Astrophysics III, Dr. Yves Revaz | EPFL

4th year physics | Exercises week 6
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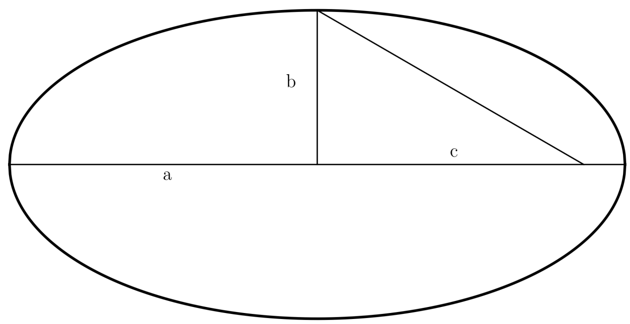
Astrophysics III: Stellar and galactic dynamics

Solutions

Problem 1:



The ellipse equation is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

the focii are at

$$c = \pm\sqrt{a^2 - b^2}$$

and the eccentricity is defined as

$$e = \frac{c}{a}$$

Using these relations, we write

$$e^2 = \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2}$$
$$y^2 = b^2 - \frac{b^2}{a^2}x^2 = \frac{b^2}{a^2}(a^2 - x^2) = (1 - e^2)(a^2 - x^2)$$

We apply a coordinate transformation now: Let $x = x' + ae\ (= x' + c)$. This gives

$$y^2 = (1 - e^2)\left(a^2 - (x' + ae)^2\right) \tag{2}$$

Now we show that the equation of Keplerian orbits (3) can be written in the same form as (2). The Keplerian orbits are defined as

$$r(\varphi) = \frac{a(1-e^2)}{1+e\cos(\varphi)} \tag{3}$$

with $x' = r\cos(\varphi)$, $y = r\sin(\varphi)$

$$\begin{aligned}
r(1+e\cos(\varphi)) &= r + er\cos(\varphi) = r + ex' \\
&= a(1-e^2) \\
r^2 &= a^2(1-e^2)^2 + e^2x'^2 - 2a(1-e^2)ex' \\
&= x'^2 + y^2 \\
y^2 &= a^2(1-e^2) + x'^2(e^2-1) - 2a(1-e^2)ex' \\
&= (1-e^2)[a^2(1-e^2) - x'^2 - 2aex'] \\
&= (1-e^2)[a^2 - a^2e^2 - (x'+ae)^2 + a^2e^2] \\
&= (1-e^2)[a^2 - (x'+ae)^2]
\end{aligned}$$

which is exactly equation (2) again.

Problem 2:

First law : The orbit of a planet is an ellipse with the Sun at one of the two foci. This was shown in question 1.

Second law : A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. Consider the Sun to be at the centre of the coordinate system and a planet at the position $\vec{x}(t)$ with a velocity $\vec{v}(t)$. Consider first the areas sweeps out during an infinitesimal time dt. This area will be:

$$\delta A = \frac{1}{2}\left|\vec{x}(t) \times \mathrm{d}\vec{x}(t)\right|, \tag{4}$$

where $\mathrm{d}\vec{x} = \vec{v}\mathrm{d}t$. So,

$$\delta A = \frac{1}{2}\,\mathrm{d}t\left|\vec{x}(t) \times \vec{v}(t)\right| = \frac{1}{2}\,\mathrm{d}t|\vec{L}|, \tag{5}$$

with $\vec{L}$, the angular momentum (consider a body of unit mass). As the latter is conserved in a spherical potential, δA is independent of the time and of the position along the orbit. We can thus write for any interval time ΔT such that $\Delta T = t_2 - t_1$:

$$A = \int_{t_1}^{t_2} \delta A = \frac{1}{2}\,|\vec{L}| \int_{t_1}^{t_2} \mathrm{d}t = \frac{1}{2}\,|\vec{L}|\,\Delta T, \tag{6}$$

which demonstrates the law.

Third law : The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit. From the previous law, we got a result of the form

$$A = \frac{1}{2}L\Delta T,$$

with L the magnitude of the angular momentum of a test particle of unit mass. For a full orbit, $\Delta T \equiv T$ is the period, and A is the area of the ellipse:

$$A = \pi ab = \pi a^2 \sqrt{1-e^2}.$$

Let us now turn our attention to L. There are different ways of calculating it, but we will use the Vis-Viva equation:

$$v^2(r) = GM\left(\frac{2}{r} - \frac{1}{a}\right).$$

Let's take , e.g., $r = r_{\text{min}}$:

$$v^2(r_{\text{min}}) = GM\left(\frac{2}{r_{\text{min}}} - \frac{1}{a}\right) = GM\left(\frac{2a - r_{\text{min}}}{r_{\text{min}} a}\right)$$

but $2a - r_{\text{min}}$ is r_{max}, and we also have $r_{\text{min}} r_{\text{max}} = b^2$. Together we get:

$$v^2(r_{\text{min}}) = \frac{GM}{a}\left(\frac{b}{r_{\text{min}}}\right)^2$$

So we have

$$L = L(r_{\text{min}}) = \sqrt{\frac{GM}{a}}b = \sqrt{\frac{GM}{a}}a\sqrt{1-e^2}$$

Thus the period is

$$T = \frac{2A}{L} = 2\frac{\pi a^2 \sqrt{1-e^2}}{\sqrt{\frac{GM}{a}}a\sqrt{1-e^2}} = 2\frac{\pi a^{3/2}}{\sqrt{GM}},$$

or

$$T^2 = \frac{4\pi^2}{GM}a^3.$$

Throughout this exercise, we took a test particle of unit mass to make dealing with the units easier. (Usually, $L = mrv$ and not only $L = rv$ which we used here.)