Introduction to Differentiable Manifolds
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Exercise Series 6 - Submanifolds (more!) **2022–10–25**

Exercise 6.1. Consider the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the projection map.

(a) Give $\mathbb{T}^n$ a natural smooth structure so that π is a local diffeomorphism.
(b) Show that a map $f : \mathbb{T}^n \to M$ (where M is a smooth manifold) is smooth if and only if the composite $f \circ \pi$ is smooth.
(c) Show that $\mathbb{T}^n$ is diffeomorphic to the product of n copies of the circle $\mathbb{S}^1$.

Exercise 6.2. If S is an embedded submanifold of M, show that there is a unique topology and smooth structure on S such that the inclusion map $S \to M$ is an embedding.

Exercise 6.3. For a subset S of a smooth manifold M, show that the following are equivalent:

(a) S is a closed embedded k-submanifold of M.
(b) For each point $p \in M$ there exists a chart (V, φ) that is k-sliced by S, i.e. we have
$$S \cap V = \{q \in V : \phi^k(q) = \cdots = \phi^{n-1}(q) = 0\}.$$

Exercise 6.4. Show that every submersion is an open map.

Exercise 6.5. If M is a smooth manifold and $\pi : N \to M$ is a covering map, show that N has a unique smooth structure such that π is a local diffeomorphism.

Exercise 6.6. Let $\iota : N \to M$ be a smooth embedding of smooth manifolds.

(a) If ι is a closed map, show that for every smooth function $f \in C^\infty(N)$ there exists a smooth function $g \in \mathcal{C}^\infty(M)$ such that $f = g \circ \iota$.
(b) Is this still true if we omit the assumption that ι is a closed map ?

Exercise 6.7 (To hand in). (a) Show that the map $f : \mathbb{P}^2 \to \mathbb{R}^3$ defined by
$$f([x, y, z]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy).$$
is smooth, and has injective differential except at 6 points.

(b) Show that the map $g : \mathbb{P}^2 \to \mathbb{R}^4$ defined by
$$g([x, y, z]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy, x^2 - z^2)$$
is a smooth embedding.

Exercise 6.8. *

(a) A smooth map $f : M \to N$ is **transverse** to a closed embedded submanifold $S \subseteq N$ if for all points $p \in f^{-1}(S)$ we have $\mathrm{T}_{f(p)}S + \mathrm{Img}(\mathrm{T}_p f) = \mathrm{T}_{f(p)}M$. If this happens, show that $f^{-1}(S)$ is a closed embedded submanifold of M. What is its dimension ? What is its tangent space ?
(b) Two smooth maps $f_0 : M_0 \to N$ and $f_1 : M_1 \to N$ are **transverse** to each other if for any pair of points $p_0 \in M_0$, $p_1 \in M_1$ such that $f_0(p_0) = f_1(p_1) =: q \in N$ we have $\mathrm{Img}(\mathrm{T}_{p_0} f_0) + \mathrm{Img}(\mathrm{T}_{p_1} f_1) = \mathrm{T}_q N$. If this happens, prove that the set
$$S := \{(p_0, p_1) \in M_0 \times M_1 \mid f_0(p_0) = f_1(p_1)\}$$
is a closed submanifold of $M_0 \times M_1$. What is its dimension ?