Astrophysics III, Dr. Yves Revaz — EPFL

4th year physics — Exercises week 5

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Astrophysics III: Stellar and galactic dynamics

Solutions

Preface: Gauss's law Many of the exercises of this series are much easier to deal with when using an integrated version of our usual Poisson's equation: Gauss's law for gravity. Starting from Poisson's equation, we integrate on an arbitrary volume:

$$4\pi G\rho = \nabla^2\Phi$$

$$4\pi G \int_V \mathrm{d}^3\mathbf{x}\,\rho = \int_V \mathrm{d}^3\mathbf{x}\,\nabla^2\Phi$$

The integral of the left hand side is the mass enclosed in the volume V, M. The right hand side can be rewritten as a surface integral using the divergence theorem:

$$4\pi\, GM = \int_{\partial V} \mathrm{d}\vec{S}\cdot\nabla\Phi,$$

where ∂V is the surface encapsulating V.

Problem 1:

Using the following relations for spherical systems, derived during the lectures : the Poisson equation in Spherical coordinates :

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\,\pi\,G\,\rho(r) \tag{1}$$

the mass inside a radius r due to a spherical distribution of matter $\rho(r')$:

$$M(r) = 4\pi\int_0^r \mathrm{d}r'\,r'^2\,\rho(r'), \tag{2}$$

the gravitational field due to a spherical distribution of matter $\rho(r')$

$$\vec{g}(r) = -\frac{G\,M(r)}{r^2}\cdot\vec{e}_r, \tag{3}$$

the potential due to a spherical distribution of matter $\rho(r')$

$$\Phi(r) = -\frac{G\,M(r)}{r} - 4\pi G\int_r^\infty \rho(r')r'\mathrm{d}r', \tag{4}$$

the gradient of the potential due to a spherical distribution of matter $\rho(r')$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G\,M(r)}{r^2}, \tag{5}$$

we can express $\rho(r)$, $\Phi(r)$, $M(r)$ and $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ as a function of respectively $\rho(r)$, $\Phi(r)$, $M(r)$ and $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$:

$\rho(r)$

- as a function of $\rho(r)$: -
- as a function of $\Phi(r)$: use the Poisson equation Eq. (1)
- as a function of $M(r)$: use Eq. (2)
- as a function of $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$: compute the first derivative of $M(r)$ from Eq. (2)

$\Phi(r)$

- as a function of $\rho(r)$: use Eq. (4)
- as a function of $\Phi(r)$: -
- as a function of $M(r)$: integrate Eq. (5)
- as a function of $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$: integrate $\Phi(r)$

$M(r)$

- as a function of $\rho(r)$: use Eq. (2)
- as a function of $\Phi(r)$: use Eq. (5)
- as a function of $M(r)$: -
- as a function of $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$: use Eq. (5)

$\frac{\mathrm{d}\Phi}{\mathrm{d}r}$

- as a function of $\rho(r)$: use Eq. (5) and express $M(r)$ with Eq. (2)
- as a function of $\Phi(r)$: compute the first derivative of $\Phi(r)$
- as a function of $M(r)$: use Eq. (5)
- as a function of $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$: -

Problem 2:

We set up our coordinates such that the slab lays on the $z = 0$ plane. As the mass distribution is discontinuous, we cannot easily rely on the Poisson equation to derive the corresponding potential. We instead use Gauss's law:

$$\int_S \vec{\nabla}\Phi \cdot \mathrm{d}\vec{S} = 4\pi G\, M_S, \tag{6}$$

where S is any surface and M_S is the mass enclosed by the surface S. Let us define S to be the surface of a cylinder perpendicular to the plane $z = 0$. By symmetry (the surface density of the plane is constant) :

$$\vec{\nabla}\Phi = \frac{\partial}{\partial z}\Phi(z) \cdot \vec{e}_z \qquad \text{and} \qquad \frac{\partial}{\partial z}\Phi(z) = -\frac{\partial}{\partial z}\Phi(-z). \tag{7}$$

Thus, in the integral (6) the surface perpendicular to the plane $z = 0$ does not contribute and we get :

$$\int_S \vec{\nabla}\Phi \cdot \mathrm{d}\vec{S} = 2\frac{\partial}{\partial z}\Phi(z)\,\Delta s. \tag{8}$$

where Δs is the surface of the cylinder parallel to the plane $z = 0$. The mass enclosed in the cylinder is :

$$M_S = \Delta s\,\Sigma_0 \tag{9}$$

and (8) with (9) and (6) give :

$$2\frac{\partial}{\partial z}\Phi(z)\,\Delta s = 4\pi G\,\Delta s\,\Sigma_0. \tag{10}$$

This leads to :

$$\frac{\partial}{\partial z}\Phi(z) = 2\pi G\,\Sigma_0, \tag{11}$$

and after integration :

$$\Phi(z) = 2\pi G\,\Sigma_0\,z + \text{const.} \tag{12}$$

Problem 3:

We consider a wire aligned with the x axis. As the mass distribution is discontinuous, we cannot rely on the Poisson equation to derive the corresponding potential. We instead rely on the Gauss Theorem :

$$\int_S \vec{\nabla}\Phi \cdot \mathrm{d}\vec{S} = 4\pi G\,M_S, \tag{13}$$

where S is any surface and M_S is the mass enclosed by the surface S. Let us define S to be the surface of a cylinder of length Δx and radius R, with its symmetry axis being the axis x, i.e., the wire. The surface Δs parallel to the x axis is:

$$\Delta s = 2\pi R\,\Delta x, \tag{14}$$

and the enclosed mass is :

$$M_S = \lambda_0\,\Delta x. \tag{15}$$

By symmetry (the linear density of the wire is constant) :

$$\vec{\nabla}\Phi = \frac{\partial}{\partial R}\Phi(R)\,\vec{e}_R, \tag{16}$$

where $\vec{e}_R$ is perpendicular to the axis x. With (14), (15) and (16), the Gauss theorem becomes :

$$\int_S \vec{\nabla}\Phi \cdot \mathrm{d}\vec{S} = 2\pi R\,\Delta x\,\frac{\partial}{\partial R}\Phi(R) = 4\pi G\,\lambda_0\,\Delta x, \tag{17}$$

which leads to :

$$\frac{\partial}{\partial R}\Phi(R) = 2\,G\,\frac{\lambda_0}{R}, \tag{18}$$

and after integrating over the radius R :

$$\Phi(R) = 2G\,\lambda_0\,\ln(R) + \text{const}, \tag{19}$$

Problem 4:

From Poisson's equation in spherical coordinates we get:

$$\nabla^2\Phi = 4\pi G\rho$$

$\nabla^2\Phi$ written in spherical coordinates, and considering a spherical potential we get:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)$$

a lot of straight-forward algebra follows, but finally we get

$$\rho = \frac{v_s^2}{4\pi G r_s^2}\frac{1}{(r/r_s)(1+r/r_s)^2}$$

The circular velocity also follows simply:

$$\begin{aligned}
v_c^2 &= r\,\frac{\partial\Phi}{\partial r} = r\,v_s^2\left[-\frac{1}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)} + \frac{\ln\left(1+\frac{r}{r_s}\right)}{r_s\left(\frac{r}{r_s}\right)^2}\right] = v_s^2\left[\frac{\ln\left(1+\frac{r}{r_s}\right)}{\frac{r}{r_s}} - \frac{1}{\left(1+\frac{r}{r_s}\right)}\right] \\
&= v_s^2\left[\frac{\left(1+\frac{r}{r_s}\right)\ln\left(1+\frac{r}{r_s}\right) - \frac{r}{r_s}}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)}\right] = v_s^2\,\frac{(r_s+r)\;r_s\,\ln\left(1+\frac{r}{r_s}\right) - r\,r_s}{r\;(r_s+r)}
\end{aligned}$$

Problem 5:

As per problem 4, the isochrone ρ is straightforward to derive, taking the form:

$$\rho = M\left[\frac{3(b+a)a^2 - r^2(b+3a)}{4\pi(b+a)^3a^3}\right] \quad \text{with} \quad a \equiv \sqrt{b^2+r^2}$$

The circular velocity is

$$v_c^2 = \frac{GMr^2}{(b+a)^2a}$$

Problem 6:

Let's define a unit surface on the disk, corresponding to a mass Σ, which is then the surface density. Defining a slab enclosing the unit surface and making its thickness tend to a vanishing value ($\varepsilon \to 0$, see Fig. 1), the surface integral reduces to twice the gradient of the potential:

$$4\pi\,G\,\Sigma = \int \mathrm{d}^2\mathbf{S}\,\nabla\Phi_{\mathrm{K}} = 2\frac{\partial\Phi_{\mathrm{K}}}{\partial z}$$

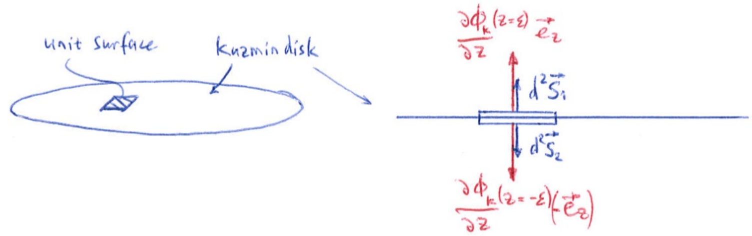

Figure 1: The Kuzmin disk with the unit surface (left) and seen edge-on (right), with the 2ε thick slab, on the surface of which the integration is made.

We have

$$\begin{aligned}\frac{\partial \Phi_{\rm K}}{\partial z} &= \frac{\partial}{\partial z}\left[-GM\left[R^2+(a+|z|)^2\right]^{-1/2}\right]\\ &= GM\left[R^2+(a+|z|)^2\right]^{-3/2}(a+|z|)\end{aligned}$$

With $|z| \to 0$, we then have:

$$\begin{aligned}4\pi\, G\Sigma_{\rm K} &= 2\frac{\partial \Phi_{\rm K}}{\partial z} = 2aGM\left[R^2+a^2\right]^{-3/2}\\ \Rightarrow \Sigma_{\rm K} &= \frac{aM}{2\pi\left(R^2+a^2\right)^{3/2}}\end{aligned}$$

<u>Problem 7</u>:

The velocity curve may be obtained from the formula (see course: result from a razor-thin homeoid since we cannot use Gauss law here):

$$v_{\rm c}^2(R) = -4G\int_0^R {\rm d}a\frac{a}{\sqrt{R^2-a^2}}\frac{\rm d}{{\rm d}a}\int_a^\infty {\rm d}R'\frac{R'\Sigma(R')}{\sqrt{R'^2-a^2}} \tag{20}$$

Replacing $\Sigma(R')$ using the Mestel's surface density we get:

$$\begin{aligned}\int_a^\infty {\rm d}R'\frac{R'\Sigma(R')}{\sqrt{R'^2-a^2}} &= \frac{v_0^2}{2\pi G}\int_a^\infty {\rm d}R'\frac{1}{\sqrt{R'^2-a^2}}\\ &= \frac{v_0^2}{2\pi G}\int_a^{R_{\rm max}} {\rm d}R'\frac{1}{\sqrt{(R'/a)^2-1}}\frac{1}{a}\\ &= \frac{v_0^2}{2\pi G}\int_a^{R_{\rm max}} {\rm d}R'\frac{\rm d}{{\rm d}R}\left(\text{arccosh}(R/a)\right)\\ &= \frac{v_0^2}{2\pi G}\left[\text{arccosh}(R_{\rm max}/a)-\text{arccosh}(1)\right]\\ &= \frac{v_0^2}{2\pi G}\text{arccosh}(R_{\rm max}/a)\end{aligned} \tag{21}$$

The derivative with respect to a of this latter result writes:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{v_0^2}{2\pi G}\operatorname{arccosh}(R_{\mathrm{max}}/a)\right) &= \frac{v_0^2}{2\pi G}\frac{\mathrm{d}}{\mathrm{d}a}\operatorname{arccosh}(R_{\mathrm{max}}/a) \\ &= -\frac{v_0^2}{2\pi G}\frac{R_{\mathrm{max}}}{\sqrt{R_{\mathrm{max}}^2-a^2}}\frac{1}{a}\end{aligned} \tag{22}$$

which, in the limit $R_{\mathrm{max}} \to \infty$ gives:

$$-\frac{v_0^2}{2\pi G a} \tag{23}$$

This leads to the circular velocity:

$$\begin{aligned}v_{\mathrm{c}}^2(R) &= \frac{2v_0^2}{\pi}\int_0^R \mathrm{d}a \frac{1}{\sqrt{R^2-a^2}} \\ &= v_0^2\end{aligned} \tag{24}$$