Stellar orbits

2nd part

Outlines

Examples of orbits in spherical potentials

- Keplerian orbits
- orbits in an homogeneous sphere
- important remarks

Orbits in axisymmetric potentials

- orbits in the equatorial plane
- orbits outside the equatorial plane
- equations of motion
- orbits in the meridian plane
- examples

Nearly circular orbits

- Epicycle frequences

Stellar orbits

Spherical Systems

Examples

Examples

① **Kepler potential** (potential of a mass point)

$$\begin{cases} \phi(r) = -\dfrac{GM}{r} \\[4mm] \dfrac{\partial \phi}{\partial r}(r) = \dfrac{GM}{r^2} = GMu^2 \end{cases}$$

$$\boxed{\dfrac{d^2 u}{d\varphi^2} + u = \dfrac{1}{L^2 u^2}\, \dfrac{\partial \phi}{\partial r}\left(\dfrac{1}{u}\right)}$$

$\Longrightarrow$

$$\boxed{\dfrac{d^2 u}{d\varphi^2} + u = \dfrac{GM}{L^2}}$$

Harmonic equation, with frequency 1

General solution

$$\frac{d^2u}{d\varphi^2} + u = \frac{GM}{L^2}$$

$$u(\varphi) = C \cos(\varphi - \varphi_0) + \frac{GM}{L^2}$$

free parameter   free parameter
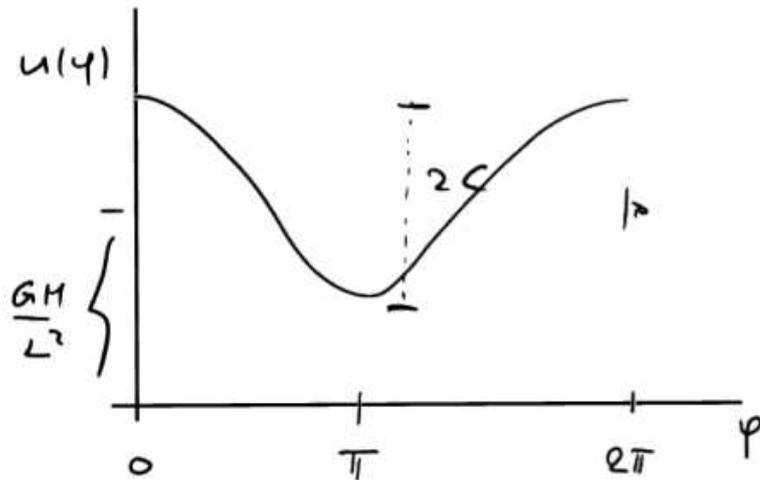
period $= 2\pi$

In term of r

$$r(\varphi) = \frac{1}{C \cos(\varphi - \varphi_0) + \frac{GM}{L^2}}$$

Introducing

$$e = \frac{cL^2}{GM} \qquad \text{eccentricity}$$

$$a = \frac{L^2}{GM(1-e^2)} \qquad \text{semi-major axis}$$

evaluate u and $\frac{du}{dt}$ for $\varphi = \varphi_0$ $\quad \left(u(\varphi) \overset{\varphi=\varphi_0}{=} c + \frac{GM}{L^2} \qquad \frac{du}{dt}(\varphi) \overset{\varphi=\varphi_0}{=} 0 \right)$

+ using $\quad \dfrac{d^2u}{d\varphi^2} + u = \dfrac{1}{L^2 u^2} \dfrac{\partial \phi}{\partial r}\left(\dfrac{1}{u}\right)$

$$r(\varphi) = \frac{a(1-e^2)}{1 + e\cos(\varphi - \varphi_0)}$$

$$\bar{E} = -\frac{GM}{2a}$$

↳ from the energy equation
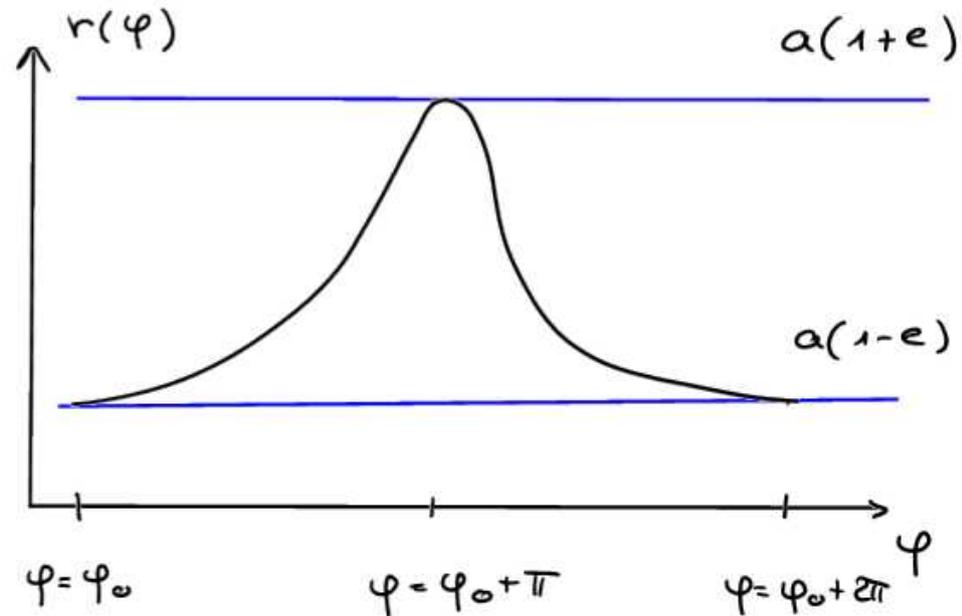
Cases

$$r(\varphi) = \frac{a(1 - e^2)}{1 + e\cos(\varphi - \varphi_0)}$$

e ≥ 1

unbound orbit as $1 + e\cos(\varphi - \varphi_0)$ can be $= 0$

$$\Rightarrow \quad r \to \infty$$

e < 1

bound orbit (ellipse)

EXERCICE

pericenter / apocenter

$a(1+e)$ $a(1-e)$

a

$$r_{min} = \frac{a(1 - e^2)}{1 + e} = a(1 - e)$$

$$r_{max} = \frac{a(1 - e^2)}{1 - e} = a(1 + e)$$

e = 0
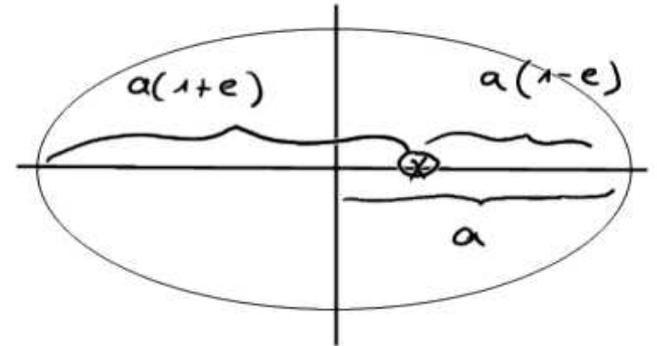
$$r_{min} = r_{max} = a \qquad (\text{circular orbit})$$

Kepler laws (1609-1619) :

- The orbit of a planet is an ellipse with the Sun at one of the two foci.

- A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.

- The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

Radial and azimuthal periods:

$$T_r = T_\phi$$

Kepler's laws of planetary motion

Second law

A radius vector joining any planet to the Sun sweeps out equal areas in equal lengths of time.

orbit

Sun

1 month A_1 A_2 1 month

$A_1 = A_2$

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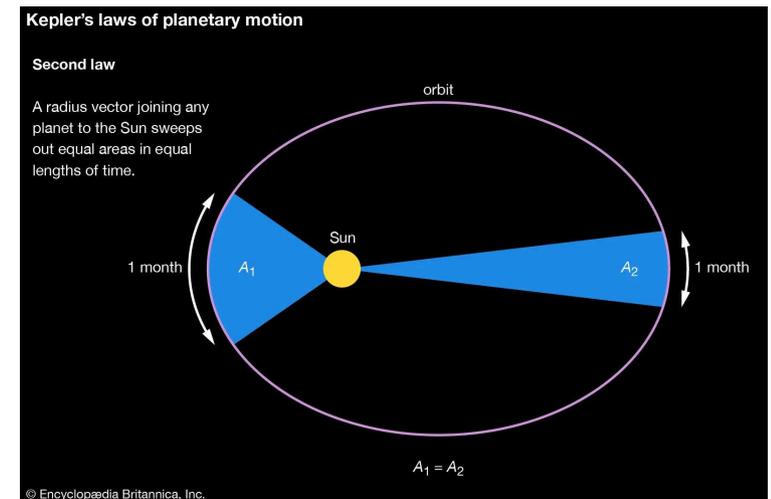
Keplerian orbits (point mass)
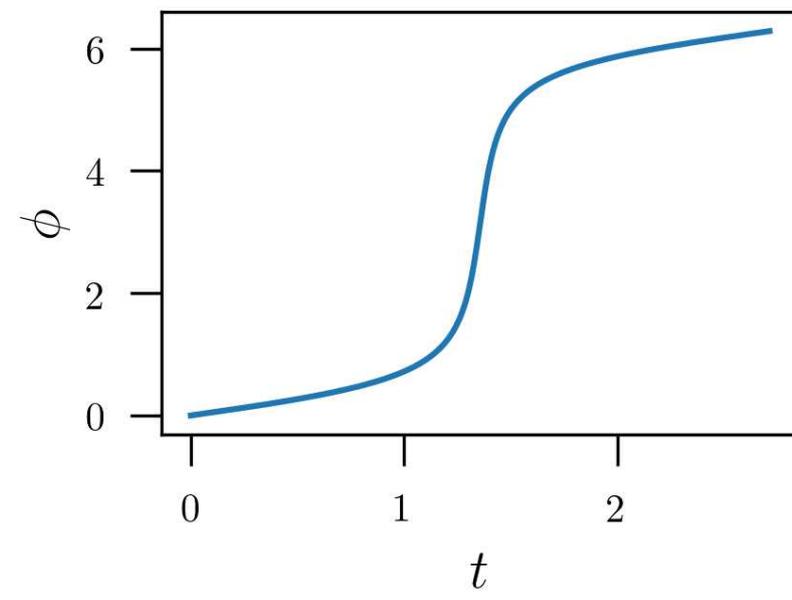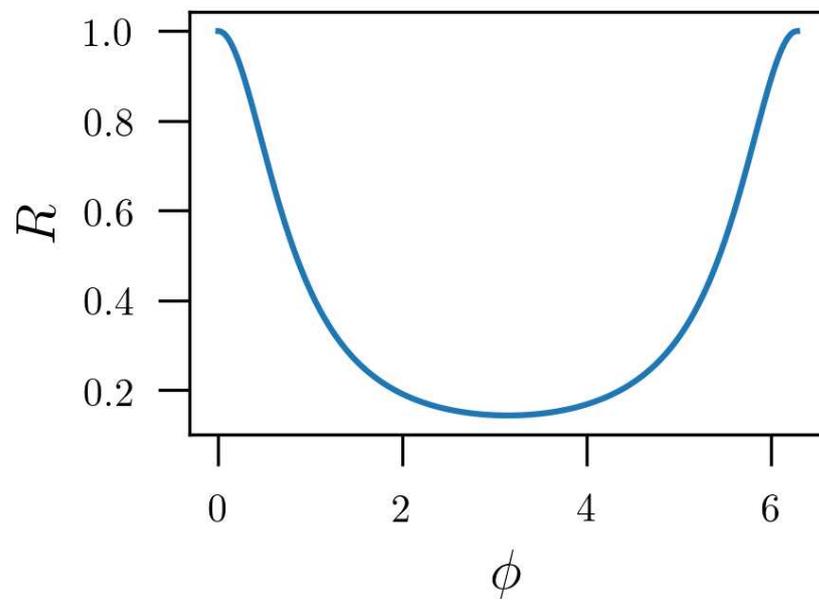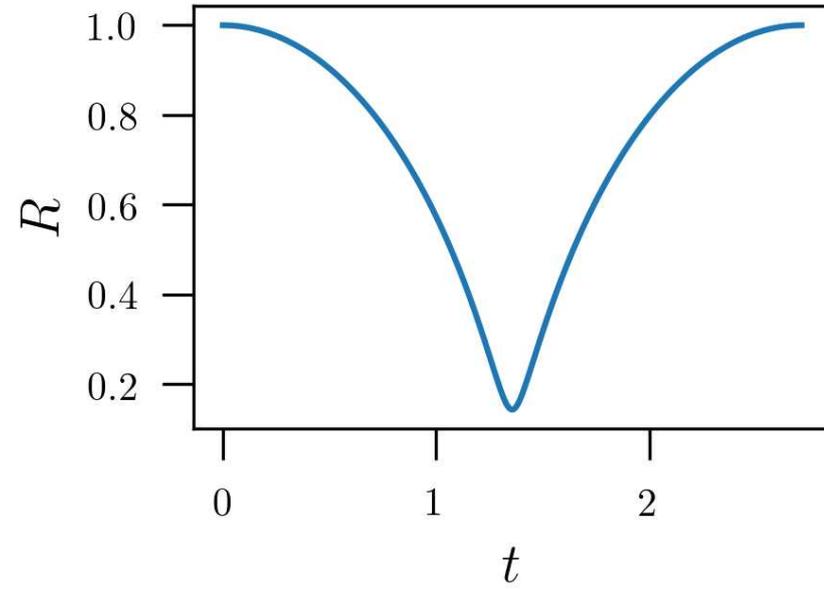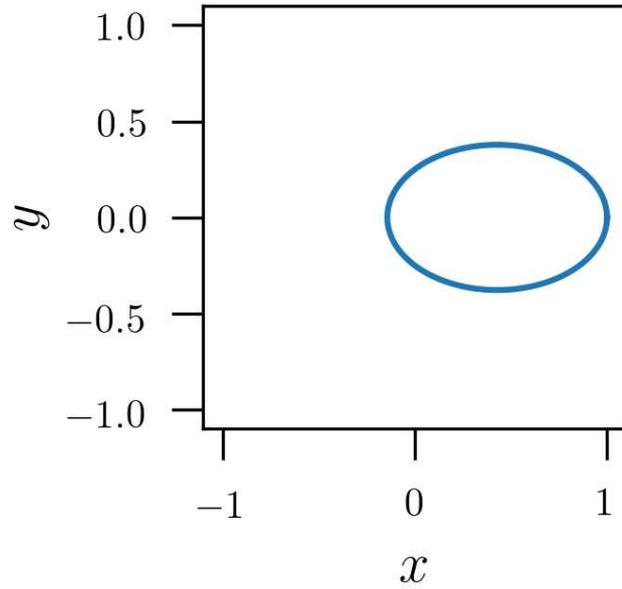
Keplerian orbits (point mass)
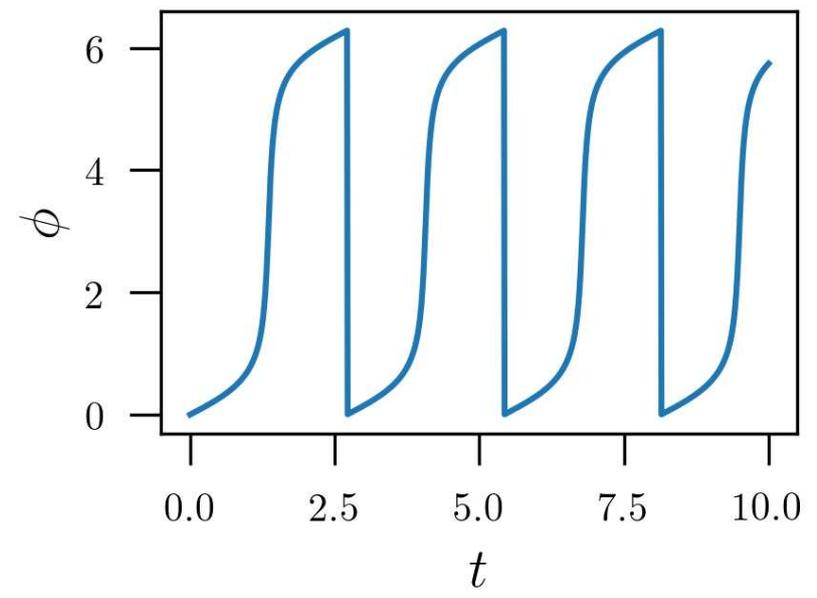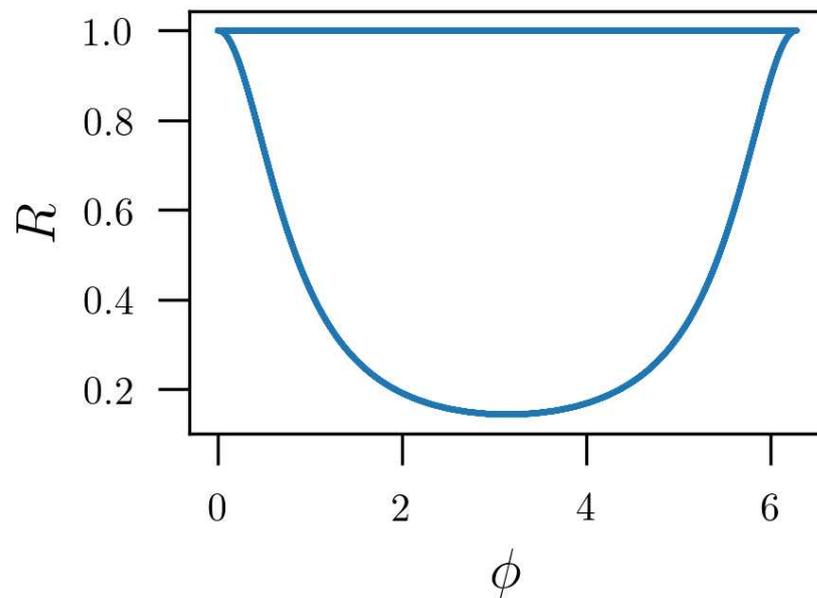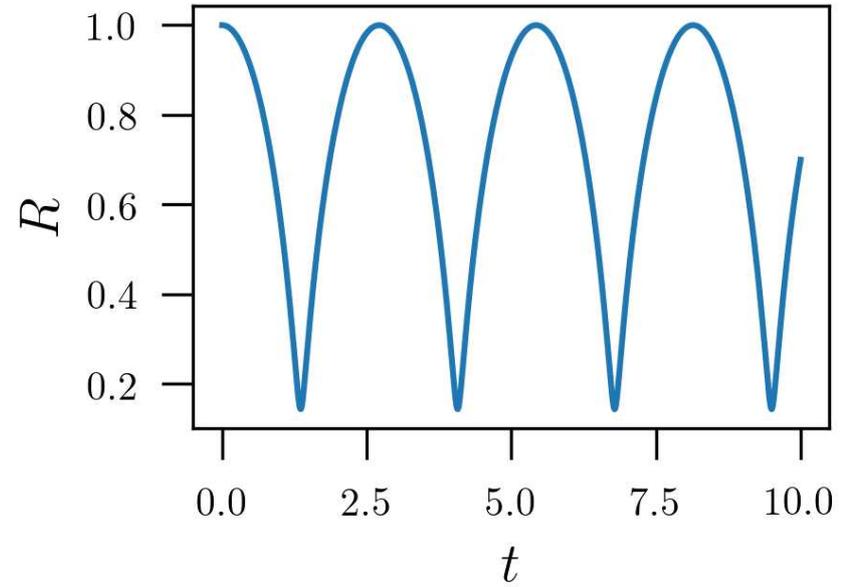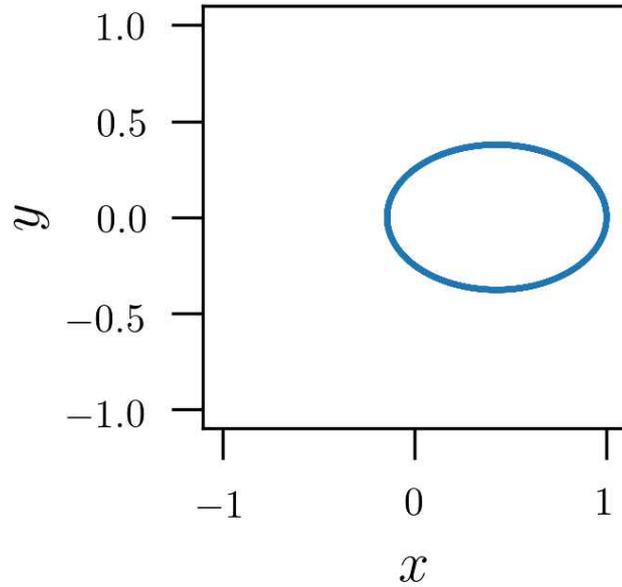
Keplerian orbits (point mass)
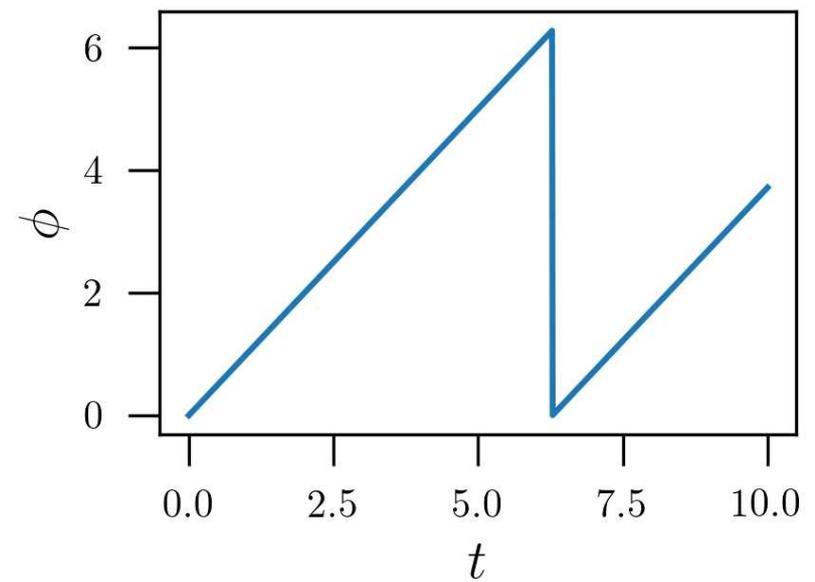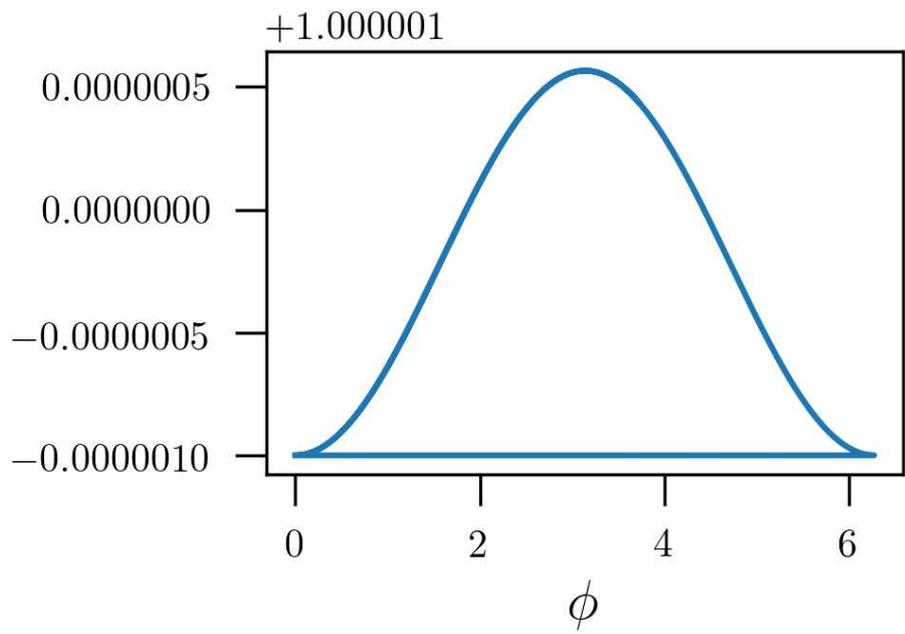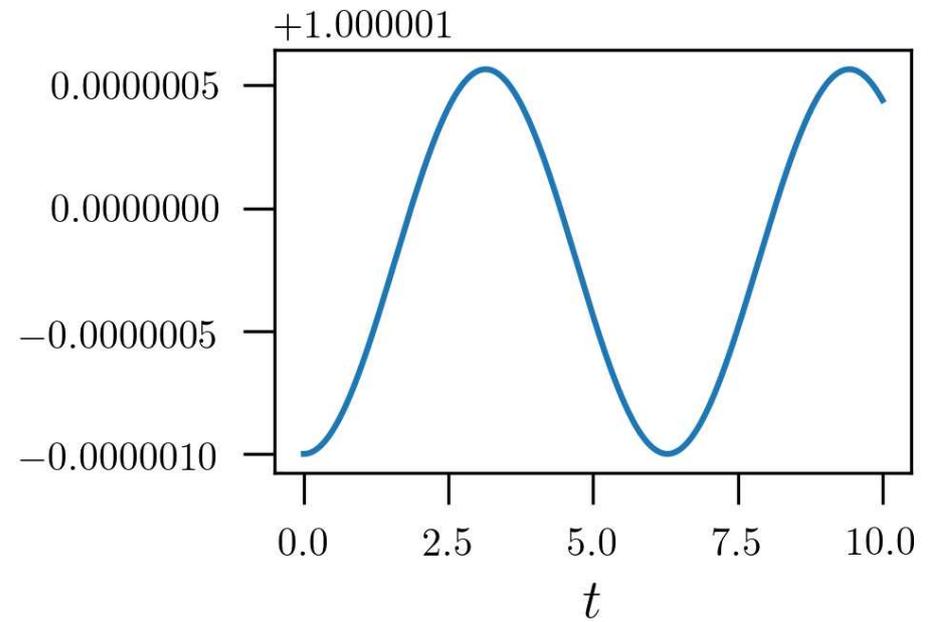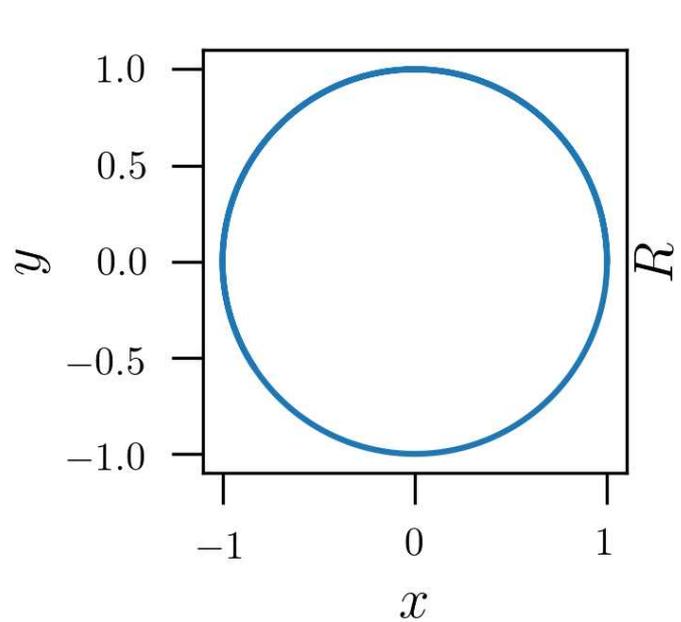
(2) Homogeneous sphere $\quad \rho_0, R_0 \quad$ (Harmonic oscillations)

$$\phi(r) = \underbrace{- 2\pi G \rho_0 R_0^2}_{cte \quad \to \; 0} + \frac{2}{3}\pi G \rho_0 \, r^2$$

$$\phi(r) = \frac{1}{2}\Omega^2 r^2 \qquad\qquad \text{with} \quad \Omega = \sqrt{\frac{4}{3}\pi G \rho_0}$$

Equations of motion $\qquad$ (in carthesian coordinates)

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}\Omega^2(x^2 + y^2)$$

$$\begin{cases} \ddot{x} = -\Omega^2 x \\[2mm] \ddot{y} = -\Omega^2 y \end{cases} \qquad \begin{cases} x(t) = X\cos(\Omega t + \varepsilon_x) \\[2mm] y(t) = Y\cos(\Omega t + \varepsilon_y) \end{cases}$$

$X, Y, \varepsilon_x, \varepsilon_y \quad$ constants fixed by the initial conditions

$x(t)$

$y(t)$
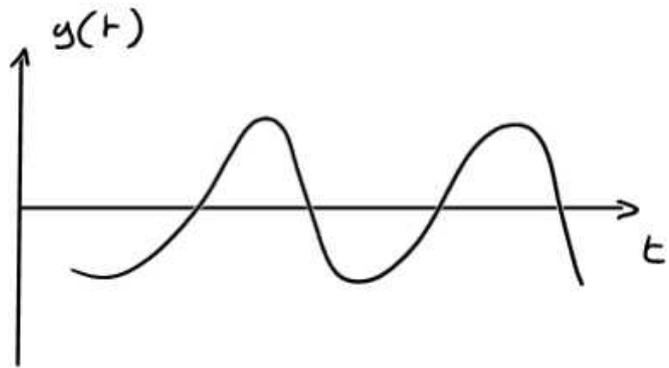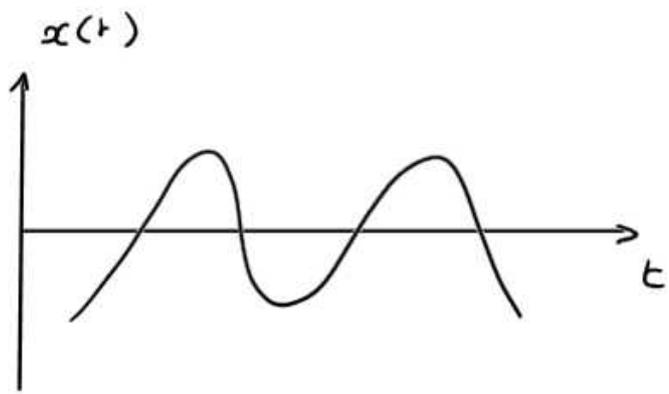
same period

$\Rightarrow$ closed orbits (ellipse)

r_{min} r_{max}

r_{max} r_{min}

y

x
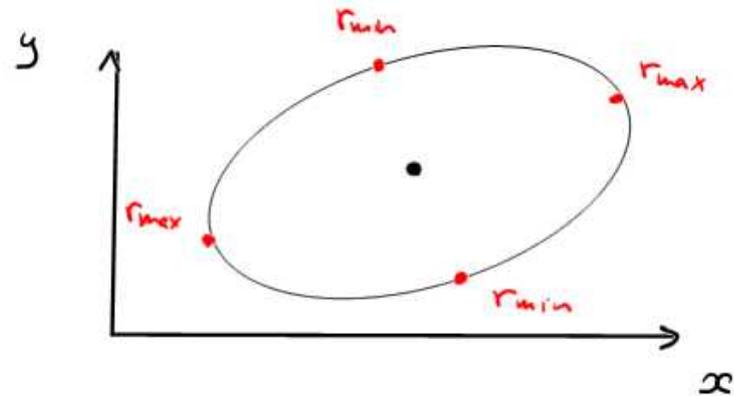
Periods
———

$$T_\varphi = \frac{2\pi}{\Omega}$$

$$T_r = \frac{1}{2} T_\varphi = \frac{\pi}{\Omega}$$

Homegeneous sphere (harmonic)
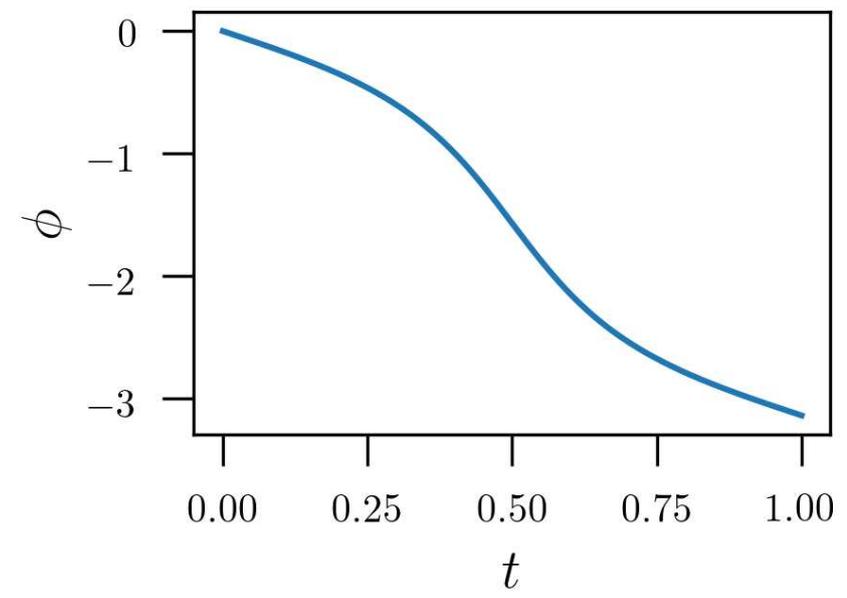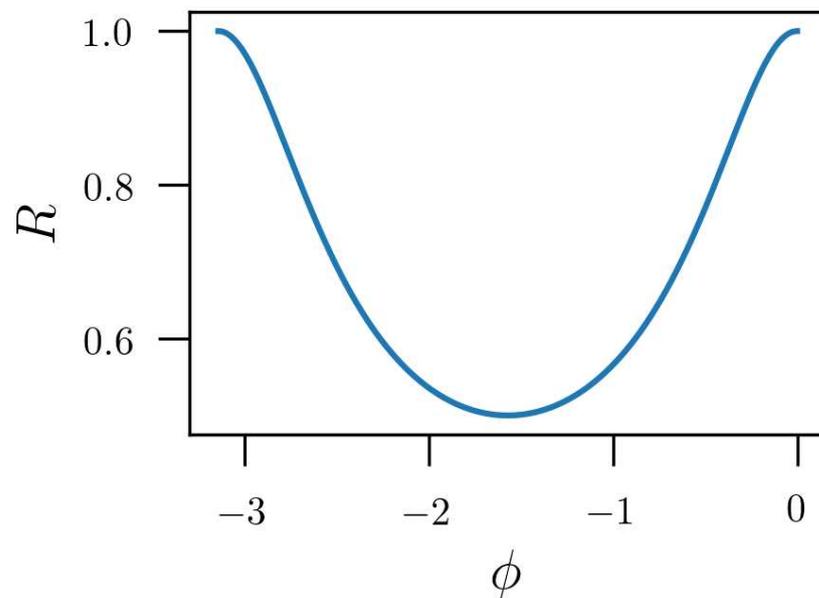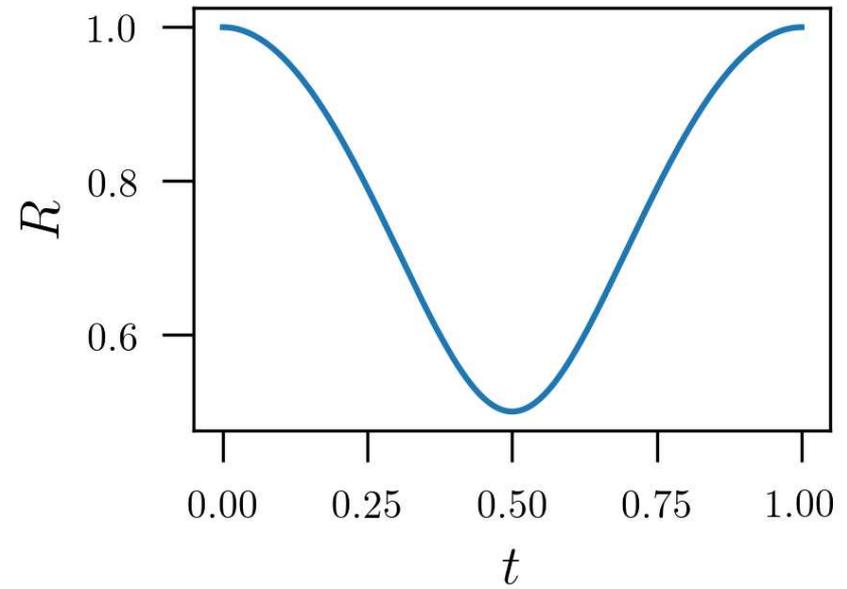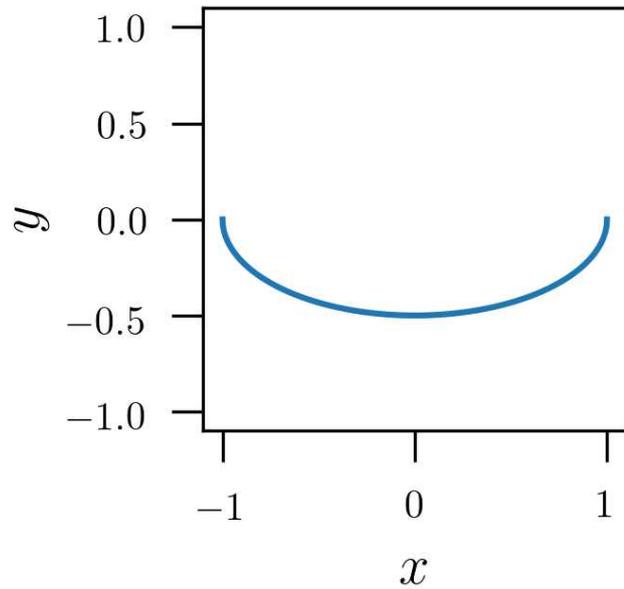
Homegeneous sphere (harmonic)
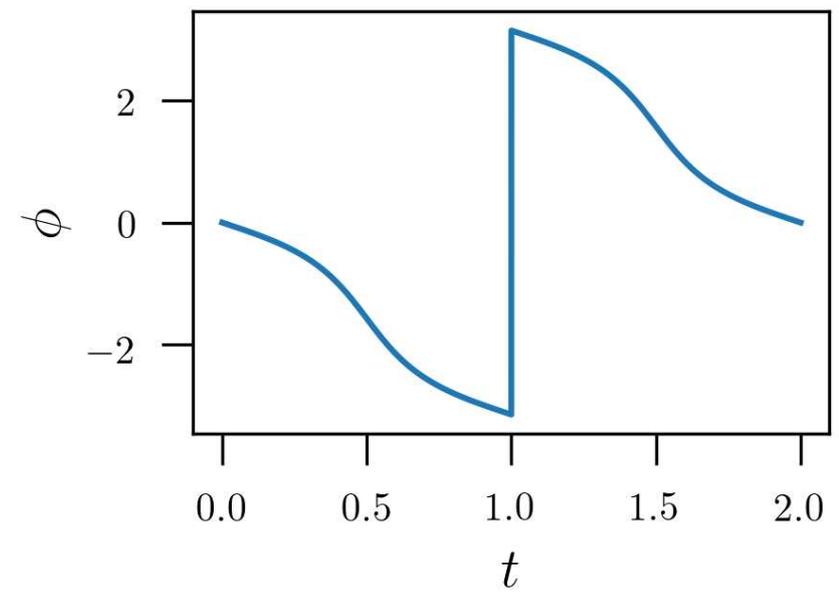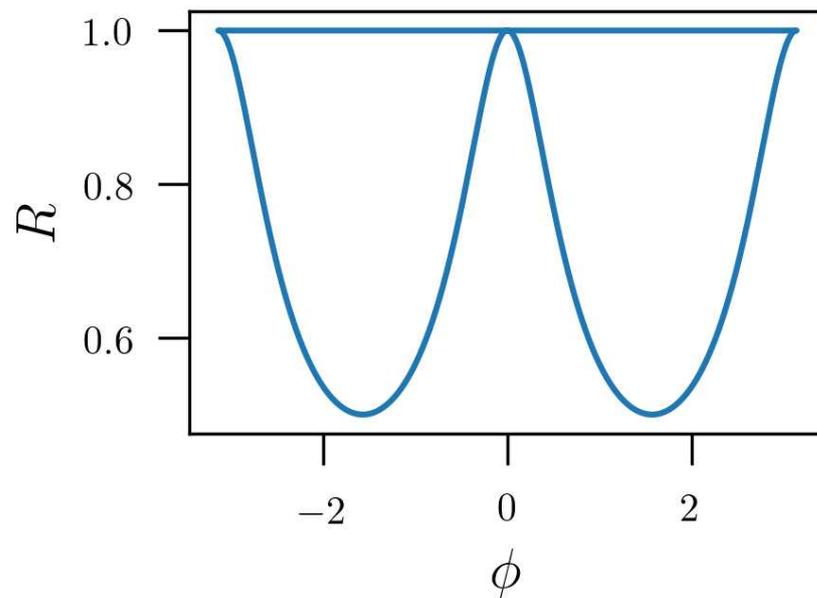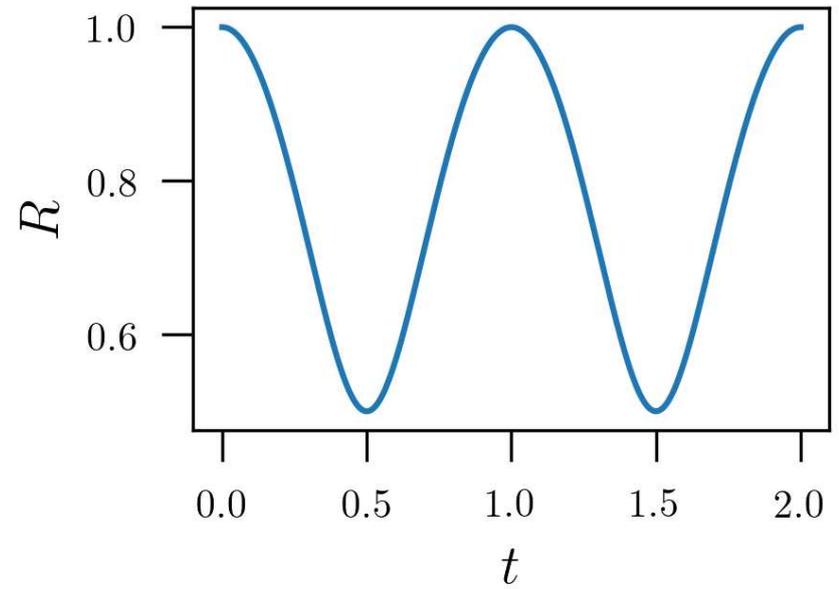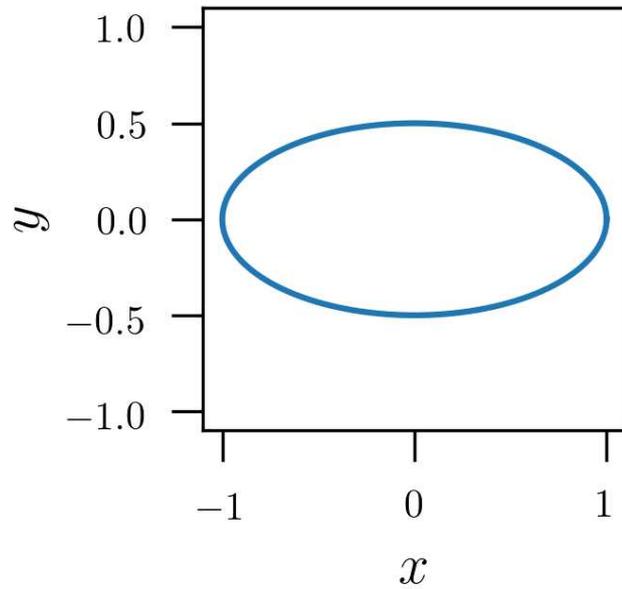
Homegeneous sphere (harmonic)

Important Remarks

Homogeneous sphere

$$T_r = \frac{1}{2} T_\varphi$$

$u(\varphi)$

Important Remarks

Homogeneous sphere

$$T_r = \frac{1}{2} T_\varphi$$

$u(\varphi)$

Keplerian potential

$$T_r = T_\varphi$$

$u(\varphi)$

Important Remarks

Homogeneous sphere

$$T_r = \frac{1}{2} T_\varphi$$

$u(\varphi)$

$$\frac{1}{2} T_\varphi < T_r < T_\varphi$$

Galaxy

$u(\varphi)$

Keplerian potential

$$T_r = T_\varphi$$

$u(\varphi)$
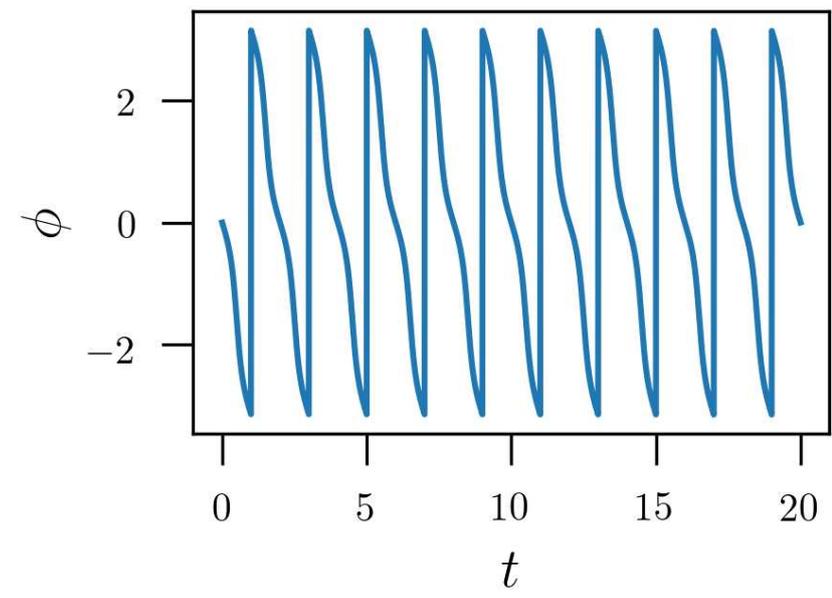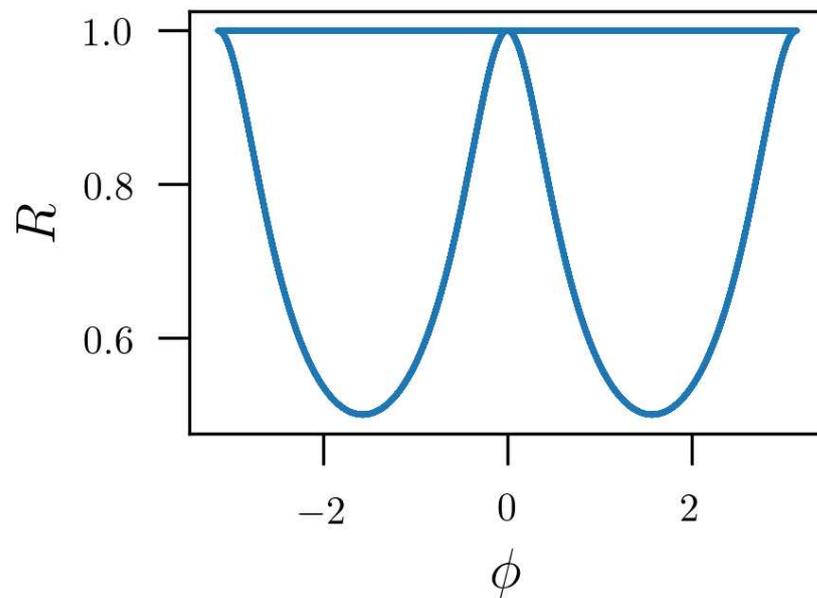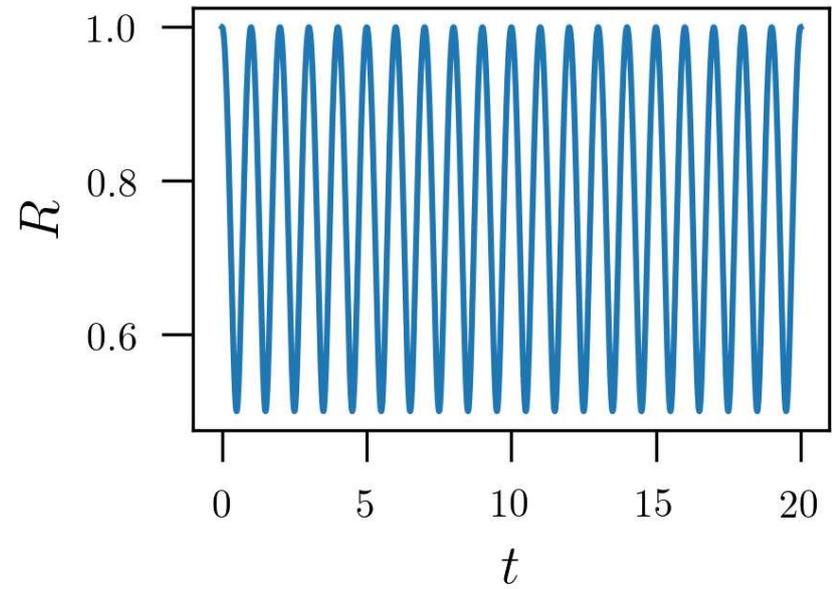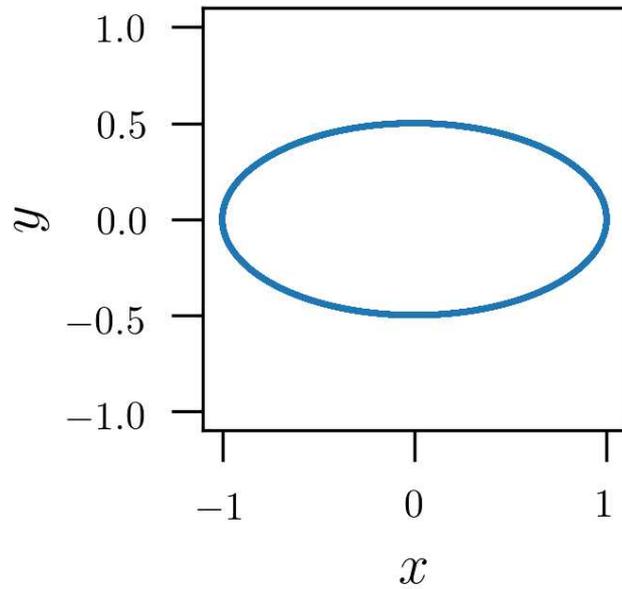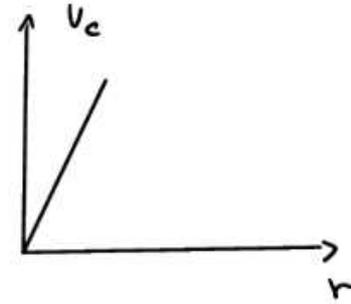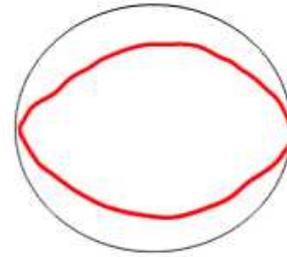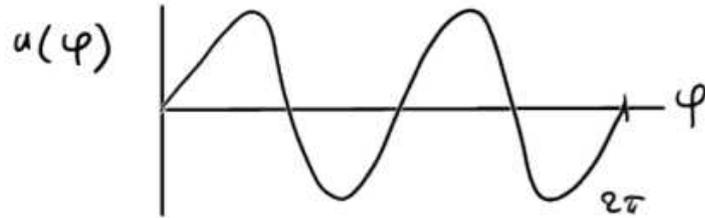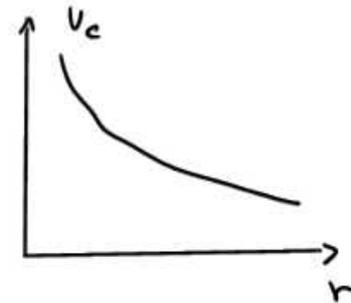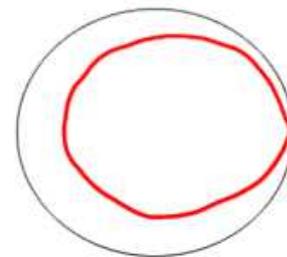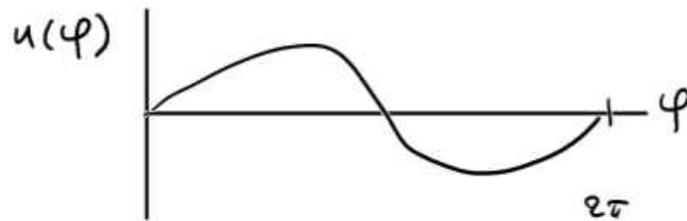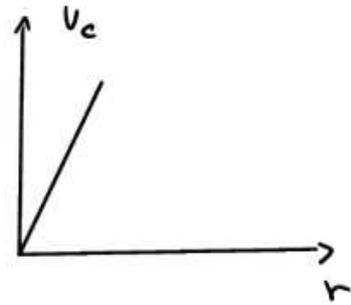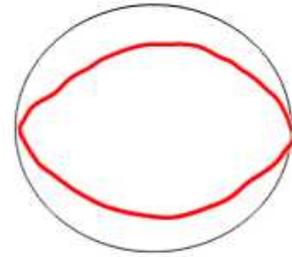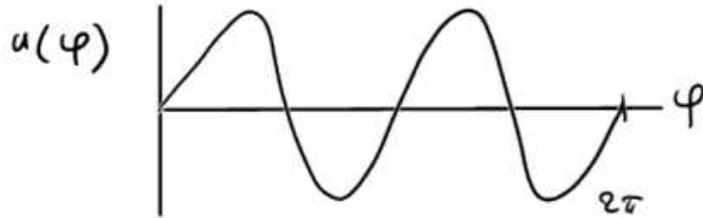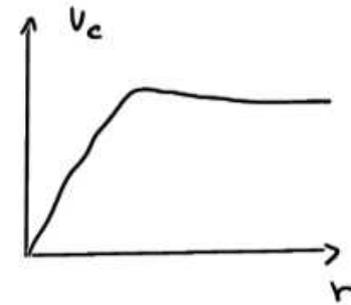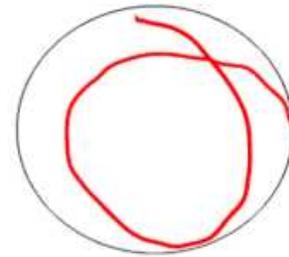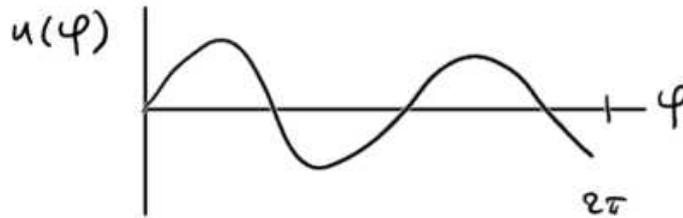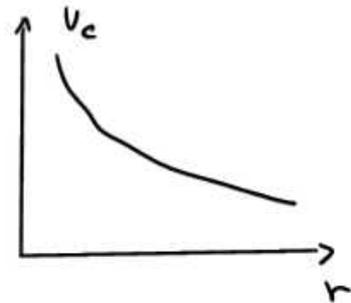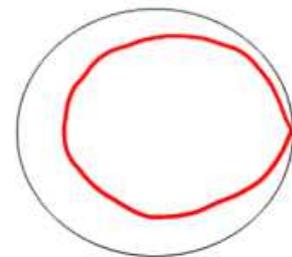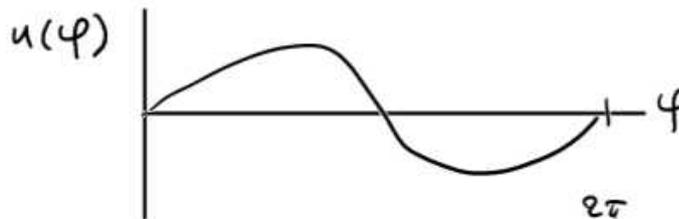
Axisymmetric Systems

Orbits in axisymmetric potentials

Axisymmetric potential $\qquad \phi(\bar{x}) = \phi(R, |z|)$

- symmetry of revolution around z

- reflection symmetry with respect
  to the $z = 0$ plane
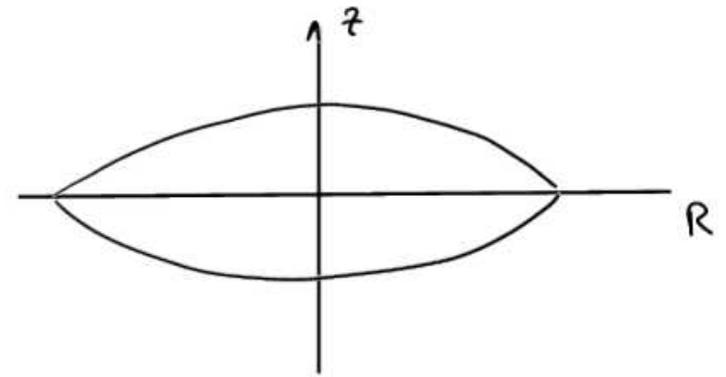
Definitions

Oblate systems : c, the semi-minor axis
  is parallel to $\vec{z}$

Prolate systems : a, the semi-major axis
  is parallel to $\vec{z}$
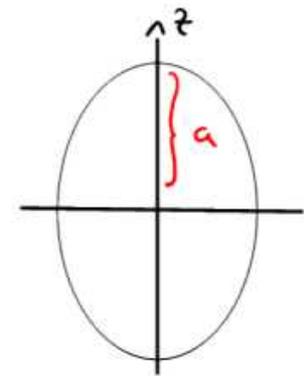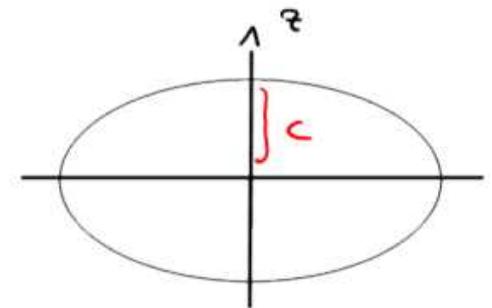
Description of the dynamics

Cylindrical coordinates (R, Θ, z)

Equatorial plane

Orbits in the equatorial plane

$\forall t, \quad \underline{z = 0}$

$$\boxed{\phi(R, |z| = 0) \equiv \phi(R)}$$
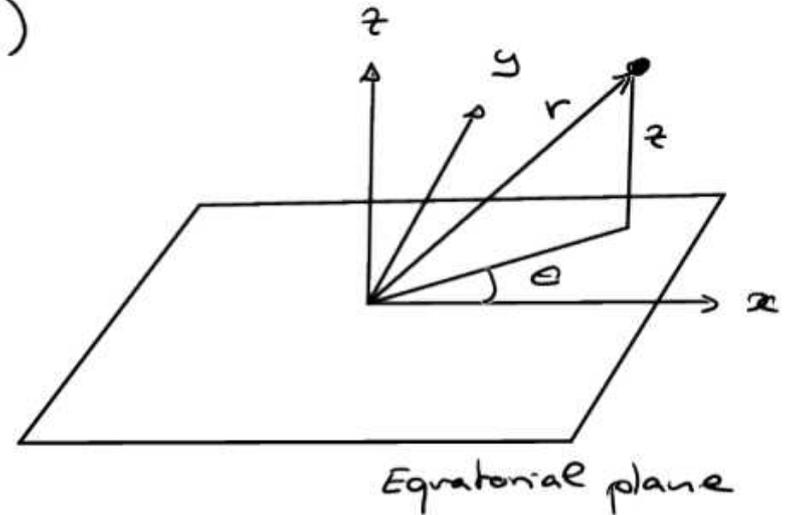
The potential seen by the stars is similar to a spherical potential

- description of the orbits in polar coordinates r, φ
- recycle all results developped for spherical potentials

Tork and angular momentum

$$\begin{cases} \dot{\vec{e}}_R = \dot{\theta}\,\vec{e}_\theta \\ \dot{\vec{e}}_\theta = -\dot{\theta}\,\vec{e}_R \\ \dot{\vec{e}}_z = 0 \end{cases}$$
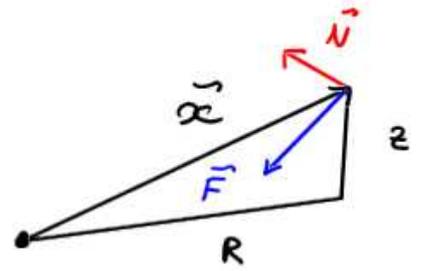
$$\begin{cases} \vec{x} = R\,\vec{e}_R + z\,\vec{e}_z \\ \vec{v} = \dot{R}\,\vec{e}_R + R\dot{\theta}\,\vec{e}_\theta + \dot{z}\,\vec{e}_z \\ \vec{F} = -\vec{\nabla}\phi = -\dfrac{\partial\phi}{\partial R}\,\vec{e}_R - \cancel{\dfrac{1}{R}\dfrac{\partial}{\partial\theta}\phi\,\vec{e}_\theta} - \dfrac{\partial\phi}{\partial z}\,\vec{e}_z \end{cases}$$

Tork

$$\vec{N} = \vec{x}\times\vec{F} = \left(R\,\vec{e}_R + z\,\vec{e}_z\right)\times\left(-\dfrac{\partial\phi}{\partial R}\,\vec{e}_R - \dfrac{\partial\phi}{\partial z}\,\vec{e}_z\right)$$

$$= \left(z\,\dfrac{\partial\phi}{\partial R} - R\,\dfrac{\partial\phi}{\partial z}\right)\vec{e}_\theta$$

Angular momentum

$$\vec{L} = \vec{x}\times\vec{v} = \left(R\,\vec{e}_R + z\,\vec{e}_z\right)\times\left(\dot{R}\,\vec{e}_R + R\dot{\theta}\,\vec{e}_\theta + \dot{z}\,\vec{e}_z\right)$$

$$= -z R \dot{\theta} \, \vec{e}_R + (z\dot{R} - R\dot{z}) \, \vec{e}_\theta + R^2 \dot{\theta} \, \vec{e}_z$$

$$\underbrace{\qquad}_{L_R} \qquad \underbrace{\qquad}_{L_\theta} \qquad \underbrace{\qquad}_{L_z}$$

$$\frac{d\vec{L}}{dt} = \left(\dot{L}_R - L_\theta \dot{\theta} \right) \vec{e}_R + \left(\dot{L}_\theta + L_R \dot{\theta} \right) \vec{e}_\theta + \dot{L}_z \, \vec{e}_z$$

$$\frac{d\vec{L}}{dt} = \vec{N}$$

$$= \left(z \frac{\partial \phi}{\partial R} - R \frac{\partial \phi}{\partial z} \right) \vec{e}_\theta$$

So:
- $\vec{e}_R$: $\dot{L}_R - L_\theta \dot{\theta} = 0 = \frac{d}{dt}(-zR\dot{\theta}) - (z\dot{R} - R\dot{z})\dot{\theta} = 0$

$$\Rightarrow \quad R^2 \dot{\theta} = cte$$

- $\vec{e}_z$: $\dot{L}_z = 0 = R^2 \dot{\theta}$

$$\Rightarrow \quad R^2 \dot{\theta} = cte$$

The z-component of the angular momentum is conserved

Orbits that moves outside the equatorial plane

Cylindrical coordinates

$$
\begin{cases}
x = R\cos\Theta \\
y = R\sin\Theta \\
z = z
\end{cases}
\qquad
\begin{aligned}
\dot{x} &= \dot{R}\cos\Theta - R\sin\Theta\,\dot{\Theta} \\
\dot{y} &= \dot{R}\sin\Theta + R\cos\Theta\,\dot{\Theta} \\
\dot{z} &= \dot{z}
\end{aligned}
$$

$$
\underline{\dot{x}^2 + \dot{y}^2 = \dot{R}^2 + R^2\dot{\Theta}^2}
$$

Lagrangian (specific) in cylindrical coordinates

$$
\mathcal{L} = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \phi(\sqrt{x^2+y^2}) = \tfrac{1}{2}(\dot{R}^2 + R^2\dot{\Theta}^2 + \dot{z}^2) - \phi(R, z)
$$

Lagrange equations

$$
\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0}
$$

$$
\vec{q} = \begin{cases} R \\ \Theta \\ z \end{cases}
\qquad
\dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{\Theta} \\ \dot{z} \end{cases}
$$

Lagrange equations

$$
\begin{cases}
\ddot{R} = R\dot{\theta}^2 & - \dfrac{\partial \phi}{\partial R} & ① \\[2em]
\dfrac{d}{dt}\left(R^2\dot{\theta}\right) = & \left(- \dfrac{\partial \phi}{\partial \theta}\right) = 0 & ② \\[2em]
\ddot{z} = & - \dfrac{\partial \phi}{\partial z} & ③
\end{cases}
$$

② $\qquad R^2\dot{\theta} = cte = L_z$

The z-component of the angular momentum is conserved

Solution

$$
\theta(t) = L_z \int_{t_0}^{t_1} \frac{1}{R^2(r)}\, dt
$$

① + ③ two coupled through $\phi(R, z)$ equations for R and z

Hamiltonian/Energy

$$H(\vec{q}, \vec{p}, t) := \vec{p} \cdot \dot{\vec{q}} - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$$

$$\vec{q} = \begin{cases} R \\ \theta \\ z \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{\theta} \\ \dot{z} \end{cases} \qquad \vec{p} = \begin{cases} \dfrac{\partial \mathcal{L}}{\partial \dot{R}} = \dot{R} \\[2mm] \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} = R^2 \dot{\theta} \\[2mm] \dfrac{\partial \mathcal{L}}{\partial \dot{z}} = \dot{z} \end{cases}$$

$$\boxed{P_\theta = R^2 \dot{\theta} = L_z}$$

$$H(R, \dot{R}, \theta, \dot{\theta}, z, \dot{z}) = \frac{1}{2}\left(\dot{R}^2 + R^2 \dot{\theta}^2 + \dot{z}^2 \right) + \phi(R, z) = E$$

E (Energy) is conserved
as $\mathcal{L}$ is time independant
ϕ

Effective potential

with $L_z = R^2 \dot{\theta}$

$$L_z^2 = R^4 \dot{\theta}^2$$

Definition

$$\phi_{eff}(R, z) = \phi(R, z) + \frac{L_z^2}{2 R^2}$$

$$\begin{cases} \dfrac{\partial \phi_{eff}}{\partial R} = \dfrac{\partial \phi}{\partial R} - \dfrac{L_z^2}{R^3} \equiv \dfrac{\partial \phi}{\partial R} - R\dot{\theta}^2 \\[4mm] \dfrac{\partial \phi_{eff}}{\partial z} = \dfrac{\partial \phi}{\partial z} \end{cases}$$

The equations of motion ① + ③ becomes

$$\begin{cases} \ddot{R} = -\dfrac{\partial \phi_{eff}}{\partial R}(R, z) \\[4mm] \ddot{z} = -\dfrac{\partial \phi_{eff}}{\partial z}(R, z) \end{cases}$$

==The 3D motion of a star in an axisymmetric potential is reduced to a 2D motion in the meridian plane (R, z)==

phase space 6D → 4D

Hamiltonian in the meridian plane

Those equations of motion may be derived from the lagrangian

$$\mathcal{L}(R, \dot{R}, z, \dot{z}) = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 - \phi_{eff}(R, z)$$

The corresponding Hamiltonian writes $\quad (p_R = \dot{R}, \; p_z = \dot{z})$

$$
\begin{aligned}
H(R, \dot{R}, z, \dot{z}) &= \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \phi_{eff}(R, z) \\
&= \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \phi(R, z) + \frac{L_z^2}{2R^2} \\
&= \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \phi(R, z) + \frac{1}{2}R^2\dot{\theta}^2 = E
\end{aligned}
$$

Kinetic energy in the orbital plane

E is conserved as ϕ_{eff} is time independent

orbit's total energy

Illustration in the ==$z = 0$== plane

---

for $R \to \infty$

$$\phi_{\text{eff}} = \phi + \frac{L_z^2}{2} \frac{1}{R^2} \sim \phi$$

$$\underbrace{\quad}_{\to 0}$$

for $R \to 0$

$$\phi_{\text{eff}} = \phi + \frac{L_z^2}{2} \frac{1}{R^2} \sim \frac{1}{R^2}$$

$$\underbrace{\phi}_{\text{bounded}} \quad \underbrace{\frac{L_z^2}{2} \frac{1}{R^2}}_{\text{diverges}}$$
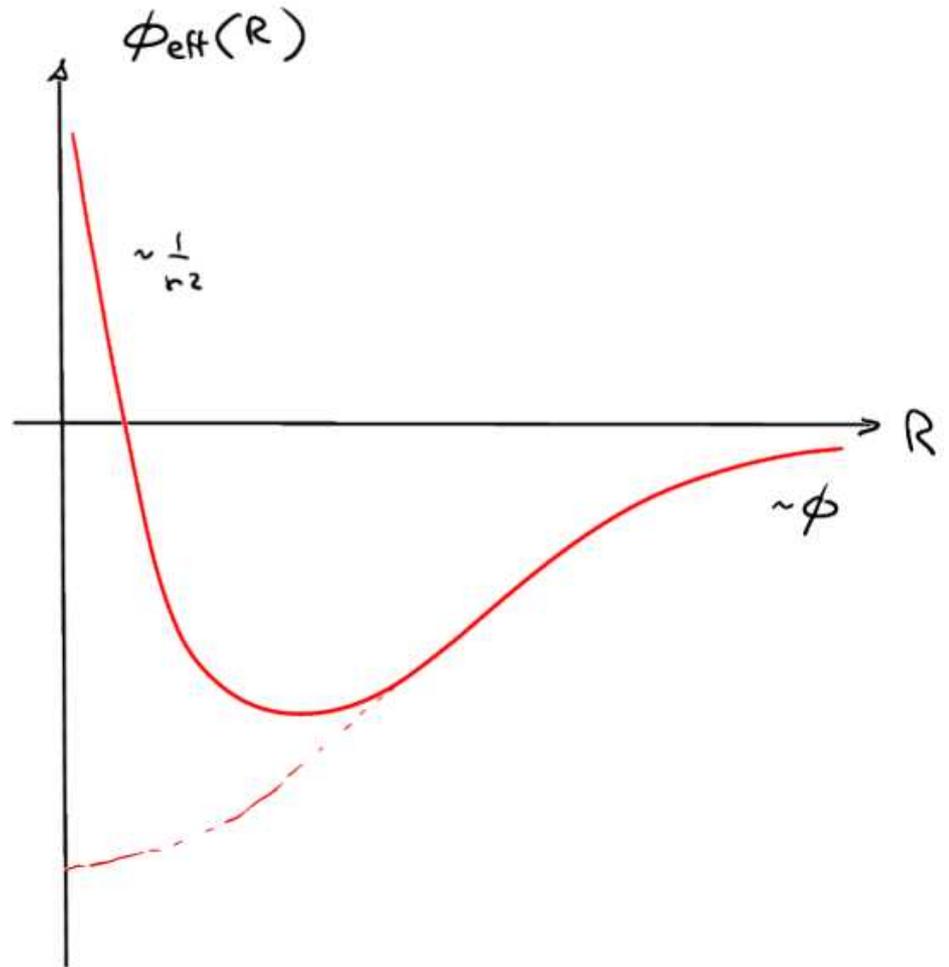
Illustration in the ==$z = 0$== plane

$$E = \frac{1}{2}\dot{R}^2 + \phi_{eff}(R)$$

4 cases

① $E > \phi_{eff}(\infty)$ except at $E = \phi_{eff}$

       $\dot{R} \neq 0$ unbound orbits

② $E < \min(\phi_{eff}(R))$ $\dot{R}^2 < 0$

       impossible

③ $\min(\phi_{eff}(R)) < E < \phi_{eff}(\infty)$

     orbit bounded between

     R_1 and R_2 (where $\dot{R} = 0$)

④ $E = \min(\phi_{eff}(R))$ (stationary point)

     $R_1 = R_2$ (circular orbit)

$\phi_{eff}(R)$

E_1

$\sim \frac{1}{r^2}$

R

E_3 $\sim \phi$

E_4

E_2
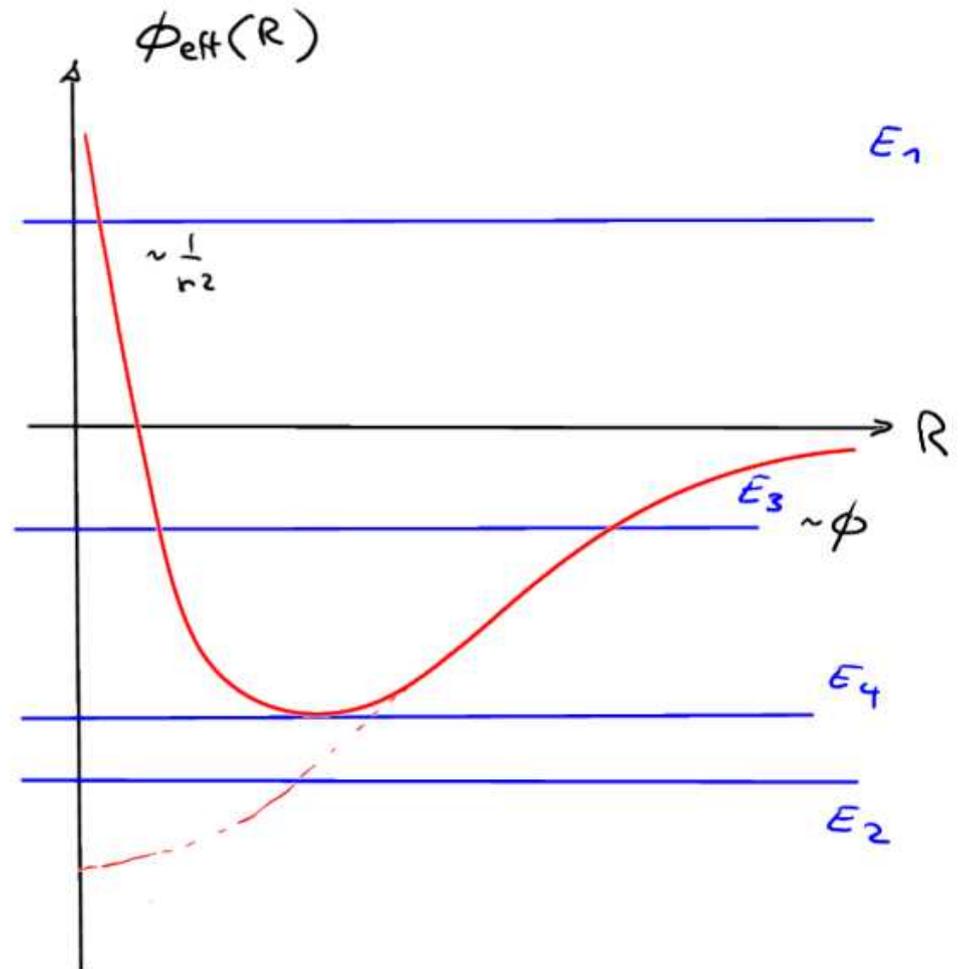
Stationary point $\qquad R = \dot{R} = 0 \qquad z = \dot{z} = 0$

from

$$
\begin{cases}
\ddot{R} & = -\dfrac{\partial \phi_{eff}}{\partial R}(R,z) \\[4mm]
\ddot{z} & = -\dfrac{\partial \phi_{eff}}{\partial z}(R,z)
\end{cases}
$$

ϕ_{eff}

R_g
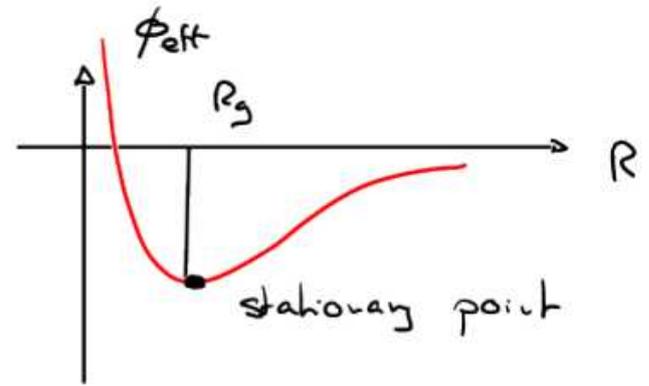
stationary point

$$
\begin{cases}
\dfrac{\partial \phi_{eff}}{\partial R} = 0 = \dfrac{\partial \phi}{\partial R} - \dfrac{L_z^2}{R^3} = 0 \\[4mm]
\dfrac{\partial \phi_{eff}}{\partial z} = 0 = \dfrac{\partial \phi}{\partial z} = 0
\end{cases}
\qquad\qquad \longrightarrow \quad
\begin{array}{l}
\text{by symmetry} \\
\text{where } z = 0
\end{array}
$$

$V_\theta = R\dot{\theta}$

$$
\boxed{R_g \quad \text{such that} \quad \left.\dfrac{\partial \phi}{\partial R}\right)_{R_g,0} = \dfrac{L_z^2}{R_g^3} = R_g \dot{\theta}^2 = \dfrac{V_\theta^2(R_g)}{R_g} = \dfrac{V_c^2(R_g)}{R_g}}
$$

$V_c^2 = R \left.\dfrac{\partial \phi}{\partial R}\right)_{R,0}$

R_g : guiding center

<mark>The stationary point in R_g in the meridional plane corresponds to a circular orbit</mark>

Examples

① Miyamoto - Nagai potential

$$\phi(R, z) = - \frac{GM}{\sqrt{R^2 + \left(a + \sqrt{z^2 + b^2}\right)^2}}$$

$$\phi_{eff}(R, z=0) = - \frac{GM}{\sqrt{R^2 + (a+b)^2}} + \frac{L_z^2}{2R^2}$$

Miyamoto Nagai Potential

Miyamoto Nagai Potential

Examples

① **Migamoto - Nagai potential**

$$\phi(R, z) = -\frac{GM}{\sqrt{R^2 + \left(a + \sqrt{z^2 + b^2}\right)^2}}$$

$$\phi_{\text{eff}}(R, z=0) = -\frac{GM}{\sqrt{R^2 + (a+b)^2}} + \frac{L_z^2}{2R^2}$$

② **Logarithmic potential**

$$\phi(R, z) = \frac{1}{2} V_o^2 \ln\left(R^2 + \frac{z^2}{q^2}\right)$$

$$\phi_{\text{eff}}(R, z=0) = \frac{1}{2} V_o^2 \ln(R^2) + \frac{L_z^2}{2R^2}$$

Logarithmic Potential
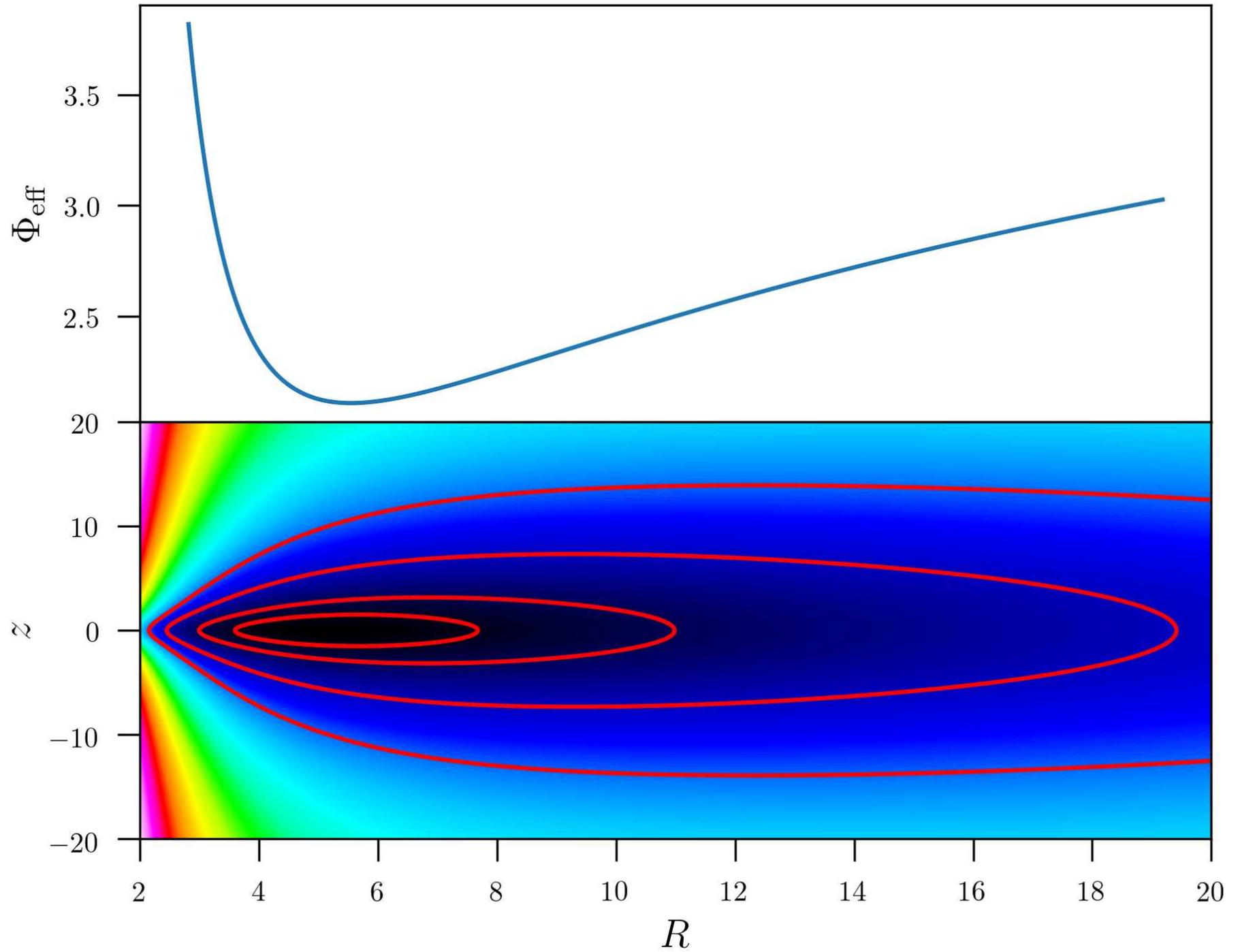
Logarithmic Potential
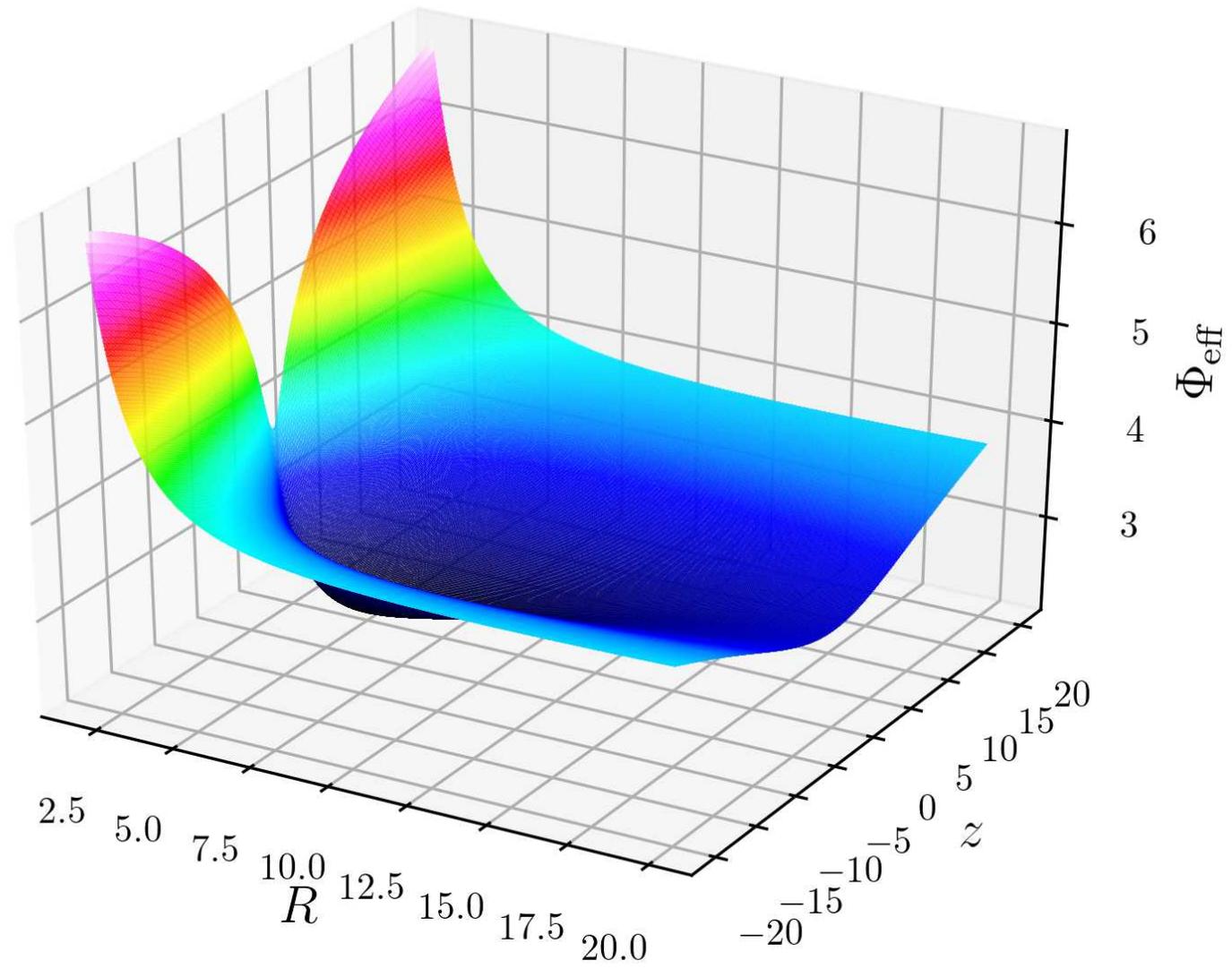
Circular orbits

angular speed $\qquad \dot{\theta} = \dfrac{L_z}{R_g^2}$

angular momentum $\qquad L_z$

energy $\qquad \phi_{eff} + \dfrac{L_z^2}{2 R_g}$

Note

> For a given angular momentum L_z, the circular orbit is the one that minimize the energy.

① $E_0 = \frac{1}{2} \dot{R}^2 + \frac{1}{2} \dot{z}^2 + \phi + \dfrac{L_z^2}{2R}$

② Dissipate energy $\qquad L_z = cte$

$\qquad \dot{z} \searrow \quad \dot{R} \searrow$

③ circular orbit
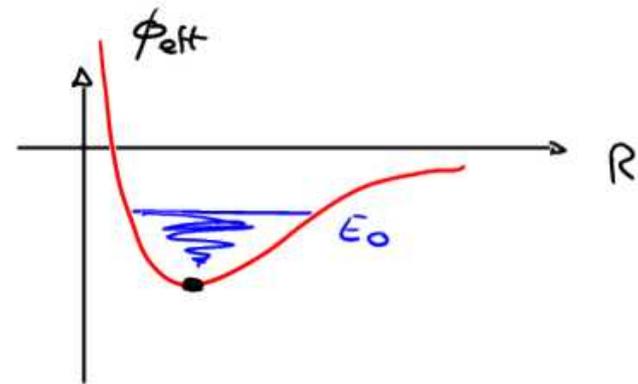
General solutions for the equations of motion

$$\begin{cases} \ddot{R} = -\dfrac{\partial \phi_{\text{eff}}}{\partial R}(R, z) \\[4mm] \ddot{z} = -\dfrac{\partial \phi_{\text{eff}}}{\partial z}(R, z) \end{cases}$$

no simple solutions 😦

need numerical integration

Hamilton's Equations

$$\vec{q} = \begin{cases} R \\ z \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{z} \end{cases} \qquad \vec{p} = \begin{cases} \dot{R} \\ \dot{z} \end{cases}$$

$$\begin{cases} \dot{q}_R = P_R & \equiv \dot{R} \\[4mm] \dot{q}_z = P_z & \equiv \dot{z} \\[4mm] \dot{P}_R = -\dfrac{\partial \phi_{\text{eff}}}{\partial q_R}(q_R, q_z) & \equiv -\dfrac{\partial \phi_{\text{eff}}}{\partial R}(R, z) \\[4mm] \dot{P}_z = -\dfrac{\partial \phi_{\text{eff}}}{\partial q_z}(q_R, q_z) & \equiv -\dfrac{\partial \phi_{\text{eff}}}{\partial z}(R, z) \end{cases}$$

Miyamoto − Nagai : ΔE = 0.2
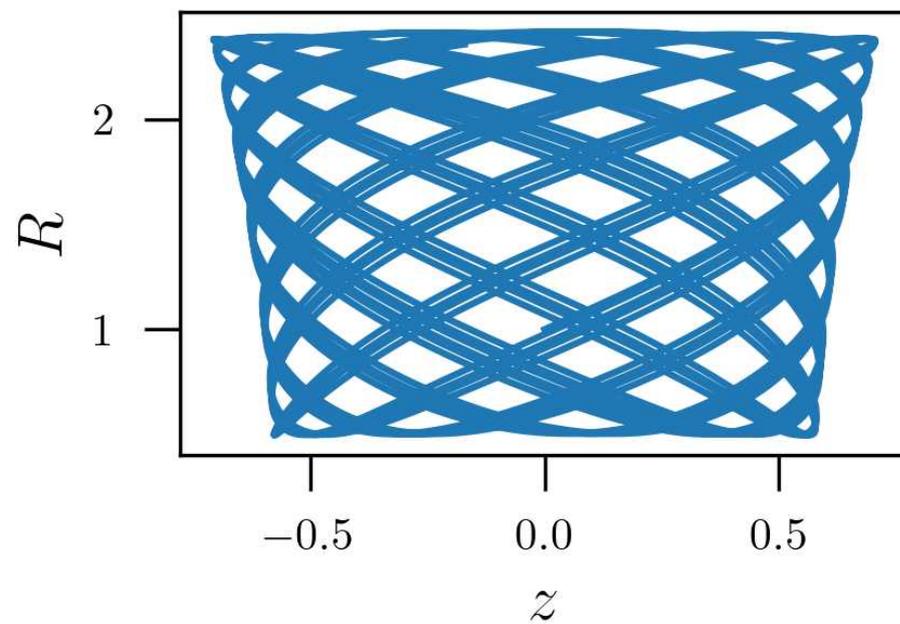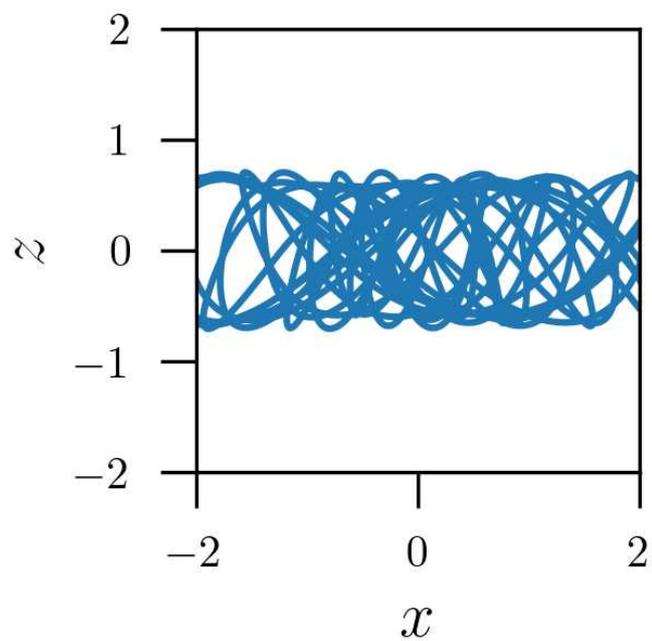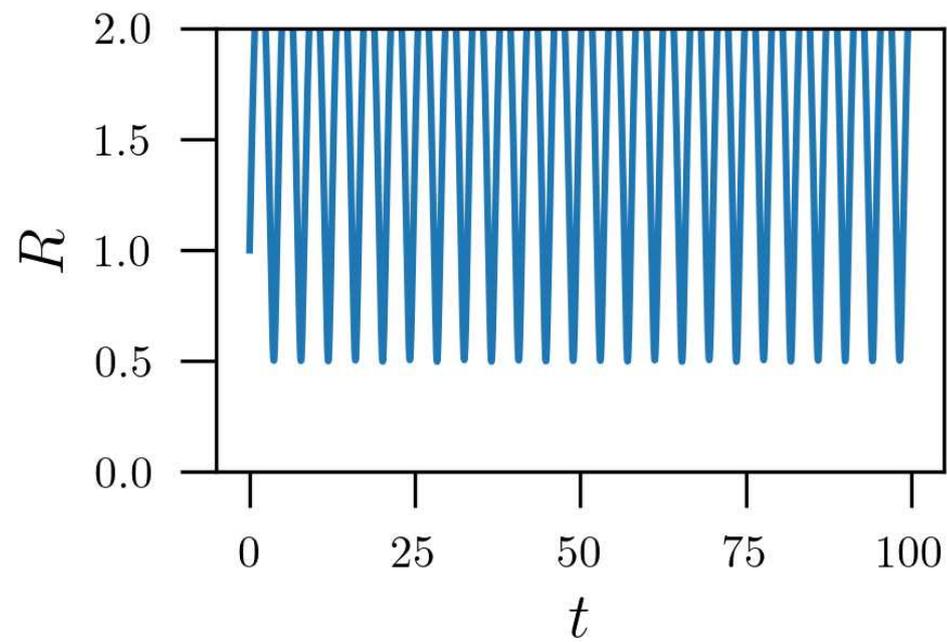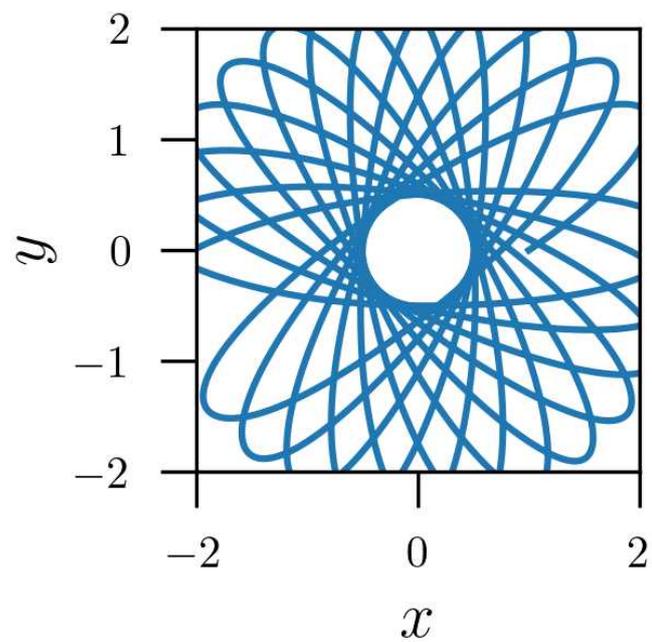
Miyamoto − Nagai : ΔE = 0.1
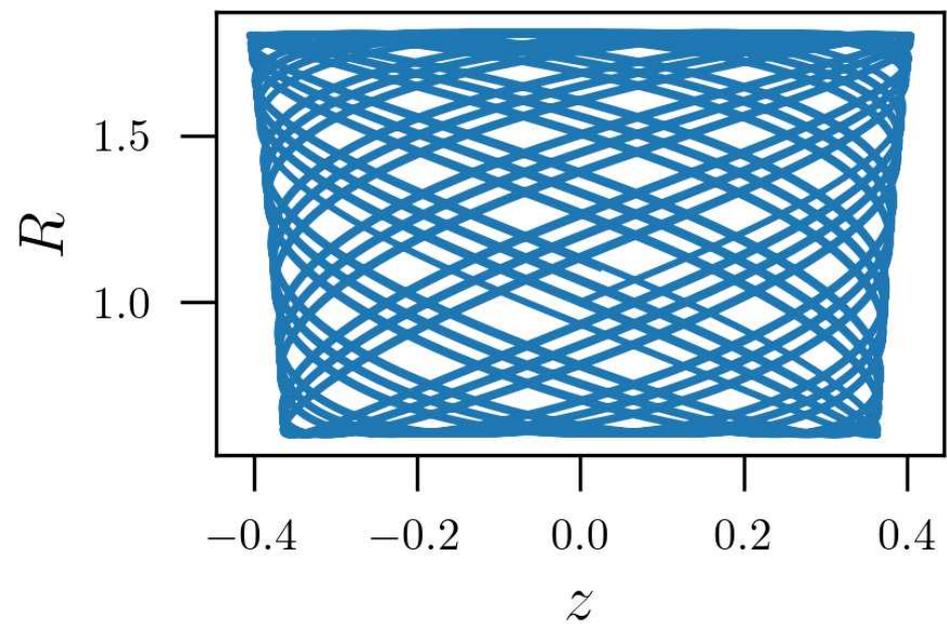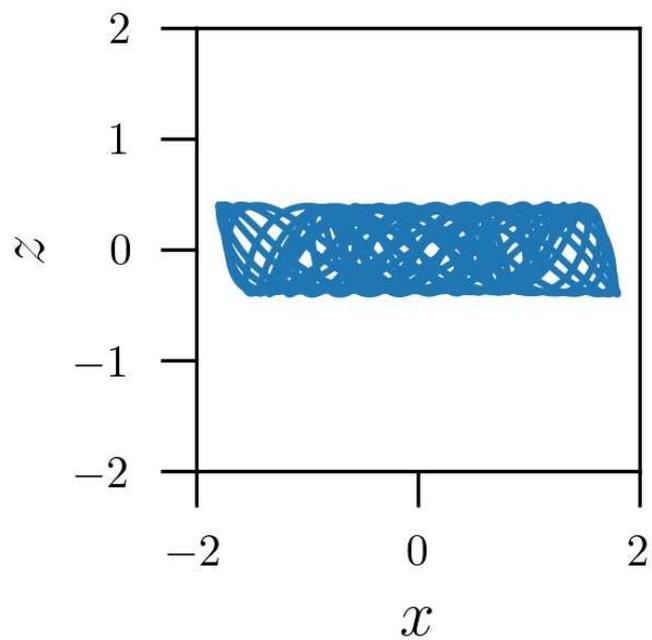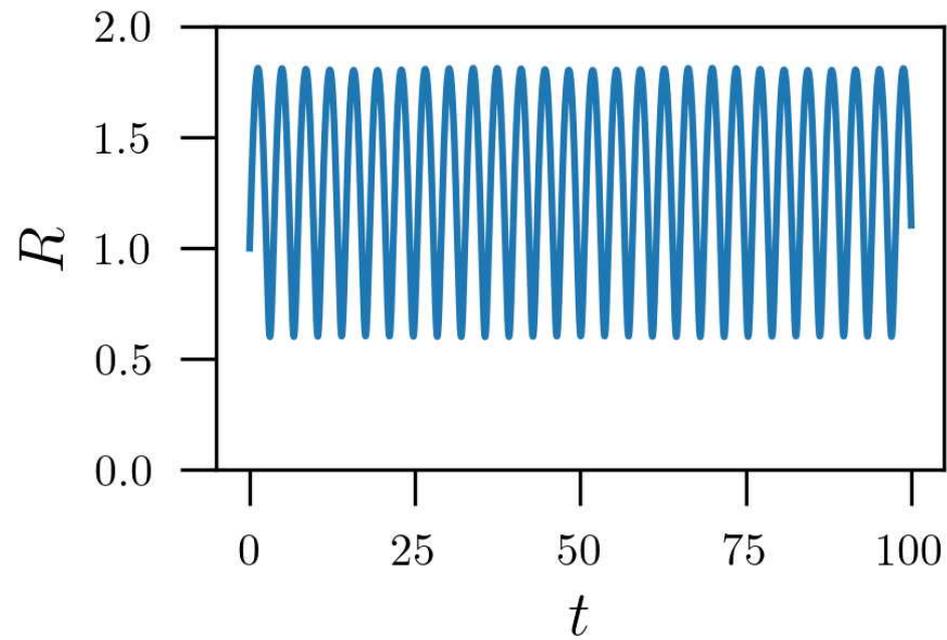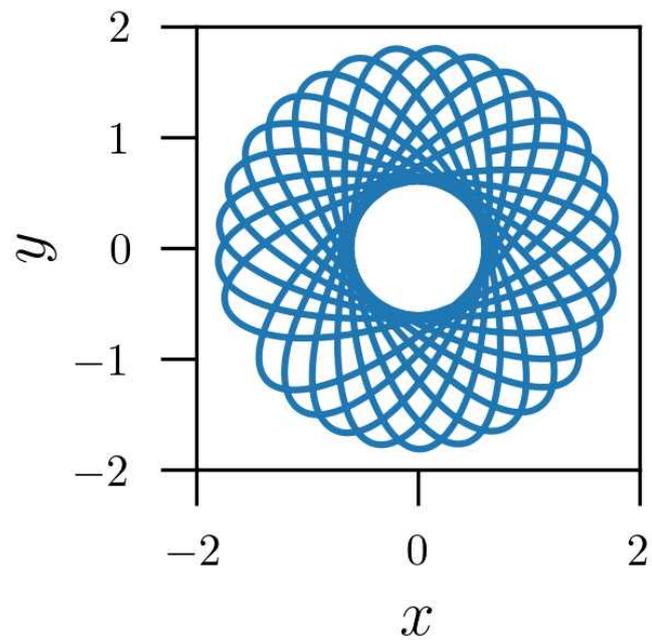
Miyamoto − Nagai : ΔE = 0.01
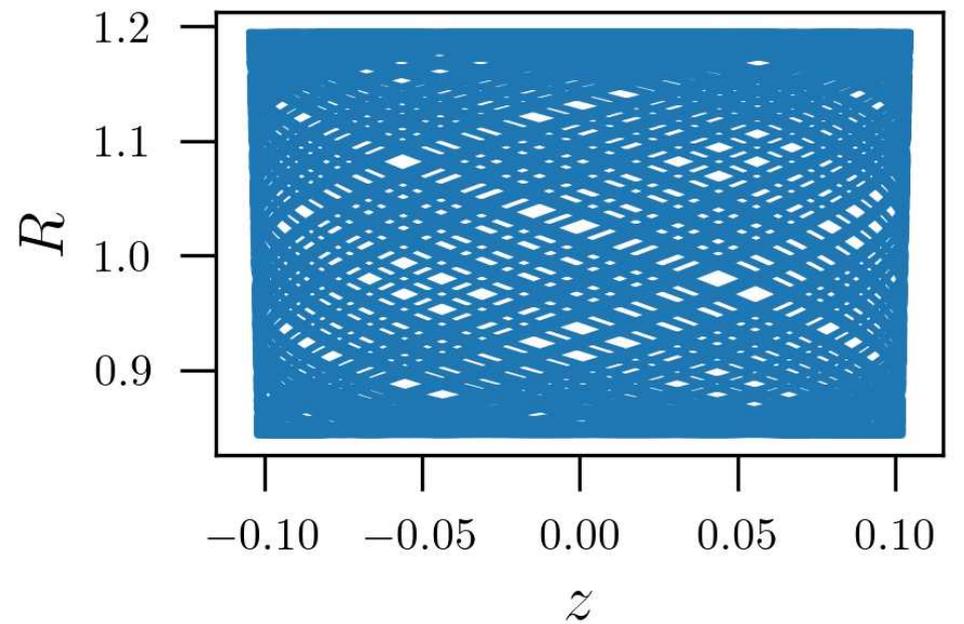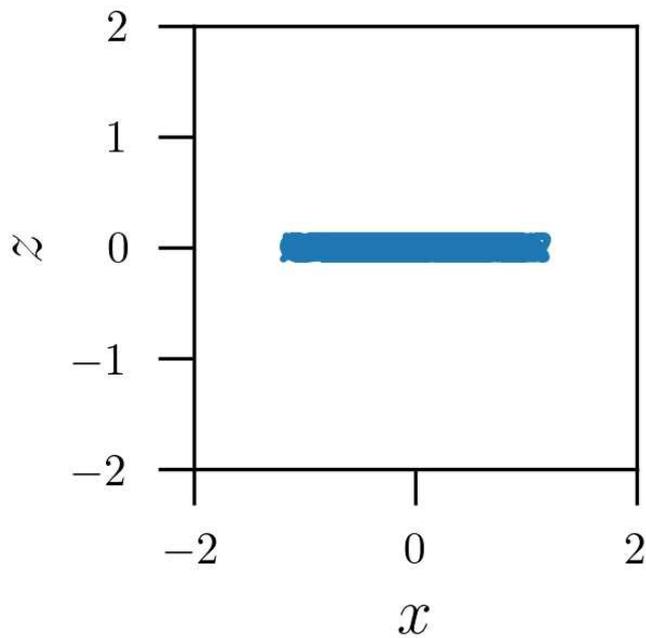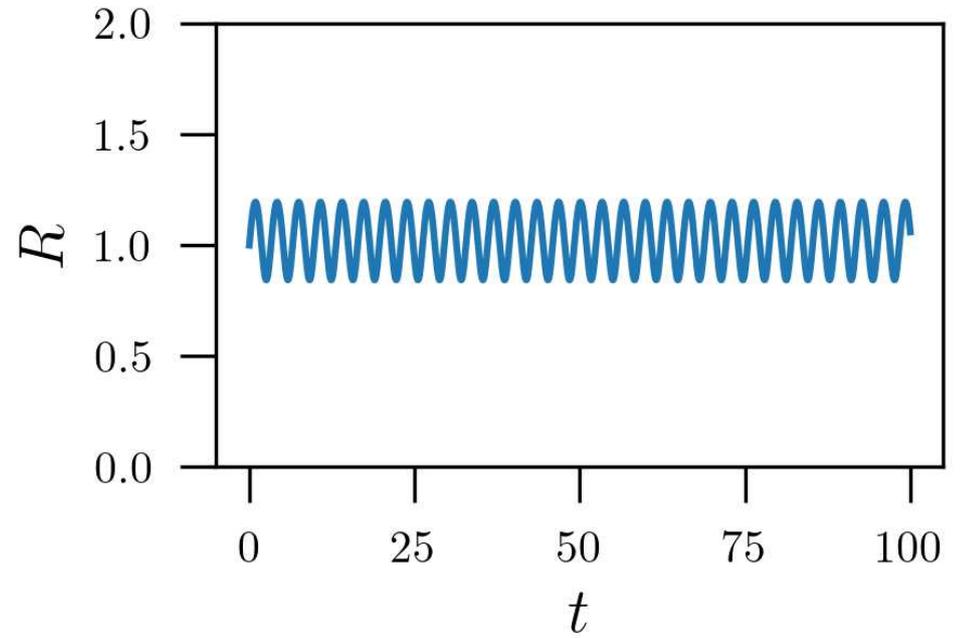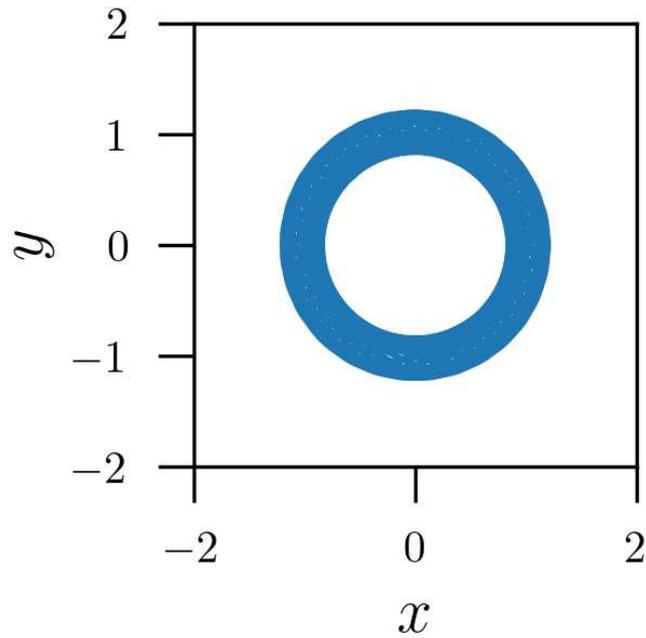
Miyamoto − Nagai : ΔE = 0.001

Miyamoto − Nagai : $\Delta E = 0$
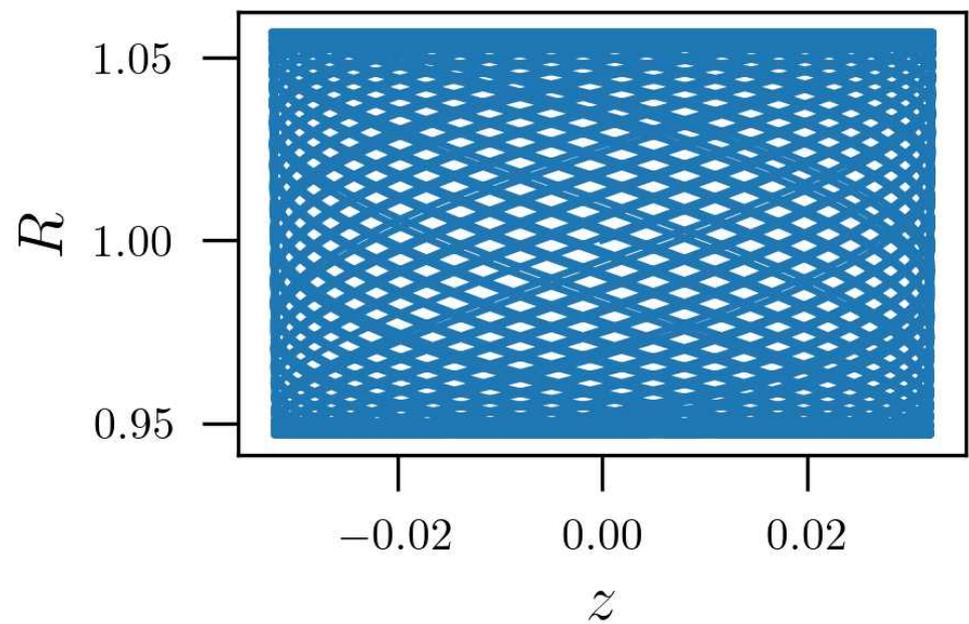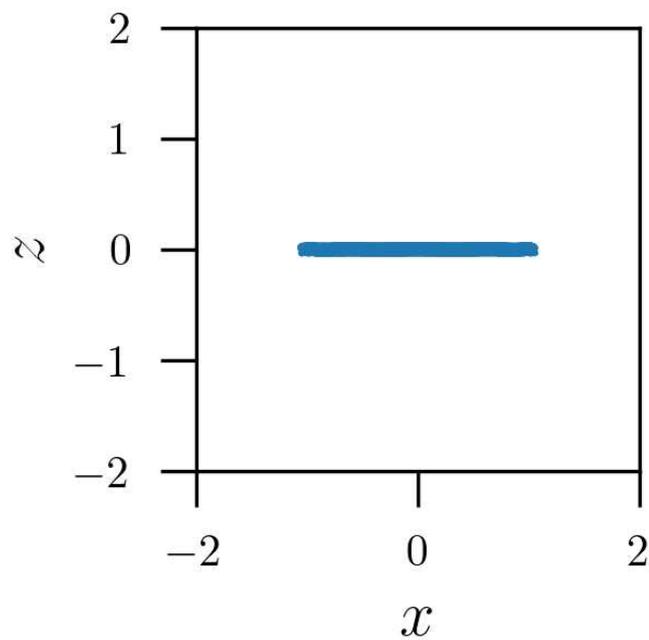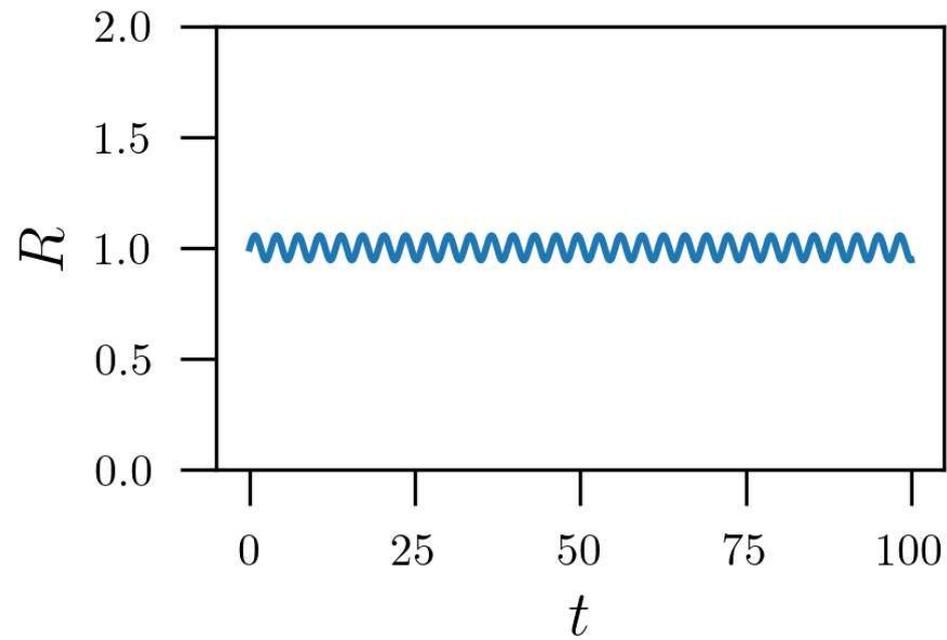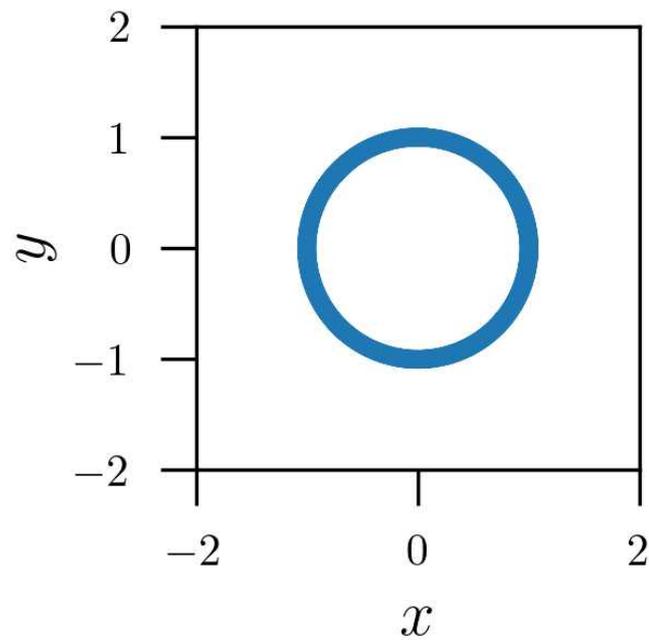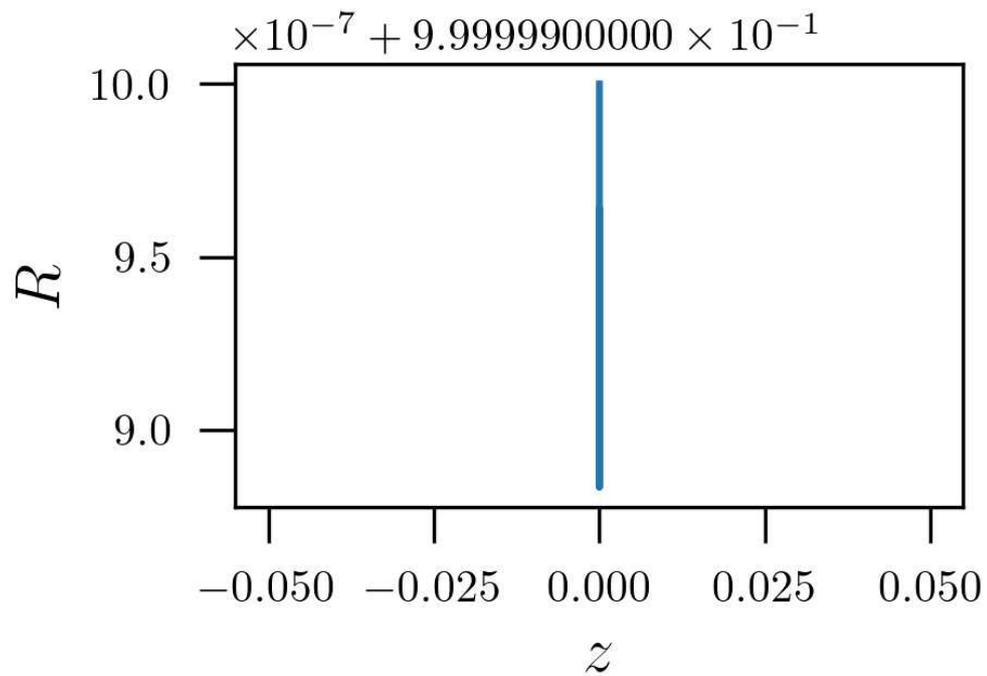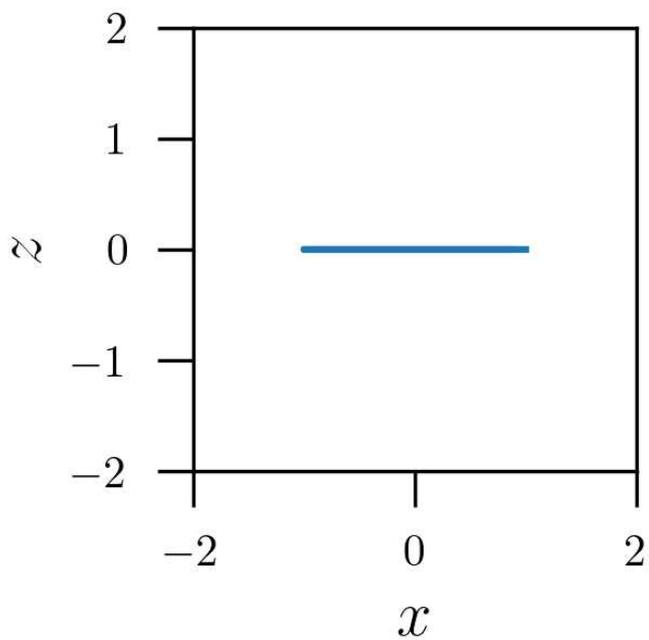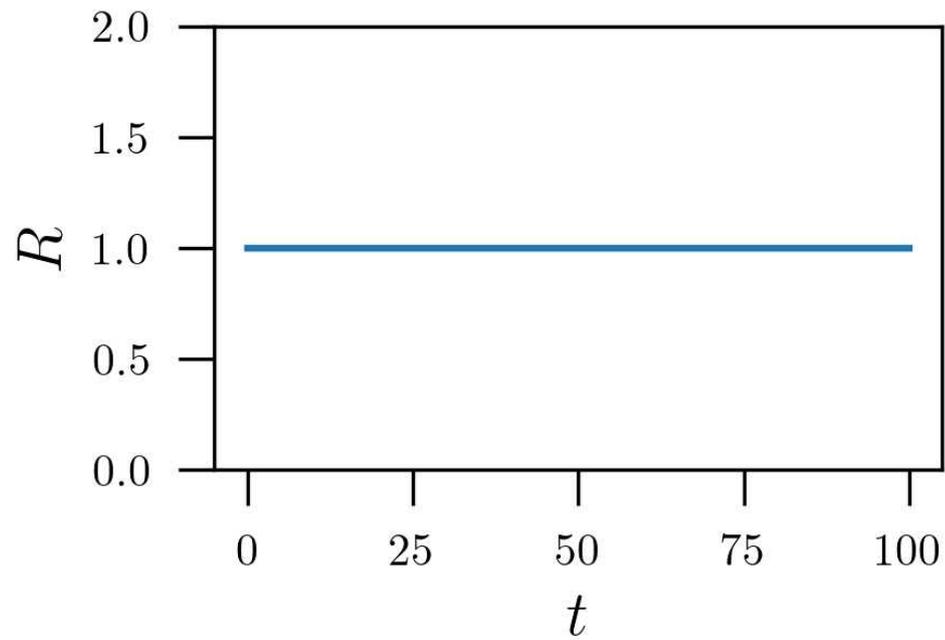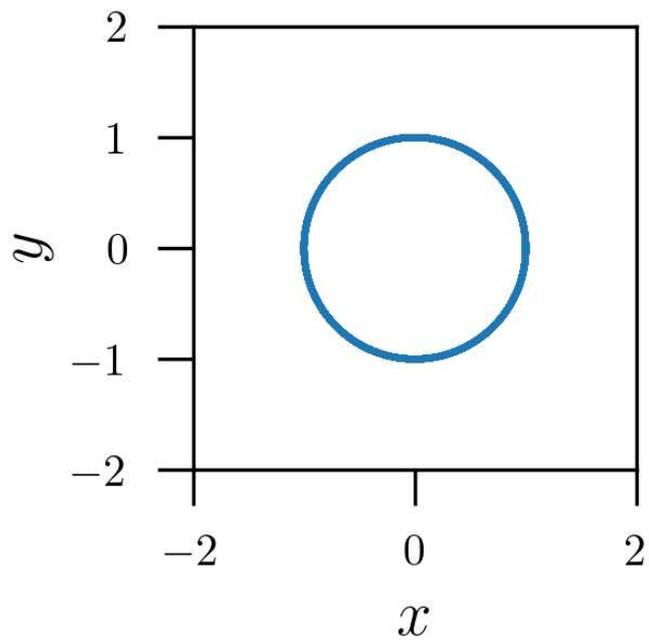
Nearly circular orbits

Nearly circular orbits

From the previous study of orbits in axisymmetric potentials

ϕ_{eff}

R_g

R

stationary point $\equiv$ circular orbits
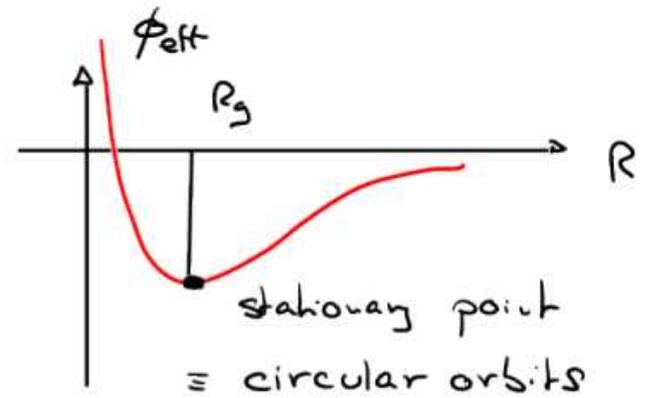
Goal   Study orbits in the neighbourhood of circular orbits

Justifications   In a disk galaxy, many stars are found in nearly circular orbits

Recall R_g : the guiding center

$$R_g \text{ such that } \left.\frac{\partial \phi}{\partial R}\right)_{R_g, 0} = \frac{L_z^2}{R_g^3} = R_g \dot{\theta}^2$$

We define $\qquad$ $x := R - R_g$ $\qquad$ the distance to the guiding center R_g
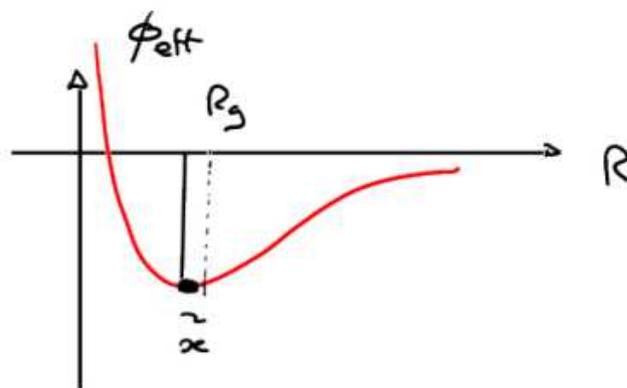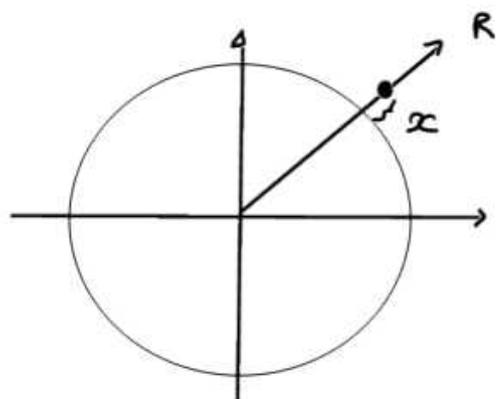
Taylor expansion of ϕ_{eff} around $R = R_g$, $z = 0$

$$\phi_{eff}(R, z) \cong \phi_{eff}(R_g, 0) + \underbrace{\frac{\partial \phi_{eff}}{\partial R}(R_g, 0)}_{= 0 \quad \text{min}}(R - R_g) + \underbrace{\frac{\partial \phi_{eff}}{\partial z}(R_g, 0)}_{= 0 \quad \text{sym.}} z$$

$$+ \frac{1}{2}\frac{\partial^2 \phi_{eff}}{\partial R^2}(R_g, 0)(R - R_g)^2 + \frac{1}{2}\frac{\partial^2 \phi_{eff}}{\partial z^2}(R_g, 0) z^2$$

$$+ \frac{1}{2}\underbrace{\frac{\partial^2 \phi_{eff}}{\partial z \partial R}(R_g, 0)}_{= 0}(R - R_g) z + O\left(((R - R_g)z)^3\right)$$

$= 0$ $\qquad$ $\phi_{eff}(R, z)$ must be sym. with respect to $z = 0$

$$\phi_{eff}(R, z) \cong \phi_{eff}(R_g, 0) + \frac{1}{2} \frac{\partial^2 \phi_{eff}}{\partial R^2}(R_g, 0) x^2 + \frac{1}{2} \frac{\partial^2 \phi_{eff}}{\partial z^2}(R_g, 0) z^2$$

Definition
$$\begin{cases} \varkappa^2(R_g) = \left(\dfrac{\partial^2 \phi_{eff}}{\partial R^2} \right)_{(R_g, 0)} \\[4mm] \nu^2(R_g) = \left(\dfrac{\partial^2 \phi_{eff}}{\partial z^2} \right)_{(R_g, 0)} \end{cases}$$

$$[\phi] = \left(\frac{m}{s} \right)^2$$

$$\left[\left(\frac{\partial^2 \phi}{\partial R^2} \right)^{\frac{1}{2}} \right] \equiv \left[\left(\frac{\partial^2 \phi}{\partial z^2} \right)^{\frac{1}{2}} \right] = \frac{1}{s}$$

frequency

Equations of motion near R_g

$$\begin{cases} \ddot{R} = - \dfrac{\partial \phi_{eff}}{\partial R}(R, z) \\[4mm] \ddot{z} = - \dfrac{\partial \phi_{eff}}{\partial z}(R, z) \end{cases} \Rightarrow \begin{cases} \ddot{x} = - \varkappa^2(R_g)\, x \\[4mm] \ddot{z} = - \nu^2(R_g)\, z \end{cases}$$

$$
\begin{cases}
\ddot{x} = -\varkappa^2(R_g)\, x \\[2em]
\ddot{z} = -\nu^2(R_g)\, z
\end{cases}
$$
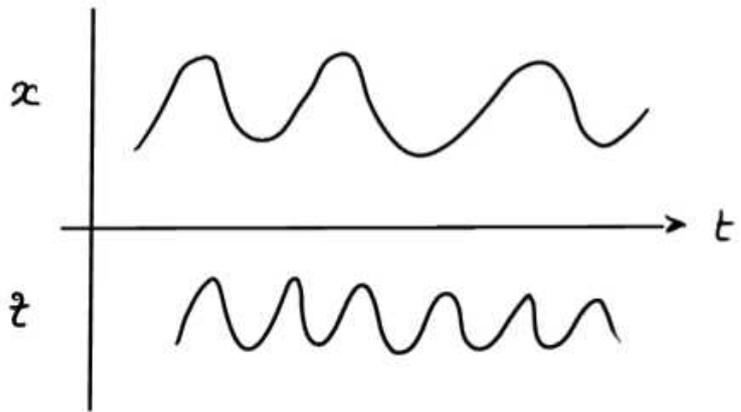
Two decoupled harmonic oscillators
with frequencies $\varkappa$ and ν

$\varkappa$: epicycle (radial) frequency

ν : vertical frequency
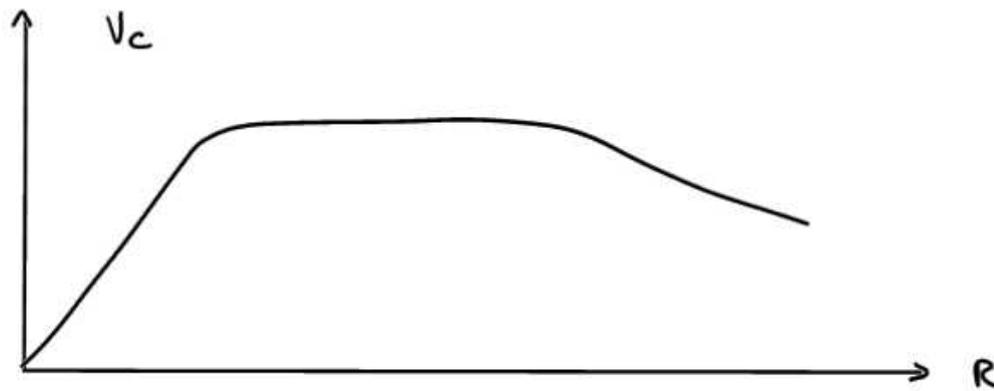
Expressions of κ and ν from the total potential

$$\kappa^2(R_g) = \left.\frac{\partial^2 \phi_{eff}}{\partial R^2}\right)_{(R_g,0)} = \left.\frac{\partial^2 \phi}{\partial R^2}\right)_{(R_g,0)} + 3\frac{L_z^2}{R_g^4}$$

(in red, right) $L_z^2 = V_c^2 R_g^2$
$= \left. R_g^3 \frac{\partial \phi}{\partial R}\right)_{R_g}$

$$= \left.\frac{\partial^2 \phi}{\partial R^2}\right)_{(R_g,0)} + \frac{3}{R_g}\left.\frac{\partial \phi}{\partial R}\right)_{(R_g,0)}$$

(in red) circ. frequency $\quad \Omega^2 = \left.\frac{1}{R_g}\frac{\partial \phi}{\partial R}\right)_{R_g}$

$$= \left.\frac{\partial^2 \phi}{\partial R^2}\right)_{(R_g,0)} + 3\Omega^2$$

(in red) $\Omega^2 = \dfrac{V_c^2}{R^2}$

$$= \left(R\frac{\partial(\Omega^2)}{\partial R} + 4\Omega^2\right)_{(R_g,0)}$$

$$= \left(\frac{1}{R}\frac{\partial(V_c^2)}{\partial R} + 2\Omega^2\right)_{(R_g,0)} = \left(\frac{1}{R}\frac{\partial(V_c^2)}{\partial R} + 2\frac{V_c^2}{R^2}\right)_{(R_g,0)}$$

$$\nu^2(R_g) = \left.\frac{\partial^2 \phi_{eff}}{\partial z^2}\right)_{(R_g,0)} = \left.\frac{\partial^2 \phi}{\partial z^2}\right)_{(R_g,0)}$$

Note : $\varkappa$ depends only on V_c

Periods :

$$
\begin{cases}
\text{radial} & T_R := \dfrac{2\pi}{\varkappa} \\[2em]
\text{vertical} & T_z := \dfrac{2\pi}{\nu} \\[2em]
\text{azimuthal} & T_\theta := \dfrac{2\pi}{\Omega}
\end{cases}
$$

Radial dependency of $\mathbf{æ}$, $\mathbf{r}$ for a typical galaxy

$$\Omega = \frac{V_c}{R}$$

$V_c = cte$ ②
$\Omega \sim \frac{1}{R}$
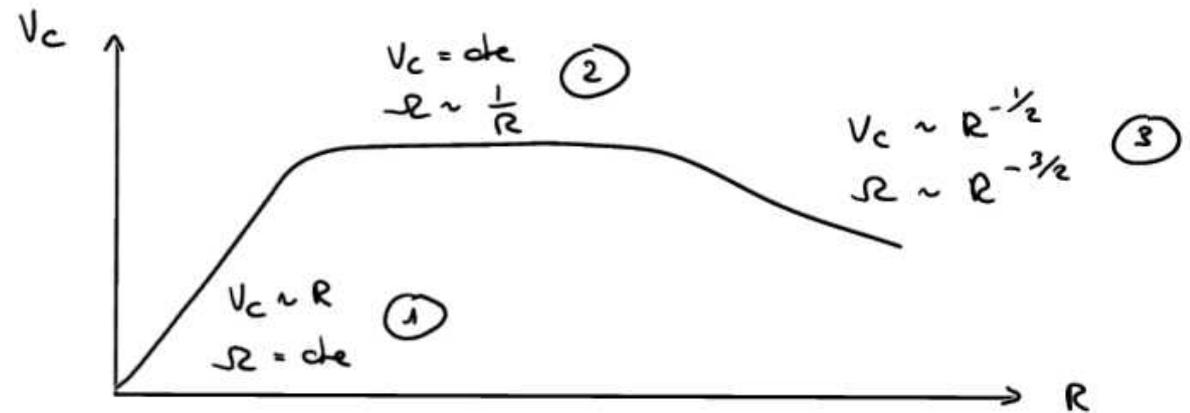
$V_c \sim R^{-\frac{1}{2}}$ ③
$\Omega \sim R^{-\frac{3}{2}}$

$V_c \sim R$ ①
$\Omega = cte$

- ① near the center

___

$V_c \sim R$ (rigid rotation) $\Rightarrow$ $\Omega = cte$

$æ^2 = R \frac{d}{dR}(\Omega^2) + 4\Omega^2$ $\Rightarrow$ $æ^2 = 4\Omega^2$

$$\boxed{æ \sim 2\Omega}$$

Radial dependency of $\varkappa$, ν for a typical galaxy
_____

$$\Omega = \frac{V_c}{R}$$

Near the graph:
- $V_c = cte$, $\Omega \sim \frac{1}{R}$ ②
- $V_c \sim R$, $\Omega = cte$ ①
- $V_c \sim R^{1/2}$, $\Omega \sim R^{-3/2}$ ③
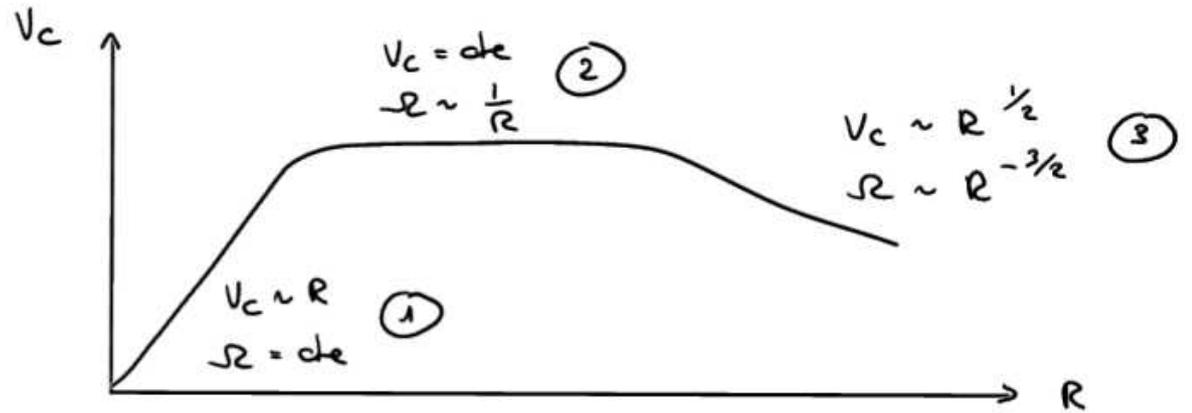
• ② flat rotation part
_____

$$V_c = cte \qquad \qquad = \qquad \Omega \sim \frac{1}{R}$$

$$\varkappa^2 = \frac{1}{R}\frac{\partial}{\partial R}(V_c^2) + 2\Omega^2 \qquad \longrightarrow \qquad \varkappa^2 = 2\Omega^2$$

$$\boxed{\varkappa \sim \sqrt{2}\,\Omega}$$

Radial dependency of $\varkappa$, ν for a typical galaxy

$$\Omega = \frac{V_c}{R}$$
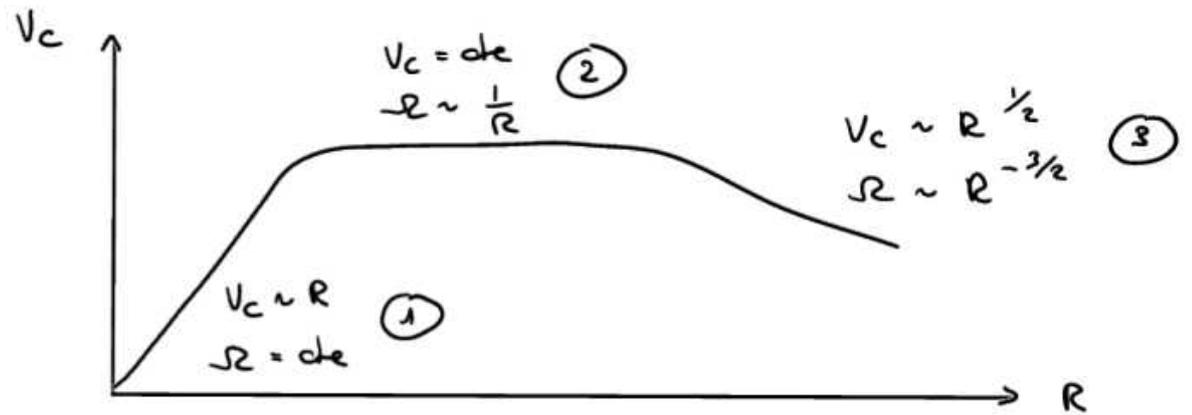
- ③ further out

$$V_c \sim R^{-1/2} \quad (\text{Keplerian decrease})$$

$$\Omega = \frac{V_c}{R} \sim R^{-3/2}$$

$$\varkappa^2 = \frac{1}{R} \frac{\partial}{\partial R}(V_c^2) + 2\frac{V_c^2}{R^2} \sim R^{-3}$$

$$\boxed{\varkappa = \Omega}$$

Thus, in general

$$\Omega \leq \mathcal{H} \leq 2\Omega$$
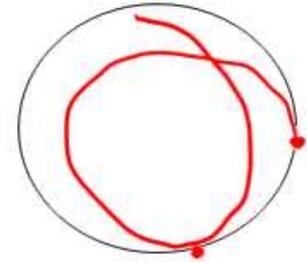
Integrals of motions

$$
\begin{cases}
\ddot{x} = -\varkappa^2(R_g)\, x \\[2mm]
\ddot{z} = -\nu^2(R_g)\, z
\end{cases}
$$

$\Rightarrow$ Two integrals of motion

(one for each oscillator)

1) $H_R = \dfrac{1}{2}\dot{x}^2 + \dfrac{1}{2}\varkappa^2 x^2$

2) $H_z = \dfrac{1}{2}\dot{z}^2 + \dfrac{1}{2}\nu^2 z^2$

Thus, if a star oscillates near a circular orbit :

3 integrals of motions $\qquad L_z,\ H_R,\ H_z$

Total Hamiltonian (near a circular orbit of radius R_g)

$$H\left(R, \dot{R}, \theta, \dot{\theta}, z, \dot{z}\right) = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2\right) + \phi(R, z)$$

$$= \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \phi(R, z) + \frac{L_z^2}{2R^2} \qquad \textcolor{red}{L_z = R^2\dot{\theta}}$$

$$\underbrace{\phantom{\frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \phi(R, z) + \frac{L_z^2}{2R^2}}}_{\phi_{eff}(R, z)}$$

$$= \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \phi_{eff}(R_s, 0) + \frac{1}{2}\varkappa^2(R - R_s)^2 + \frac{1}{2}\nu^2 z^2$$

$$H(R, p_R, z, p_z) = H_R(R, p_R) + H_z(z, p_z) + \phi_{eff}(R_s, 0)$$

Orbital motions

$$\begin{cases} \ddot{x} = -\mathscr{x}^2(R_g)\, x \\[2mm] \ddot{z} = -\nu^2(R_g)\, z \end{cases} \qquad + \qquad R^2\dot{\theta} = L_z$$

Solutions

① motion in z

$$z(t) = Z \cos(\nu t + \xi)$$

② motion in x

$$x(t) = X \cos(\mathscr{x} t + \alpha)$$

Note valid only for small oscillations

as long as $\nu^2 = \dfrac{\partial^2 \phi}{\partial z^2} \simeq$ cte

ie $\rho_{disk} \simeq$ cte $\qquad \left(\nu^2 = \dfrac{\partial^2 \phi}{\partial z^2} = 4\pi G \rho \right)$

→ $z <$ disk scale lenght

~ 300 pc

③ motion in θ

$$L_z = R^2 \dot{\theta}$$

$$\theta(t) = L_z \int_{t_0}^{t} dt' \frac{1}{R^2(t)} = L_z \int_{t_0}^{t} dt' \frac{1}{(R_g + x(t))^2}$$

$$\simeq \frac{L_z}{R_g^2} \int_{t_0}^{t} dt' \frac{1}{\left(\frac{x}{R_g} + 1\right)^2} \overset{\text{Taylor}}{\simeq} \Omega_g \int_{t_0}^{t} dt' \left(1 - \frac{2x(t)}{R_g}\right)$$

$$\textcolor{red}{\Omega_g = \frac{L_z}{R_g^2}}$$

introducing $x(t) = X \cos(\varkappa t + \alpha)$

$$\boxed{\theta(t) = \Omega_g \cdot t - \frac{2 \Omega_g}{\varkappa} \frac{X}{R_g} \sin(\varkappa t + \alpha) + \theta_0}$$

$\underbrace{\qquad}$ motion of the guiding center along the circular orbit

$\underbrace{\qquad}$ oscillations

New carthesian system x, y, z with an origin that
follows the guiding center

$s = \infty \cdot R_g$

R_g
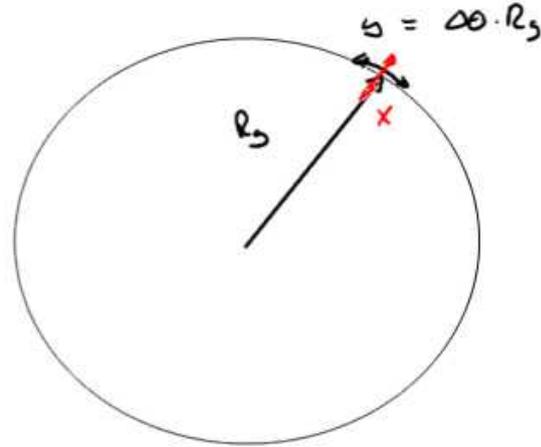
$$\begin{cases} R(t) = R_g \\ \Theta(t) = \Omega_g t + \Theta_0 \end{cases}$$

Then, from

$$\Theta(t) = \Omega_g \cdot t - \frac{2\Omega_g}{\omega} \frac{X}{R_g} \sin(\omega t + \alpha) + \Theta_0$$

$$\underbrace{\hspace{5cm}}_{\Delta\Theta = \frac{y}{R_g}}$$

$$y = -\frac{2\Omega_g}{\omega} X \sin(\omega t + \alpha)$$

$$\boxed{y(t) = -Y \sin(\omega t + \alpha)}$$ $Y := \frac{2\Omega_g}{\omega} X$

Complete solution

$$\begin{cases} x(t) = X \cos(\varkappa t + \alpha) \\[6pt] y(t) = -Y \sin(\varkappa t + \alpha) \\[6pt] z(t) = Z \cos(\nu t + \xi) \end{cases}$$

$\Big\}$ ellipse

$$Y = \frac{2 \Omega_y}{\varkappa} X$$
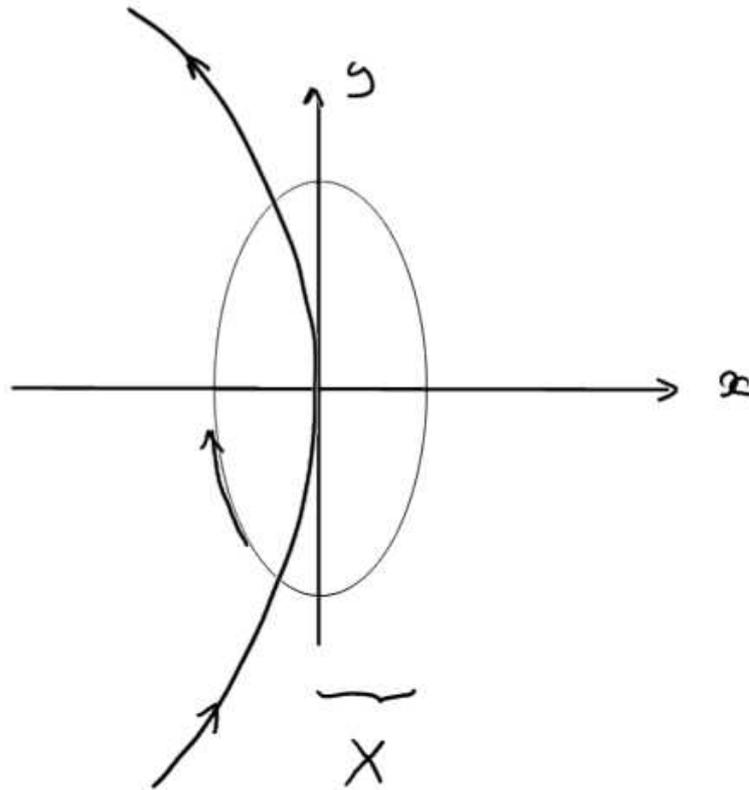
Radial dependency for a typical galaxy
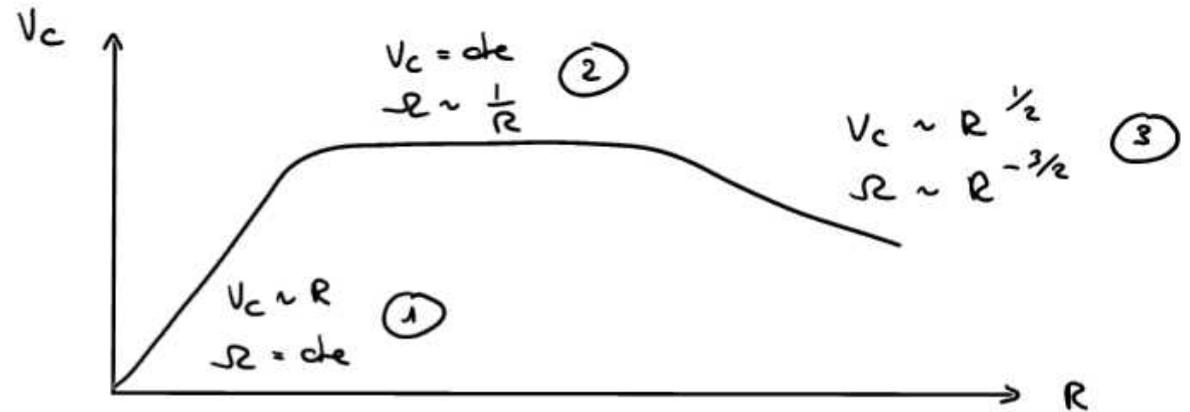
① **near the center**

$x = 2\Omega$ $\dfrac{X}{Y} = 1$ circle

② **flat rotation part**

$x = \sqrt{2}\,\Omega$ $\dfrac{X}{Y} = \dfrac{\sqrt{2}\,\Omega}{2\Omega}$ $X < Y$

③ **further out**

$x = \Omega$ $\dfrac{X}{Y} = \dfrac{\Omega}{2\Omega}$ $X < Y$

$\kappa/\Omega = 2.0$

$$\kappa/\Omega = 1.5$$

$$\kappa/\Omega = 1.0$$

$\kappa/\Omega = 10.0$

The End