Solution 11
Quantum Information Processing

Exercise 1 *Useful identity for the realisation of CNOT*

Remark: Here all matrices that must be exponentiated are diagonal. In this case it is easier to do computaions in components.

- $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is diagonal so

$$R_1 = R_2 = \begin{pmatrix} \exp\left(-i\frac{\pi}{4}\right) & 0 \\ 0 & \exp\left(i\frac{\pi}{4}\right) \end{pmatrix} = e^{-i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

- The Hadamard gate is $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

- For the Hamiltonian we have

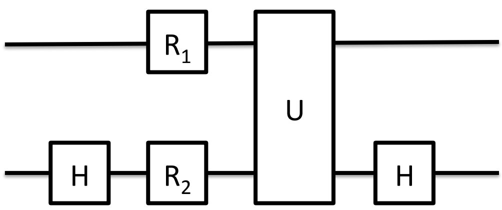
$$\mathcal{H} = \hbar J \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hbar J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- If we let evolve for a time $t = \frac{\pi}{4J}$ we find

$$U = \exp\left(-\frac{it}{\hbar}\mathcal{H}\right) = \exp\left(-\frac{i\pi}{4J\hbar}\mathcal{H}\right) = \begin{pmatrix} \exp\left(-i\frac{\pi}{4}\right) & 0 & 0 & 0 \\ 0 & \exp\left(i\frac{\pi}{4}\right) & 0 & 0 \\ 0 & 0 & \exp\left(i\frac{\pi}{4}\right) & 0 \\ 0 & 0 & 0 & \exp\left(-i\frac{\pi}{4}\right) \end{pmatrix},$$

$$\Rightarrow U = \exp\left\{(-i\frac{\pi}{4})\right\}. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Remark: The product of matrices corresponds to the circuit:



The input state is on the left side $|\psi\rangle$ and the output state on the right $(I_{2\times 2}\otimes H)U(R_1\otimes R_2)(I_{2\times 2}\otimes H)|\psi\rangle$.

We compute the product:

$$R_1 \otimes R_2 = e^{-i\frac{\pi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$
$$= e^{-i\frac{\pi}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and

$$U(R_1 \otimes R_2) = e^{-i\frac{3\pi}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$= e^{-i\frac{3\pi}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Also

$$I_{2\times 2} \otimes H = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

and

$$\frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right) = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right),$$

and then

$$\frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right) \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) = \frac{1}{2} \left(\begin{array}{cc|cc} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ \hline 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right),$$

finally we find

$$
(I_{2\times 2}\otimes H)U(R_1\otimes R_2)(I_{2\times 2}\otimes H) = e^{-i\frac{3\pi}{4}}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ \hline 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{array}\right).
$$

This matrix is basically equivalent to a CNOT gate and is equal to

$$
\begin{aligned}
& e^{-i\frac{3\pi}{4}}\begin{pmatrix} \sigma_x & 0\\ 0 & -\mathbb{1}\end{pmatrix}\\
&= e^{-i\frac{3\pi}{4}}\left\{\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\otimes\sigma_x - \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\otimes\mathbb{1}\right\}\\
&= e^{-i\frac{3\pi}{4}}\{|0\rangle\langle 0|\otimes\sigma_x - |1\rangle\langle 1|\otimes\mathbb{1}\}.
\end{aligned}
$$

Cette operation flip un bit si le bit de controle est dans l'etat $|o\rangle$ et change la phase du bit si le bit de controle est dans l'état $|1\rangle$.

Remark: To get the standard CNOT standard we must use rotations with other signs:

$$
R_1 = \exp\left(+i\frac{\pi}{2}\frac{\sigma_1^2}{2}\right) \text{ et } R_2 = \exp\left(+i\frac{\pi}{2}\frac{\sigma_2^2}{2}\right).
$$

and we get

$$
\begin{aligned}
(I_{2\times 2}\otimes H)U(R_1\otimes R_2)(I_{2\times 2}\otimes H) &= e^{i\frac{3\pi}{4}}\{|0\rangle\langle 0|\otimes\mathbb{1} + |1\rangle\langle 1|\otimes\sigma_x\}\\
&= e^{i\frac{3\pi}{4}}\underbrace{\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{array}\right)}_{\text{standard CNOT}}
\end{aligned}
$$

Exercise 2 *Refocusing*

a) One could write matrices in component form and multiply them. More simply we can apply the identity to computational basis states and check the equality (this means the equality is valid for any vector by linearity).

   For example we for $|\psi_0\rangle = |\uparrow\uparrow\rangle$, we find (using that R_1 flips a spin; verify !)

$$
\begin{aligned}
|\psi_1\rangle &= e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}|\uparrow\uparrow\rangle = e^{-itJ}|\psi_0\rangle,\\
|\psi_2\rangle &= (R_1\otimes I)|\psi_1\rangle = e^{-itJ}|\downarrow\uparrow\rangle,\\
|\psi_3\rangle &= e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}|\psi_2\rangle = e^{-itJ}e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}|\downarrow\uparrow\rangle = e^{-itJ}e^{itJ}|\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle,\\
|\psi_4\rangle &= (R_1\otimes I)|\psi_3\rangle = (R_1\otimes I)|\downarrow\uparrow\rangle = |\uparrow\uparrow\rangle,
\end{aligned}
$$

   which shows $|\psi_4\rangle = |\psi_0\rangle = (I_1\otimes I_2)|\psi_0\rangle$. For other basis states we proceed with similar verifications $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$.

b) $J << 1$. Thus $\tau = \frac{\pi}{4J} >> \pi$. The π-pulses of NMR are much faster than the evolution of nuclear spins. Thus by injecting two π-pulses at instants $\frac{\tau}{2}$ and τ we recover the initial state and everything looks as if the spins had not evolved.