Homework 11
Quantum Information Processing

Exercise 1 *Useful identity for realisation of NOT gate by NMR or quantum circuits.*

We consider two qubits (for example: spins 1/2 or general two-level systems) and the operations:

- Rotations of angle $\frac{\pi}{2}$ around z axis for each spin

$$R_1 = \exp\left(-i\frac{\pi}{2}\frac{\sigma_1^z}{2}\right) \text{ et } R_2 = \exp\left(-i\frac{\pi}{2}\frac{\sigma_2^z}{2}\right)$$

- Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
- Unitary evolution $U = \exp\left(-i\frac{t}{\hbar}\mathcal{H}\right)$ associated to the (anitropic) Heisenberg hamiltonian

$$\mathcal{H} = \hbar J \sigma_1^z \otimes \sigma_2^z.$$

We let the system evolve for a time $t = \frac{\pi}{4J}$.

a) Calculate the product

$$(I \otimes H)\, U\, (R_1 \otimes R_2)\, (I \otimes H)$$

where I is the 2×2 identity matrix.

b) Show that this product is equivalent to a 4×4 CNOT gate defined by

$$\text{CNOT}|x\rangle \otimes |y\rangle = |x\rangle \otimes |y \oplus x\rangle$$

Here $x, y \in \{0, 1\}$ and $\oplus$ addition modulo 2. The bit $|x\rangle$ is called control bit. The second qubit is flipped iff the control bit is in the state $|1\rangle$.

Exercise 2 *Refocusing technique*

Consider the hamiltonian of Heisenberg type (here only the "z" term is present).

$$\mathcal{H} = \hbar J \sigma_1^z \otimes \sigma_2^z$$

for the interaction between two qubits. Let

$$R_1 = \exp\left(i\pi\frac{\sigma_1^x}{2}\right),$$

the π-pulse (or rotation around x) acting on the first spin. This is a one-qubit opertaion and can be realized by NMR techniques seen in class comme vu au cours (constant + rotating field).

We consider the Heisenberg evolution of the two spins for a time $\frac{t}{2}$, followed by a π-pulse, followed by the Heisenberg evolution for a time $\frac{t}{2}$, followed by a final π-pulse. The total evolution is

$$U_{tot} = (R_1 \otimes \mathbb{I}_2)\, e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}\, (R_1 \otimes \mathbb{I}_2)\, e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}$$

a) Show the general identity valid for all times t:

$$(R_1 \otimes \mathbb{I}_2)\, e^{-\frac{it}{\hbar}\mathcal{H}}\, (R_1 \otimes \mathbb{I}_2)\, e^{-\frac{it}{\hbar}\mathcal{H}} = \mathbb{I}_1 \otimes \mathbb{I}_2$$

b) In practice $J << 1$. Can you say a few words on the physical interpretation of this identity ?