Introduction to Differentiable Manifolds
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Solutions Series 4 - More on Tangent vectors and smooth embeddings
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Notes:

Local parametrizations. For some problems it is convenient to work with local parametrizations rather than charts. Recall that a **local parametrization** (or **inverse chart**) of a topological n-manifold M is a triple (V, U, ϕ), where $V \subseteq \mathbb{R}^n$ and $U \subseteq M$ are open sets, and $\phi : V \to U$ is a homeomorphism. An **inverse atlas** on M is a set $\mathcal{A}$ of local parametrizations of N whose images cover M. An inverse atlas $\mathcal{A}$ is **smooth** if the **transition map** $\psi^{-1} \circ \phi$ between any two local parametrizations (V, U, ϕ), $(V', U', \psi) \in \mathcal{A}$ is smooth. A smooth inverse atlas defines a smooth structure.

Notation for tangent vectors. In some solutions we will use the following notation for tangent vectors. Let p be a point of a smooth n-manifold M, let (V, U, φ) be a local parametrization of M such that $p \in U$, and let $\widetilde{p} = \phi^{-1}(p)$. Then for each vector $\widetilde{v} \in \mathbb{R}^n$, we denote

$$[\phi, \widetilde{v}]_p := D_{\widetilde{p}}\phi(D_{\widetilde{p}}\widetilde{v}) \in T_pM.$$

Warning: *Be careful if you read the notes of last year: this tangent vector $[\varphi, \widetilde{v}]_p$ corresponds to the vector $[p, \phi^{-1}, \widetilde{v}]$ of the notes.*

More explicitly, this tangent vector $[\phi, \widetilde{v}]_p \in T_pM$ is the derivation that maps each smooth function $h \in \mathcal{C}^\infty(M)$ to the number

$$\begin{aligned}
[\phi, v]_p(h) &= (D_{\widetilde{p}}\phi(D_{\widetilde{p}}\widetilde{v}))(h) \\
&= (D_{\widetilde{p}}\widetilde{v})(h \circ \phi) \\
&= D_{\widetilde{p}}(h \circ \phi)(\widetilde{v}) \\
&= \frac{\partial}{\partial t}\bigg|_{t=0} (h \circ \phi)(\widetilde{p} + t\,\widetilde{v}).
\end{aligned}$$

Fixed the point $p \in M$ and the local parametrization ϕ, the map

$$\begin{array}{rccc}
\widehat{\phi}_p : & \mathbb{R}^n & \to & T_pM \\
& \widetilde{v} & \mapsto & [\phi, \widetilde{v}]_p
\end{array}$$

is an isomorphism, since it is the composite of the isomorphism

$$\begin{array}{ccc}
\mathbb{R}^n & \to & T_{\widetilde{p}}\mathbb{R}^n \\
\widetilde{v} & \mapsto & D_{\widetilde{p}}v
\end{array}$$

(seen in Lecture 3) with the isomorphism $D_{\widetilde{p}}\phi : T_{\widetilde{p}}\mathbb{R}^n \to T_pM$ (whose inverse is $D_a\phi^{-1}$). In conclusion, each tangent vector $v \in T_pM$ can be written in the form $v = [\varphi, v]_p$ for some unique vector $v \in \mathbb{R}^n$.

Differential of a map. The differential of a smooth map $f : M \to N$ at a point p can be expressed by the formula

$$\mathrm{D}_pf\,([\phi, \widetilde{v}]_p) = [\psi,\, \mathrm{D}_p(\psi^{-1} \circ f \circ \varphi)(\widetilde{v})]_{f(p)},$$

where
ϕ is a local parametrization of M that covers the point p and
ψ is a local parametrization of N that covers the point $f(p)$.

Change of parametrizations. In particular, putting $f = \mathrm{id}_M$, we see that if both ϕ and ψ are parametrizations of M that cover the same point $p \in M$, then

$$[\varphi, \widetilde{v}]_p = [\psi, \widetilde{w}]_p \quad \text{if and only if} \quad \widetilde{w} = D_{\widetilde{p}}(\psi^{-1} \circ \phi)(\widetilde{v}),$$

where $\widetilde{p} = \phi^{-1}(p)$.

Tangent spaces and Tangent bundles.

Exercise 4.1 (A little heads-up regarding coordinate vectors)**.** Let φ and ψ be smooth charts on a smooth manifold M defined on the same domain U. Let $(x^1, \ldots, x^n)$ be the coordinates induced by φ and $(z^1, \ldots, z^n)$ the coordinates induced by ψ. If the first coordinate functions x^1 and z^1 agree ($x^1 = z^1$ on U), this does *not* imply $\frac{\partial}{\partial x^1}\big|_p = \frac{\partial}{\partial z^1}\big|_p$ for $p \in U$.

Work out a simple example of this fact e.g. on $M = \mathbb{R}^2$ by considering on the one hand the Cartesian coordinates (x, y) and on the other hand the chart (u, v) given by $u = x$, $v = x + y$.
This shows that $\frac{\partial}{\partial x^i}\big|_p$ *depends on the whole system* $(x^1, \ldots, x^n)$*, not only on* x^i*.*

Solution. The two coordinate charts are related by

$$\begin{cases} u = x \\ v = x + y \end{cases} \qquad \begin{cases} x = u \\ y = v - u. \end{cases}$$

By the chain rule we have

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial u} \cdot \frac{\partial}{\partial y} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

We consider for example the function $f(x, y) = xy$ on $\mathbb{R}^2$. The coordinate derivatives of f with respect to two different charts are

$$\frac{\partial}{\partial x} f = y$$
$$\frac{\partial}{\partial u} f = y - x \neq \frac{\partial}{\partial x} f$$

Thus the coordinate vectors depends on the whole system.

We consider for example a linear function $f(x, y) = ax + by$ on $\mathbb{R}^2$. The coordinate derivatives of f with respect to two different charts are

$$\frac{\partial}{\partial x} f = a$$
$$\frac{\partial}{\partial u} f = a - b \neq \frac{\partial}{\partial x} f$$

Thus the coordinate vectors depends on the whole system. □

Exercise 4.2 (The tangent space of a vector space)**.** Let V be an n-dimensional vector space, endowed with the natural smooth structure given by picking an isomorphism $\mathbb{R}^n \to V$ (via the Smooth Charts Lemma)

(a) Fix $p \in V$. To every $v \in V$ we associate the curve passing through p

$$\gamma_v : \mathbb{R} \to V : t \mapsto p + t\, v$$

Show that the map $\Phi_p : V \to T_p V : v \mapsto \gamma_v'(0)$ is an isomorphism of vector spaces.

Solution. To prove that the map $\Phi_p : V \to T_p V$ is an isomorphism, we fix a linear isomorphism $\phi : \mathbb{R}^n \to V$ and use it as a local parametrization of V. Our plan is to take profit from the fact that the linear map

$$\begin{array}{ccc} \mathbb{R}^n & \to & T_p V \\ \widetilde{v} & \mapsto & [\phi, \widetilde{v}]_p \end{array}$$

is an isomorphism. For any vector $v \in V$, the tangent vector $\Phi_p(v) \in T_pV$ is the derivation that maps each smooth function $h \in \mathcal{C}^\infty(V)$ to the number

$$\begin{aligned}
\Phi_p(v)(h) &= \gamma_v'(0)(h) \\
&= \frac{\partial}{\partial t}\bigg|_{t=0} h(\gamma_v(t)) \\
&= \frac{\partial}{\partial t}\bigg|_{t=0} h(p+tv) \\
&= \frac{\partial}{\partial t}\bigg|_{t=0} h(\phi(\widetilde{p}+t\,\phi(\widetilde{v})) \quad \text{setting } \widetilde{p}=\phi^{-1}(p) \text{ and } \widetilde{v}:=\phi^{-1}(v) \\
&= \frac{\partial}{\partial t}\bigg|_{t=0} (h\circ\phi)(\widetilde{p}+t\,\widetilde{v}) \quad \text{since } \phi \text{ is linear} \\
&= D_{\widetilde{p}}\widetilde{v}(h\circ\phi) \\
&= (D_{\widetilde{p}}\phi(D_{\widetilde{p}}\widetilde{v}))(h) \\
&= [\phi,\widetilde{v}]_p(h) \\
&= [\phi,\phi^{-1}(v)]_p(h)
\end{aligned}$$

This computation shows that $\Phi_p(v) = [\phi, \phi^{-1}(v)]_p$ for any vector $v \in V$. This means that Φ_p is the composite of the isomorphisms ϕ^{-1} and $\widetilde{v} \mapsto [\phi, \widetilde{v}]_P$. We conclude that Φ_p is an isomorphism as well. □

(b) Let $f : V \to W$ be a *linear* map between vector spaces V, W. Consider the differential $D_pf : T_pV \to T_{f(p)}W$ at any point $p \in V$. Identifying $T_pV \cong V$ and $T_{f(p)}W \cong W$ via the isomorphisms Φ_p, $\Phi_{f(p)}$, show that D_pf is identified with f. That is, show that the following diagram commutes:

$$\begin{array}{ccc}
T_pV & \xrightarrow{D_pf} & T_{f(p)}W \\
{\scriptstyle\Phi_p}\uparrow & & \uparrow{\scriptstyle\Phi_{f(p)}} \\
V & \xrightarrow[f]{} & W
\end{array}$$

Solution. To check that the diagram commutes, we take two linear isomorphisms $\phi : \mathbb{R}^m \to V$ and $\psi : \mathbb{R}^m \to W$ and employ them as parametrizations. Let us show that for any vector $v \in V$ we have

$$\begin{array}{ccc}
[\phi,\phi^{-1}(v)]_p & \xmapsto{\mathrm{D}_pf} & [\psi,\psi^{-1}(f(v))]_{f(p)} \\
{\scriptstyle\Phi_p}\uparrow & & \uparrow{\scriptstyle\Phi_{f(p)}} \\
v & \xmapsto[f]{} & f(v)
\end{array}$$

In the previous item we have shown that for any vector $\widetilde{v} \in \mathbb{R}^n$ we have $\Phi_p(v) = [\phi, \phi^{-1}(v)] \in T_pV$, and in the same way we see that $\Phi_{f(p)}(f(v)) = [\psi, \psi^{-1}(f(v))] \in T_{f(p)}W$. To finish, we verify that

$$\begin{aligned}
\mathrm{D}_pf([\phi,\phi^{-1}(v)]_p) &= [\psi,\, \mathrm{D}_{\phi(p)}(\psi^{-1}\,f\,\phi)(\phi^{-1}(v))]_{f(p)} \\
&= [\psi, (\psi^{-1}\,f\,\phi)(\phi^{-1}(v))]_{f(p)} \\
&= [\psi, \psi^{-1}(f(v))]_{f(p)}.
\end{aligned}$$

Here, we used the fact that $\mathrm{D}_{\phi(p)}(\psi^{-1}\,f\,\phi) = \psi^{-1}\,f\,\phi$ since the map $\psi^{-1}\,f\,\phi : \mathbb{R}^m \to \mathbb{R}^n$ is linear. □

Exercise 4.3 (Differential of the determinant function)**.** Consider the determinant function $\det : M_n(\mathbb{R}) \to \mathbb{R}$, where $M_n(\mathbb{R}) \simeq \mathbb{R}^{n\times n}$ is the vector space of real $n \times n$

matrices, with its natural smooth structure. We want to compute its differential transformation $D_A \det$ at any matrix $A \in \mathrm{GL}_n(\mathbb{R})$ (i.e. at any invertible matrix),

$$D_A \det : T_A M_n(\mathbb{R}) \to T_{\det(A)}\mathbb{R}$$

(Note that we may identify $T_A M_n(\mathbb{R})$ with $M_n(\mathbb{R})$ and $T_{\det(A)}\mathbb{R}$ with $\mathbb{R}$.)

(a) Verify that det is a smooth function.

Hint: Write the determinant as a sum over all n-permutations.

Solution. The determinant can be written as

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{0 \le i < n} a_{i,\sigma(i)}.$$

Each of the terms $f_\sigma(A) := \mathrm{sgn}(\sigma) \prod_{0 \le i < n} a_{i,\sigma(i)}$ is a monomial, hence a smooth function. □

(b) Show that the differential of det at the identity matrix $I \in M_n(\mathbb{R})$ is

$$D_I \det(B) = \mathrm{tr}(B).$$

where tr denotes the trace.

Solution. Let's define a curve $\gamma_B : \mathbb{R} \to GL(n) : t \to I + tB$, for $B \in GL(n)$. Using the identification $\Phi_I : GL(n) \to T_I(GL(n)) : B \to \gamma_B'(0)$ (and the usual identification $T_1\mathbb{R} \cong \mathbb{R}$) we have

$$\begin{aligned} D_I \det(B) &= D_I \det(\gamma_B'(0)) \\ &= (\det \circ \gamma_B)'(0) \\ &= \frac{d}{dt}\Big|_{t=0} (\det(I + tB)) \\ &= \sum_{\sigma \in S_n} \frac{d}{dt}\Big|_{t=0} f_\sigma(I + tB) \end{aligned}$$

Let us derivate each of the monomials f_σ.

The coefficients of the matrix $A = I + tB$ are $a_{i,j} = \delta_{i,j} + t\, b_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta. Note that at $t = 0$ all the coefficients that are not on the diagonal vanish.If $\sigma \neq \mathrm{id}_n$, then the monomial f_σ has at least two coefficients that are not on the diagonal, hence we have $\frac{d}{dt}\Big|_{t=0} (f_\sigma(I + tB)) = 0$. Thus the only term which survives is the one corresponding to the permutation $\sigma = \mathrm{id}_n$, and we have

$$\begin{aligned} D_I \det(B) &= \frac{d}{dt}\Big|_{t=0} f_{\mathrm{id}_n}(I + tB) \\ &= \frac{d}{dt}\Big|_{t=0} \sum_{0 \le i < n} (1 + t\, b_{i,i,}) \\ &= \frac{d}{dt}\Big|_{t=0} (1 + t\,\mathrm{tr}(B) + t^2 \dots) \\ &= \mathrm{tr} B \end{aligned}$$

□

(c) Show that for arbitrary $A \in \mathrm{GL}_n(\mathbb{R}), B \in M_n(\mathbb{R})$.

$$D_A \det(B) = (\det A)\,\mathrm{tr}(A^{-1}B)$$

Hint: Write $\det(A + tB) = (\det A)(\det(I + tA^{-1}B))$.

Solution. Similarly, we define $\gamma_B : \mathbb{R} \to GL(n) : t \to A + tB$, for $B \in GL(n)$. With the identification $\Phi_A : GL(n) \to T_A(GL(n)) : B \to \gamma_B'(0)$ we have

$$\begin{aligned} D_A \det(B) &= D_A \det(\gamma_B'(0)) \\ &= (\det \circ \gamma_B)'(0) \\ &= \frac{d}{dt}\Big|_{t=0} (\det(A + tB)) \\ &= \det(A) \lim_{t\to 0} \frac{1 + t\ \mathrm{tr}(A^{-1}B) + O(t^2)) - 1}{t} \\ &= \det(A)\ \mathrm{tr}(A^{-1}B) \end{aligned}$$

□

(d) Show that $D_A \det$ is the null linear transformation if $A = 0$ and $n \geq 2$.

Solution. It suffices to check that $f_B'(t) = 0$ when $t = 0$ for the function

$$f_B(t) = \det(A + tB) = \det(tB) = t^n \det(B).$$

Now, $f_B'(t) = n\,t^{n-1} \det(B)$, thus $f_B'(0) = 0$ as required. □

Exercise 4.4 (Tangent Bundles)**.** (a) Show that $T_{(p_1,p_2)} M_1 \times M_2 \cong T_{p_1} M_1 \oplus T_{p_2} M_2$. Show that in fact this extends to the tangent bundles, i.e. there is a diffeomorphism $T(M_1 \times M_2) \cong TM_1 \times TM_2$.

Solution. Let us first recall how we define the smooth structure on a tangent bundle and the smooth structure of a product manifold. After that, we will combine these concepts to study the tangent bundle of a product manifold.

Smooth structure of a tangent bundle. The smooth structure on the tangent bundle TN of a smooth n-manifold N is defined by the local parametrizations of the form

$$\begin{array}{rccc} \widehat{\phi}: & V \times \mathbb{R}^n & \to & \pi_{TN}^{-1}(U) \\ & (x, v) & \mapsto & \big(\phi(x), D_x\phi(D_x v)\big) \end{array},$$

where (V, U, ϕ) is a local parametrization of N and $\pi_{TN} : TN \to N$ is the canonic projection $(p, w) \mapsto p$.

Smooth structure of a product manifold. The smooth structure on a product manifold $\prod_i M_i$ is defined by the local parametrizations of the form

$$\begin{array}{rccc} \phi: & \prod_i V_i & \to & \prod_i U_i \\ & x = (x_i)_i & \mapsto & p = \big(\phi_i(x_i)\big)_i \end{array},$$

where (V_i, U_i, ϕ_i) is a local parametrization of M_i for each i.

Tangent bundle of a product manifold. We consider the tangent bundle of a product manifold $M = \prod_{i\in I} M_i$. For each $i \in I$, we define the i-th component of a tangent vector $z \in T_pM$ as the vector $z_i = D_p\pi_i^M(v) \in T_{p_i}M_i$, where $\pi_i^M : M \to M_i$ the i-th projection $p = (p_j)_j \mapsto p_i$. We claim that for each point $p \in M$, the linear map

$$\begin{array}{rccc} f_p: & T_pM & \to & \prod_i T_{p_i}M_i \\ & z & \mapsto & (z_i)_{i\in I} = (D_p\pi_i^M(z))_{i\in I} \end{array}$$

is an isomorphism. Moreover, we claim that the map

$$\begin{array}{rccc} f: & TM & \to & \prod_i TM_i \\ & (p, z) & \mapsto & \big(p_i, D_p\pi_i^M(z)\big)_{i\in I} \end{array}$$

is a diffeomorphism.

Let us prove the second claim first. To do so, we examine a local expression of f constructed as follows. For each $i \in I$, take a local parametrization (V_i, U_i, ϕ_i) of M_i. From these we construct, as stated above, a local parametrization $(\prod_i V_i, \prod_i U_i, \phi)$ of M, which is characterized by the commutative diagram

$$\begin{array}{ccc} \prod_j V_j & \xrightarrow{\phi} & \prod_j U_j \\ \pi_i^V \downarrow & & \downarrow \pi_i^U \\ V_i & \xrightarrow[\phi_i]{} & U_i \end{array}$$

where $\pi_i^U : \prod_j U_j \to U_i$ and $\pi_i^V : \prod_j V_j \to V_i$ are the i-th projections. From this parametrization of M we obtain the parametrization of TM

$$\begin{array}{cccc} \widehat{\phi}: & \prod_i V_i \times \prod_i \mathbb{R}^{n_i} & \to & \pi_{TM}^{-1}(\prod_i U_i) \\ & (x,v) & \mapsto & \big(p = \phi(x) = (\phi_i(x_i))_i, z = D_x\phi(D_x v)\big). \end{array}$$

On the other hand, each tangent bundle TM_i has a local parametrization

$$\begin{array}{cccc} \widehat{\phi}_i: & V_i \times \mathbb{R}^{n_i} & \to & \pi_i^{-1}(U_i) \\ & (x_i, v_i) & \mapsto & (\phi_i(x_i), D_{x_i}\phi_i(v_i)) \end{array},$$

and putting these together, we form a parametrization of $\prod_i TM_i$

$$\begin{array}{cccc} \psi: & \prod_i (V_i \times \mathbb{R}^{n_i}) & \to & \prod_i \pi_i^{-1}(U_i) \\ & (x_i, v_i)_i & \mapsto & (\phi_i(x_i), D_{x_i}\phi_i(v_i))_{i\in I}. \end{array}$$

We claim that the local expression of f with respect to the charts $\widehat{\phi}$, ψ is the map

$$\begin{array}{cccc} \widetilde{f}: & \prod_i V_i \times \prod_i \mathbb{R}^{n_i} & \to & \prod_i (V_i \times \mathbb{R}^{n_i}) \\ & (x,v) & \mapsto & (x_i, v_i)_{i\in I} \end{array}.$$

Indeed, applying f to a point

$$\widehat{\phi}(x,v) = \big(\underbrace{\phi(x)}_{p} = (\phi_i(x_i))_i, \underbrace{D_x\phi(D_x v)}_{z}\big)$$

we get

$$f\left(\widehat{\phi}(x,v)\right) = f(p,z) = \big(\phi_i(x_i), \underbrace{D_p\pi_i^M(z)}_{z_i}\big)_{i\in I},$$

where

$$\begin{aligned} z_i &= D_p\pi_i^M(z) \\ &= D_p\pi_i^U(z) \\ &= D_p\pi_i^U(D_x\phi(D_x v)) \\ &= D_x(\pi_i^U \circ \phi)(D_x v) \\ &= D_x(\phi_i \circ \pi_i^V)(D_x v) \\ &= D_{x_i}\phi_i(D_x\pi_i^V(D_x v)) \\ &= D_{x_i}\phi_i(D_{x_i} v_i), \end{aligned}$$

and we obtain the same result by computing

$$\psi\left(\widetilde{f}(x,v)\right) = \psi((x_i, v_i)_i) = (\phi_i(x_i), D_{x_i}\phi_i(v_i))_{i\in I}.$$

(To compute z_i we used the following facts. First, the map π_i^U is the restriction of π_i^M to the open set $\prod_i U_i \subseteq \prod_i M_i$, thus it has the same differential. Second, the identity $\pi_i^U \circ \phi = \phi_i \circ \pi_i^V$, which has been expressed above by a commutative diagram. Third, the identity $D_x\pi_i^V(D_x v) = D_{x_i} v_i$, valid for every vector $v \in \prod_i \mathbb{R}^{n_i}$, which can be verified directly by applying both derivations to a function $h \in \mathcal{C}^\infty(V_i)$.)

This shows that we have a commutative diagram

$$\begin{array}{ccc} A := \pi_{TM}^{-1}(\prod_i U_i) & \xrightarrow{f|_A^B} & \prod_i \pi_{TM_i}^{-1}(U_i) =: B \\ \uparrow \widehat{\phi} & & \uparrow \psi \\ \prod_i V_i \times \prod_i \mathbb{R}^{n_i} & \xrightarrow[\widetilde{f}]{} & \prod_i (V_i \times \mathbb{R}^{n_i}) \end{array},$$

which means that $\widetilde{f}$ is the local expression of f w.r.t. the charts $\widetilde{\phi}$, ψ. Since $\widetilde{f}$ is a diffeomorphism (and so are the local parametrizations $\widehat{\phi}$, ψ), we conclude that $f|_A^B$ is a diffeomorphism. This implies that f is a local diffeomorphism, since we can do the same reasoning for each parametrization $\widehat{\phi}$ of TM.

To show that f is a diffeomorphism, we need just show that f is bijective. To see this, consider the projection map $\eta : \prod_i TM_i \to M$ that sends $(p_i, z_i)_i \mapsto (p_i)_i$. Note that $\eta \circ f = \pi_{TM}$. This means that for each point $p \in M$, the function f maps the fiber

$$\pi_{TM}^{-1}(p) = \{p\} \times T_pM$$

to the fiber

$$\eta^{-1}(p) = \prod_i (\{p_i\} \times T_{p_i}M)\,.$$

We may then consider the restriction

$$\begin{array}{rccl} \overline{f_p} := f|_{\pi_{TM}^{-1}(p)}^{\eta^{-1}(p)} : & \{p\} \times T_pM & \to & \prod_i (\{p_i\} \times T_{p_i}M) \\ & (p, z) & \mapsto & (p_i, z_i = D_p\pi_i^M(z))_{i\in I} \end{array}$$

which is essentially the same thing as the linear map

$$\begin{array}{rcccl} f_p : & T_pM & \to & \prod_i T_{p_i}M_i & \\ & z & \mapsto & (z_i)_{i\in I} & = (D_p\pi_i^M(z))_{i\in I} \end{array},$$

since the point p is fixed. To show that f is bijective, it suffices to show that $\overline{f_p}$ is bijective for each point $p \in M$. An indeed, for each point $p \in M$, we can deduce that $\overline{f_p}$ is bijective from the local expression of f that we already have. Indeed, suppose we have parametrizations (V_i, U_i, ϕ_i) as above, such that $p \in \prod_i U_i$. Then the set $A = \pi_{TM}^{-1}(\prod_i U_i)$ contains the fiber $\pi_{TM}^{-1}(p)$, and the set $B = \pi_{TM_i}^{-1}(U_i) = \eta^{-1}(\prod_i U_i)$ contains the fiber $\eta^{-1}(p)$, and the fact that $f|_A^B$ is bijective implies that $\overline{f_p}$ is bijective. This finishes the proof that the map f is bijective, and therefore it is a diffeomorphism.

Finally, the fact that $\overline{f_p}$ is bijective also implies that the linear map $f_p : T_pM \to \prod_i T_{p_i}M_i$ is an isomorphism, which is the other claim that we had to prove. □

(b) Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Solution. Note that the 1-sphere $\mathbb{S}^1$ (i.e. the circle) is diffeomorphic to the 1-torus $\mathbb{T}^1$. Therefore, to solve the exercise, we may prove the following, more general fact. Consider the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, and let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the quotient map $x \mapsto [x]$. We claim that the map $f : \mathbb{T}^n \times \mathbb{R}^n \to T\mathbb{T}^n$ given by

$$f([x], v) = \big([x], D_x\pi(D_xv)\big)$$

is a diffeomorphism.

To prove this, let us first recall how we define the smooth structure on the torus $\mathbb{T}^n$. (This is a problem of Series 6, so we will not give all the details.) We say that an open set $U \subseteq \mathbb{R}^n$ is **nice** if π is injective on U. Since the quotient map π is open, it follows that the map $\phi_U := \pi|_U^{\pi(U)} : U \to \pi(U)$ is a local parametrization of $\mathbb{T}^n$. These parametrizations ϕ_U (for $U \subseteq \mathbb{R}^n$ nice) constitute an inverse atlas on $\mathbb{T}^n$ (exercise).

For each nice open set $U \subseteq \mathbb{R}^n$ we also get a local parametrization of $T\mathbb{T}^n$

$$\begin{array}{rccc} \widehat{\phi_U}: & U \times \mathbb{R}^n & \to & \pi^{-1}_{T\mathbb{T}^n}(\pi(U)) \\ & (x,v) & \mapsto & ([x]), \underbrace{D_x\phi_U}_{=D_x\pi}(D_x v)) \end{array}$$

(where $\pi_{T\mathbb{T}^n} : T\mathbb{T}^n \to \mathbb{T}^n$ is the projection), and a local parametrization of $\mathbb{T}^n \times \mathbb{R}^n$

$$\begin{array}{rccc} \psi_U: & U \times \mathbb{R}^n & \to & \pi(U) \times \mathbb{R}^n \\ & (x,v) & \mapsto & ([x],v) \end{array}.$$

The local expression of f with respect to the parametrizations ψ_U, ϕ_U is the identity map $\mathrm{id}_{U\times\mathbb{R}^n}$, since for $(x,v) \in U \times \mathbb{R}^n$ we have

$$f(\psi_U(x,v)) = f([x],v) = f([x], D_x\pi(D_x v)) = \phi_U(x,v).$$

From this we conclude that f is a diffeomorphism by reasoning as in the first part of the exercise. □

Immersions and smooth Embeddings.

Exercise 4.5. Consider the map

$$f : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (2 + \tanh t) \cdot (\cos t, \sin t).$$

Show that f is an injective immersion. Is it a smooth embedding?

Solution. First notice that f is an immersion since $f'(t) \neq 0$ for every $t \in \mathbb{R}$. To see this observe that

$$f_*\big|_t\Big(\frac{\partial}{\partial t}\Big|_t\Big) = \sum_{0\le j<2} \frac{\partial}{\partial t}\Big|_t (x^j \circ f)\ \frac{\partial}{\partial x^j}\Big|_{f(t)} = f_0'(t)\ \frac{\partial}{\partial x^0}\Big|_{f(t)} + f_1'(t)\ \frac{\partial}{\partial x^1}\Big|_{f(t)}$$

Hence if $f'(t) \neq 0$ then we have $\mathrm{Ker}\, f_*\big|_t = \{0\}$ which is equivalent to $f_*\big|_t$ injective for every $t \in \mathbb{R}$. Thus it suffices to compute

$$f_0'(t) = \left(\frac{1}{\cosh^2 t}\right)\cos t - (2 + \tanh t)\sin t$$

and

$$f_1'(t) = \left(\frac{1}{\cosh^2 t}\right)\sin t - (2 + \tanh t)\cos t$$

To see that $f'(t) \neq 0$ notice that

$$\|f'(t)\|^2 = \left(\frac{1}{\cosh^2 t}\right)^2 + (2 + \tanh t)^2 > 0$$

where $\|\cdot\|$ denotes the euclidean norm. This proves that f is an immersion. Furthermore the function f is an injection since the function $r(t) = \|f(t)\| = 2 + \tanh t$ is strictly increasing.

Note that f is an injective immersion. Let us prove that it is a smooth embedding. Consider the open set $U = \{x \in \mathbb{R}^2 : 1 < \|x\| < 3\}$. We will show that $f|^U : \mathbb{R} \to U$ is a proper map (hence a closed map; see e.g. Thm. 4.95 of Lee's book on topological manifolds). It follows that f is an embedding, since its the composite $f = \iota_U \circ f|^U$ of a closed embedding $f|^U$ and the inclusion map $\iota_U : U \to M$, which is an open embedding.

To see that $f|^U$ is proper we let $K \subseteq U$ be a compact set and verify that $f^{-1}(K) \subseteq \mathbb{R}$ is compact as well. Since K is closed (because it is a compact subset of a Hausdorff space) and f is continuous, the preimage $f^{-1}(K)$ is closed. Finally, we have to check that $f^{-1}(K)$ is bounded. Let a (resp b) be the minimum (resp. maximum) norm of a point $x \in X$. Note that $[a,b] \subseteq (1,3)$. It follows that $f^{-1}(K) \subseteq [a', b']$, where a', b' are the preimages of a, b by the monotonic map $t \mapsto 2 + \tanh t$. □

Exercise 4.6. Consider the following subsets of $\mathbb{R}^2$. Which is an embedded submanifold ? Which is the image of an immersion ?

(a) The "cross" $S := \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$.

Solution. The cross S is not an embedded submanifold, because it is the union of the lines $y = 0$ and $x = 0$, and is therefore not locally Euclidean at the origin (exercise of series 1).

On the other hand, S is the disjoint union of two embedded submanifolds: S_0 = the horizontal axis, and S_1 = the vertical axis minus the origin. Let M be the 1-manifold obtained as disjoint union of S_0 and S_1. The inclusion map of M into $\mathbb{R}^2$ is an injective immersion and has S as image. □

(b) The "corner" $C := \{(x, y) \in \mathbb{R}^2 \mid xy = 0, x \geq 0, y \geq 0\}$

Solution. We will show that C is not even an immersed submanifold of $\mathbb{R}^2$, so in particular it cannot be an embedded submanifold.

We proceed by contradiction. Suppose that C is an immersed submanifold, i.e. it has a topology τ and smooth structure such that the canonical inclusion $\iota : C \hookrightarrow \mathbb{R}^2$ is an immersion. Let (U, φ) be a smooth chart s.t. $(0, 0) \in U$, $\varphi(0, 0) = 0$ where $U \subset (C, \tau)$ is open [1]. By making the image $\varphi(U)$ smaller if necessary we can suppose that it is an open interval containing 0, $\varphi(U) = J \subset \mathbb{R}$.

Since ι is an immersion, then

$$f := \iota \circ \varphi^{-1} : J \to \mathbb{R}^2$$

is a smooth map with non-zero derivatives everywhere. Here we emphasize that on J and $\mathbb{R}^2$ we have the standard Euclidean topology and smooth structure. In particular, we find that $f'(0) \neq (0, 0)$. Hence either $f_1'(0) \neq 0$ or $f_2'(0) \neq 0$. If $f_1'(0) \neq 0$ then for any neighborhood of $0 \in J$, we can find points $t_1, t_2 \in J$ s.t. $f_1(t_1) < 0$ and $f_1(t_2) > 0$. It contradict the fact that $f_1 \geq 0$. Similarly we arrive at a contradiction if $f_2'(0) \neq 0$. □

Exercise 4.7. Let N be a embedded n-submanifold of some m-manifold M. Show that there exists an open set $U \subseteq M$ that contains N as a closed subset.

Solution. Consider a family of charts $\varphi_i : W_i \to V_i$ that cover N and are slice charts for N, meaning that $\varphi_i(x) \in \mathbb{R}^n \times \{0\}$ iff $x \in N$, or equivalently, that $N \cap W_i = \varphi_i^{-1}(\mathbb{R}^n \times \{0\})$. Therefore $N \cap W_i$ is a closed subset of W_i for all i. We conclude that N is closed in $W = \bigcup_i W_i$, which is an open subset of M. □

Exercise 4.8 (To hand in). Let $f : M \to N$ be an injective immersion of smooth manifolds. Show that there exists a closed embedding $M \to N \times \mathbb{R}$.

Hint: Recall that there exists a proper map $g : M \to \mathbb{R}$ (Exercise 3.2)

[1] Note that in the case of an embedded manifold we could assume that $U = V \cap C$ for some $V \subset \mathbb{R}^2$ open, but here a-priori we do not know the topology τ.