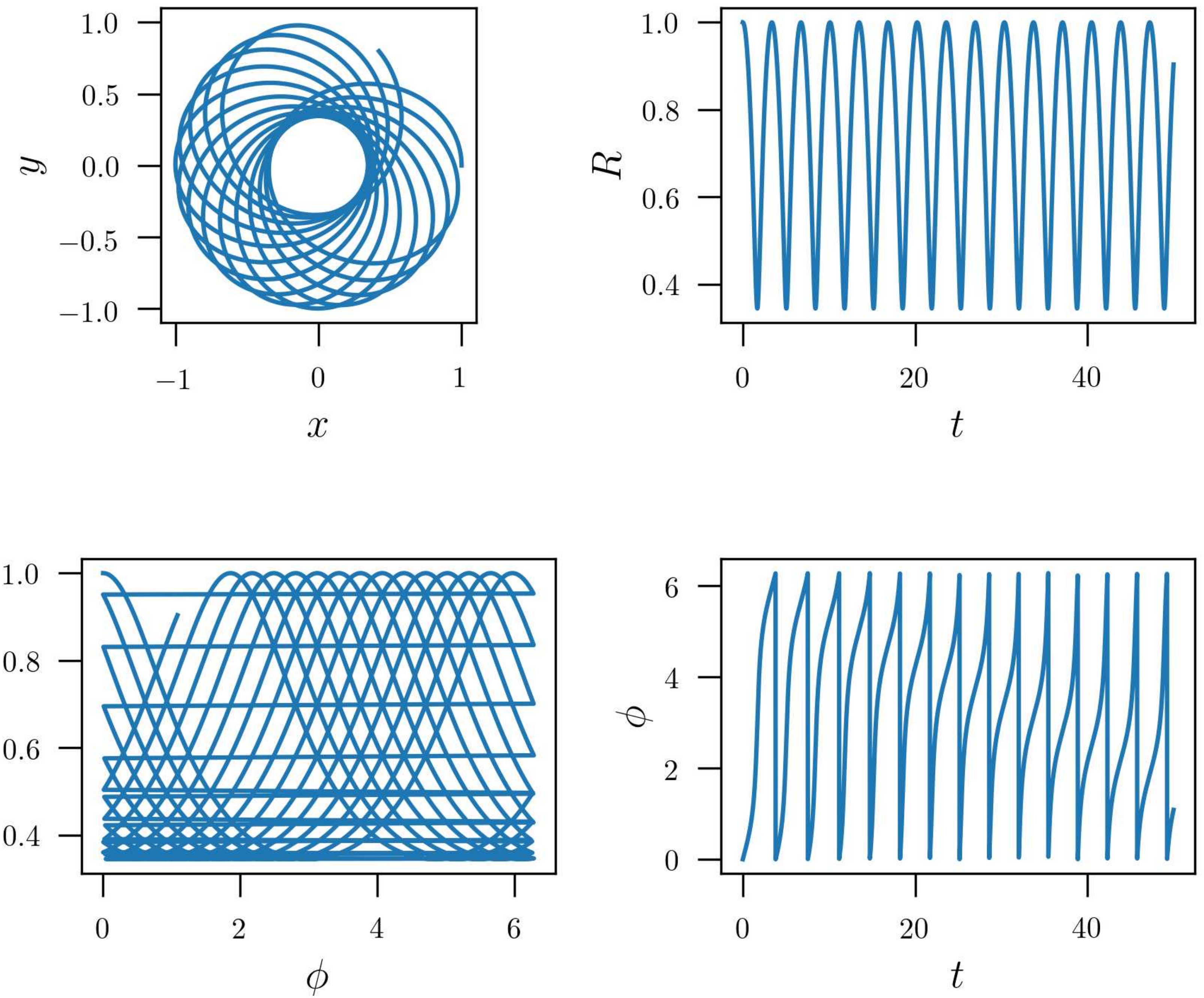


Plummer



1.0
0.5
0.0
−0.5
−1.0
y
−1
0
1
x
1.0
0.9
0.8
0.7
R
0
20
40
t
1.0
0.9
0.8
0.7
R
0
2
4
6
ϕ
6
4
2
0
ϕ
0
20
40
t

Plummer



1.0
0.5
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−0.5
−1.0
y
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x
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0.8
0.6
0.4
R
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20
40
t
1.0
0.8
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ϕ
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20
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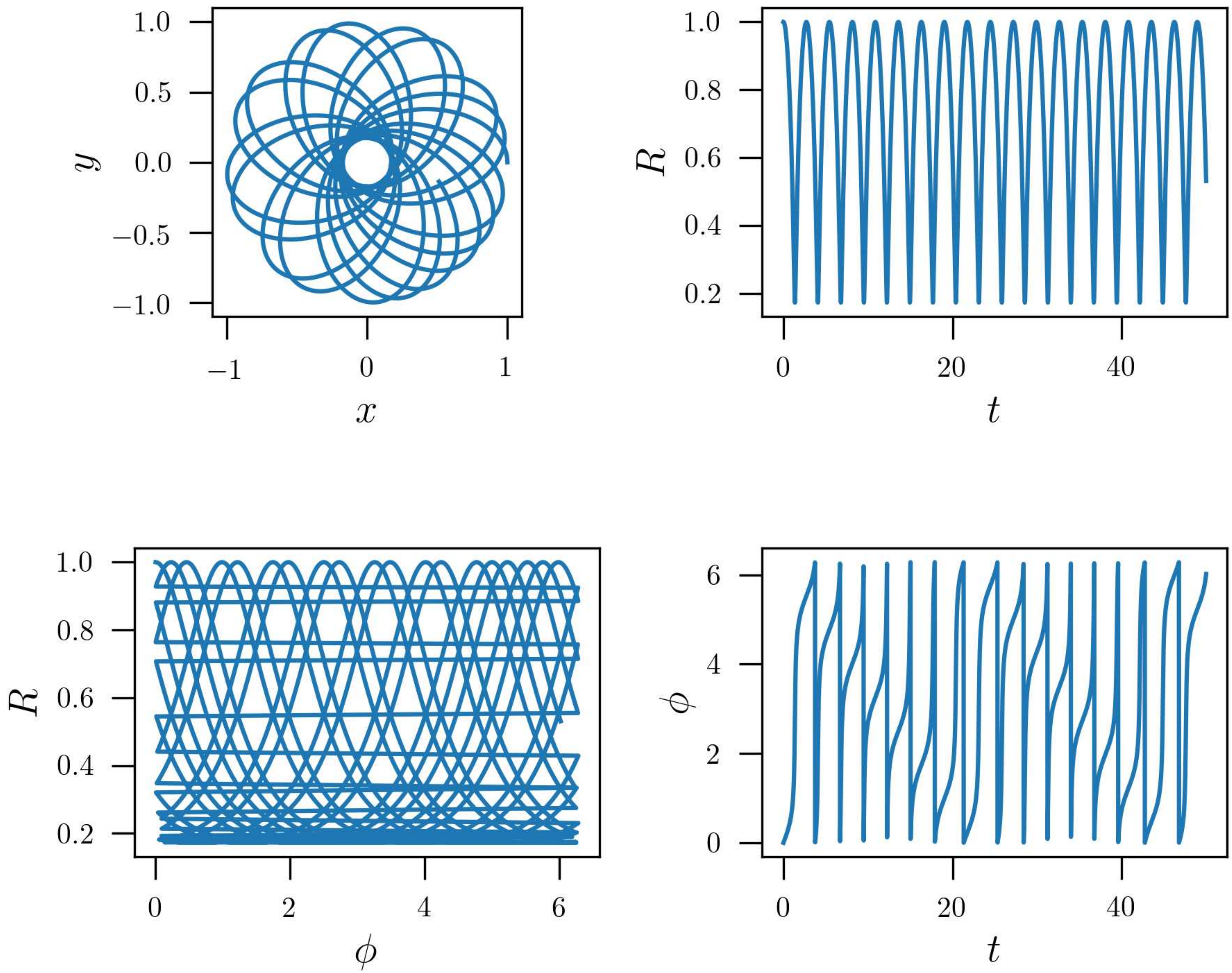
t

Plummer



Radial period

Time to travel from the apocenter to the pericenter

$$T_r = 2\int_{t_1}^{t_2} dt = 2\int_{r_1}^{r_2} \frac{dt}{dr}\, dr \qquad \begin{cases} r(t_1) = r_1 \\ r(t_2) = r_2 \end{cases}$$

From

$$E = \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r) = \frac{1}{2}\dot{r}^2 + \frac{L}{2r^2} + \phi(r)$$

$$\dot{r}^2 = 2(E - \phi(r)) - \frac{L^2}{r^2}$$

$$\frac{dr}{dt} = \sqrt{2(E-\phi(r)) - \frac{L^2}{r^2}}$$

$$\frac{dt}{dr} = \frac{1}{\sqrt{2(E-\phi(r)) - \frac{L^2}{r^2}}}$$

$$\boxed{T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E-\phi(r)) - \frac{L^2}{r^2}}}}$$

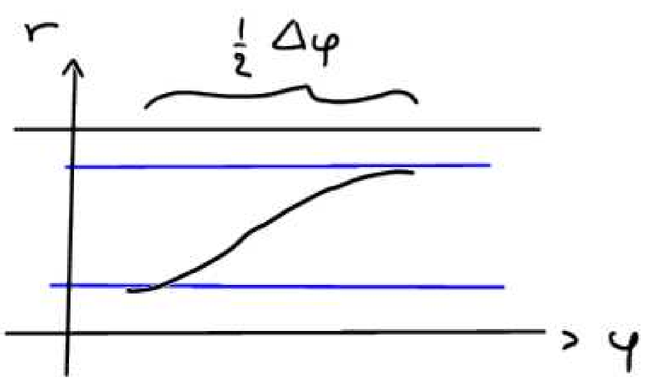
Azimuthal period

$$T_\varphi = \int_{t_1'}^{t_2'} dt \qquad \begin{cases} \varphi(t_1') = 0 \\ \varphi(t_2') = 2\pi \end{cases}$$

$\Delta\varphi$: increase of the azimuthal angle during T_r

$$\Delta\varphi = 2\int_{\varphi_1}^{\varphi_2} d\varphi = 2\int_{t_1}^{t_2} \frac{d\varphi}{dt}\, dt = 2\int_{r_1}^{r_2} \frac{d\varphi}{dt}\frac{dt}{dr}\, dr$$

$$= 2\int_{r_1}^{r_2} \frac{L}{r^2} \frac{1}{\sqrt{2(E-\Phi(r)) - \frac{L^2}{r^2}}}\, dr$$



Azimuthal period: time to increase φ by 2π

$$T_\varphi = \frac{2\pi}{\langle \dot{\varphi} \rangle} \qquad \text{mean azimuthal "velocity"} \qquad \langle \dot{\varphi} \rangle = \frac{|\Delta\varphi|}{T_r}$$

$$\boxed{T_\varphi = \frac{2\pi}{|\Delta\varphi|}\, T_r}$$

As in general $\dfrac{2\pi}{|\Delta\varphi|}$ is not a rational number

the orbit is not guarantee to be closed

The End