Nuclear Fusion and Plasma Physics

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Fluid description of plasmas

The MHD model: summary of its implications

- Flux freezing in ideal MHD and consequences for plasma confinement and stability

MHD equilibrium

- Basic system of equations
- Magnetic tension and pressure terms
- Examples of equilibrium configurations
  - the z-pinch
  - the θ-pinch
  - the force-free equilibrium
  - bending the z-pinch into a torus

MHD stability

- General discussion on stability
- Example of instabilities: sausage and kink instability of a z-pinch.
- General interchange instability, good and bad curvature regions
- Methods to stabilise a plasma

1 MHD model

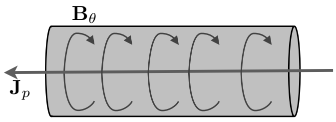
The MHD model is a single fluid description of plasma, for macroscopic, large scale, relatively slow phenomena.

Ideal MHD $\eta = 0 \Rightarrow$ flux freezing (ex. dynamo effect, solar flares).

The magnetic flux contained within any surface moving with the plasma is constant.

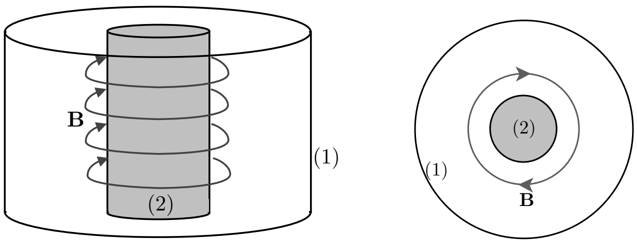
Application of flux freezing to fusion: plasma can be shaped or stabilised by magnetic fields

Example 1: Linear z-pinch



- $\mathbf{J}_p$ (plasma current) produces $\mathbf{B}_\theta$.
- Increasing $\mathbf{J}_p$ increases $\mathbf{B}_\theta$ and plasma is compressed (flux must be conserved - in this case it is the flux on any azimuthal surface).

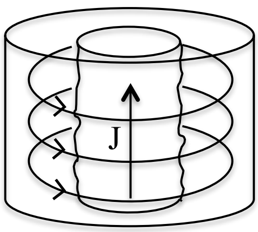
Example 2: B-field produced by external sources, plasma inside conductor



(1) conductor (2) plasma

If **B** is increased, the plasma must be compressed as field lines cannot move through it. Conductor must be solid enough mechanically!

Example 3: Stabilisation of plasma instability by a conducting wall



- Plasma has current densitiy **J**.
- An instability develops.
- If plasma moves toward the wall, flux conservation tends to keep the plasma in the center.

Resistive MHD $\eta \neq 0 \Rightarrow$ **B**-field can diffuse with respect to the plasma

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}. \tag{1.1}$$

The characteristic time for diffusion of magnetic field is $\tau = \frac{\mu_0 L^2}{\eta}$, where L is the typical scale length. This can be very long (several seconds) for fusion plasmas, as η is small due to their high temperature, and L is generally large. Finite resistivity of the plasma (and of the conductor outside) limits the beneficial effects of flux freezing for confinement and stabilization to diffusion times.

1.1 Ideal MHD equilibrium

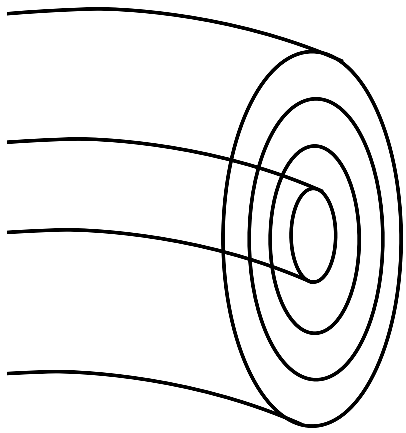
Static equilibrium: $\frac{\mathrm{d}}{\mathrm{d}t} = 0$, $\mathbf{u} = 0$. The ideal MHD system becomes

$$\begin{cases} \mathbf{J} \times \mathbf{B} = \nabla p & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \\ \nabla \cdot \mathbf{B} = 0 & \nabla \cdot \mathbf{J} = 0 \end{cases} \tag{1.2}$$

Notes.

1.
$$\mathbf{B} \cdot (\nabla p) = \mathbf{B} \cdot (\mathbf{J} \times \mathbf{B}) = 0 \qquad (\nabla p \perp \mathbf{B}) \tag{1.3}$$
Thus, pressure is constant along magnetic field lines: **B** lies on p-constant surfaces, also called magnetic surfaces.

2.
$$\mathbf{J} \cdot (\nabla p) = \mathbf{J} \cdot (\mathbf{J} \times \mathbf{B}) = 0 \tag{1.4}$$
J also lies on p-constant surfaces
Thus, isobaric surfaces = magnetic surfaces = current surfaces (in ideal MHD equilibrium).
Note, however, that **J** and **B** are not necessarily aligned (or orthogonal).
This gives a simple way to represent the plasma equilibrium.



3. Force balance is $\mathbf{J} \times \mathbf{B} = \nabla p$, which can be expressed in a more intuitive way:

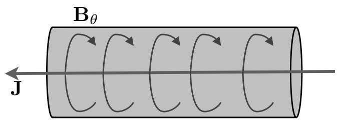
$$\nabla p = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = \left(\underbrace{\frac{B^2}{\mu_0} (\mathbf{b} \cdot \nabla) \mathbf{b}}_{\text{tension}} - \underbrace{\nabla_\perp \left(\frac{B^2}{2\mu_0} \right)}_{\text{pressure}} \right)$$

Tension = "restoring" force acting when B-lines are bent $\sim \frac{B^2}{\mu_0 R_c}$, with R_c being the radius of curvature of the field lines

where we have used $\mathbf{b} = \mathbf{B}/B$, $\mathbf{B} \times (\nabla \times \mathbf{B}) = \nabla \left(B^2/2 \right) - (\mathbf{B} \cdot \nabla) \mathbf{B}$ and $\nabla_\perp = \nabla - \mathbf{b}(\mathbf{b} \cdot \nabla)$.

Examples of MHD equilibrium configurations

Example 1: Linear z-pinch



Current flowing along the z-axis

$$\begin{cases} p = p(r) \\ \mathbf{B} = B_\theta(r)\hat{\theta} \\ \mathbf{J} = J_z(r)\hat{z} \end{cases}$$

Cylindrical coordinates

$$\begin{cases} \nabla \times \mathbf{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \\ \nabla \times \mathbf{A})_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \\ \nabla \times \mathbf{A})_z = \frac{1}{r}\frac{\partial r A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \end{cases}$$

From Ampere's law

$$\frac{1}{\mu_0 r}\frac{\mathrm{d}}{\mathrm{d}r}(rB_\theta) = J_z$$

For a complete calculation we need to assume a current density profile. Let's take the simplest case $J_z = const = \frac{I_p}{\pi a^2}$. Thus

$$\frac{1}{\mu_0 r}\frac{\mathrm{d}}{\mathrm{d}r}(rB_\theta) = \frac{I_p}{\pi a^2} \Rightarrow rB_\theta = \frac{\mu_0 r I_p}{\pi a^2}\frac{r}{2} + const \underset{B_\theta(0)=0}{\Rightarrow} B_\theta(r) = \frac{\mu_0 r}{2\pi a^2} I_p, \text{for } r \leq a.$$

Note that for $r > a$, it's the usual application of Ampere's law

$$B_\theta(r) = \frac{\mu_0 I_p}{2\pi r}, \text{for } r > a$$

From force balance:

$$\frac{dp}{dr} = -J_z B_\theta$$

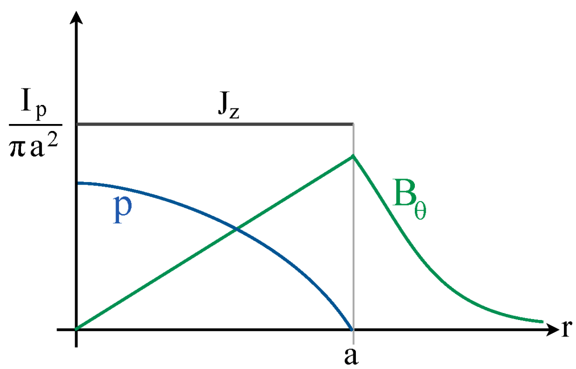
Thus for $r \leq a$:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -J_z \frac{\mu_0 I_p}{2\pi a^2} r \Rightarrow p(r) = -\frac{\mu_0 I_p^2}{2\pi^2 a^4}\frac{r^2}{2} + const$$

But, $p(a) = 0 \Rightarrow const = \frac{\mu_0 I_p^2}{2\pi^2 a^4}\frac{a^2}{2}$. The pressure (inside the plasma) is given by

$$p(r) = \frac{\mu_0 I_p^2}{4\pi^2 a^2}\left[1 - \left(\frac{r}{a}\right)^2\right] \tag{1.5}$$

Summary

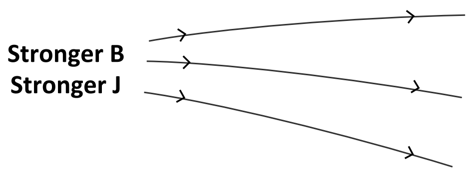


Example 2: Low-β plasma

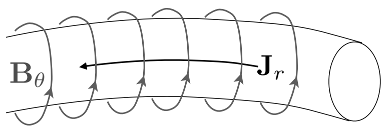
$\beta = \frac{p}{B^2/2\mu_0} \ll 1$, so pressure term ~ 0. So $\mathbf{J} \times \mathbf{B} \cong 0$ meaning $\mathbf{J}$ parallel to $\mathbf{B}$ or $\mathbf{J} = \mu(r)\mathbf{B}$ with μ scalar quantity. Current can only flow along $\mathbf{B}$, not across it. As $\nabla \cdot \mathbf{J} = 0$,

$$\nabla \cdot (\mu(r)\mathbf{B}) = \mu(r)\nabla \cdot \mathbf{B} + (\mathbf{B} \cdot \nabla)\mu(r) = 0 \Rightarrow (\mathbf{B} \cdot \nabla)\mu(r) = 0$$

μ is constant along the field line, i.e. the ratio J/B is constant. A little like a "normal" fluid in "physical" conducts.

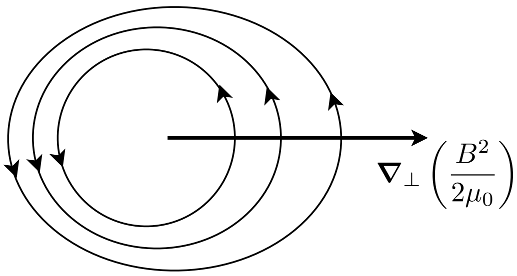


Example 3: Toroidal equilibrium



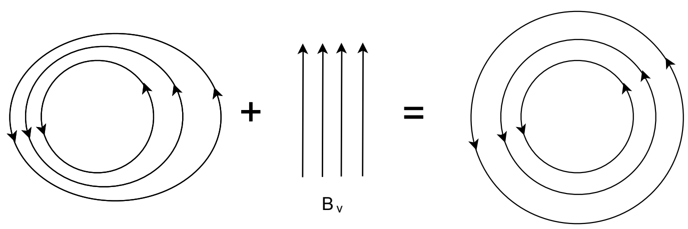
Bend the z-pinch into a torus.

B_θ is due to plasma current: it's stronger inside than outside. On a cross-section:



The pressure force is outwards: "hoop force". Can be seen as many conductors repelling each other as they carry current in the opposite direction.

How to reinforce the field to the outside, and weaken it to the outside? With a vertical field!

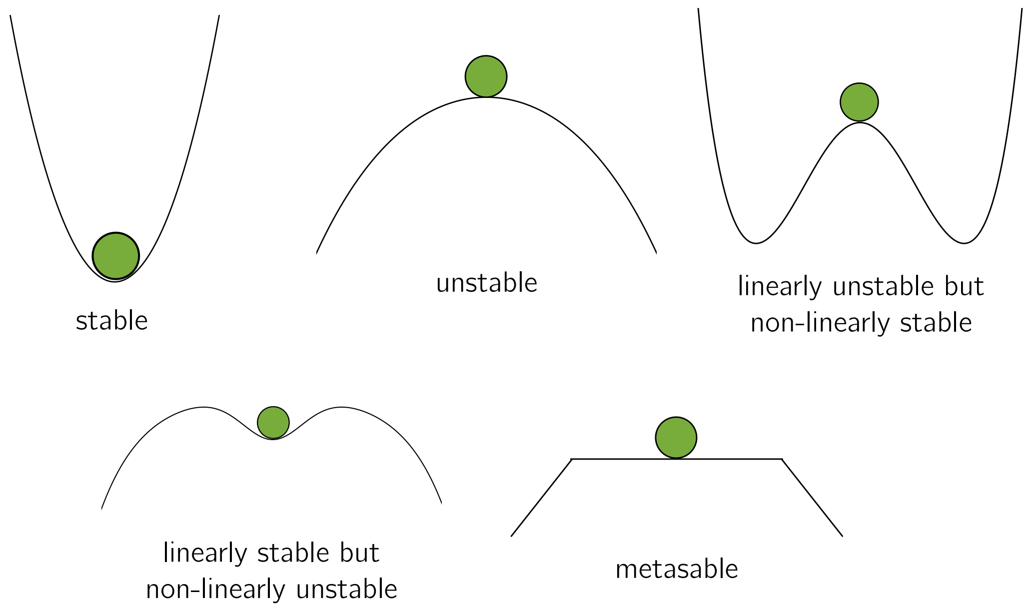


Another way of saying it, is that by $\mathbf{J} \times \mathbf{B}_v$-force the plasma is pushed back (of course, implying that we have a toroidal current too!). This is part of the tokamak concept (it needs an external vertical field for equilibrium), which will be discussed in the next lecture.

2 MHD stability

For any practical purpose, the equilibrium is a necessary condition to have a confined plasma but not sufficient to hold it for macroscopic time scales. For this we need to have a stable equilibrium.

Mechanical analogy



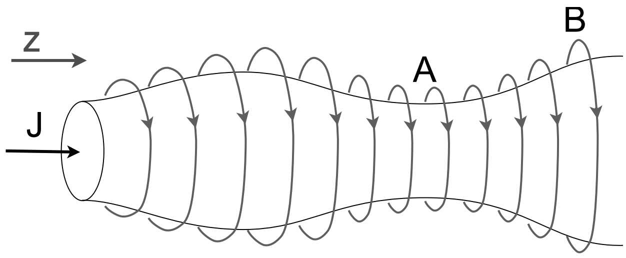
Perturbations that lower/increase the potential energy correspond to unstable/stable situations.

To study the stability in 3D (fusion relevant) situations, one needs to consider all possible perturbations to equilibrium. How to do it?

- Macroscopic "interchange" of flux tubes (we know flux is frozen-in with plasma): does energy increase or decrease when the flux tubes are "interchanged"? (stable vs. unstable, and how fast does the instability grow?)
- Fourier analysis of *small* perturbations (linearisation), or normal mode analysis: consider perturbations $\propto \exp(i\mathbf{k}\cdot\mathbf{x} - \imath\omega t)$ or, in a cylinder, $\propto \exp(ikx + im\theta - i\omega t)$, $m \in \mathbb{Z}$. This is a useful approach in uniform plasmas. $Im(\omega)$ will give us the growth rate (positive or negative) of the instability.
- In non-uniform plasmas, we take the MHD equation of motion and impose a small displacement $\boldsymbol{\xi} \rightarrow \dot{\boldsymbol{\xi}} = F(\boldsymbol{\xi})$. This is the Lagrangian point of view. Still, the issue is the sign of the change of the energy. For normal modes one gets an eigenvalue equation: $A\boldsymbol{\xi} = \omega^2\boldsymbol{\xi} \Rightarrow$ the sign of ω^2 determines the stability.

Examples of instabilities

Example 1: z-pinch with "sausage perturbation"

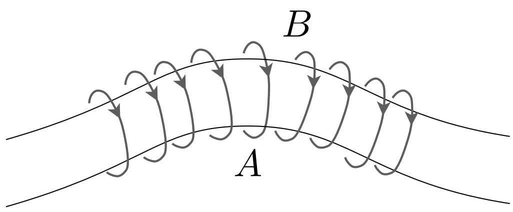


$\exp(\imath k_z z + \imath m\theta)$,
$k_z \neq 0, m = 0$

At point A, B_θ is stronger $\longrightarrow$ tension $\frac{B_\theta^2}{\mu_0 r}$ and pressure $\frac{B_\theta^2}{\mu_0}$ increase, the perturbation increases as the inward force ∇p is not balanced.

At point B, B_θ is weaker and $\frac{B_\theta^2}{\mu_0 r}$ is smaller: kinetic pressure is not balanced and pushes the plasma out $\Rightarrow$ perturbation increases. $\Longrightarrow$ **Instability**

Example 2: z-pinch with kink



$k_z \neq 0, m = 1$
B is stronger in A than in B: perturbation grows. $\Longrightarrow$ **Instability**

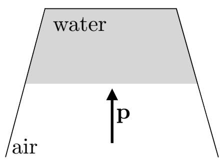
Note. The same configuration can be unstable with respect to different kinds of perturbations.

How would you act to stabilise the z-pinch? $\rightarrow$ Adding B_z (z/θ-pinch): B_z provides "tension" along z.

In general, bending field lines requires energy, so the presence of **B**-tension along a given direction is stabilising. Perturbations leading to instabilities tend to "avoid" bending field lines.

2.1 Intercharge instability (Rayleigh-Taylor)

Example: ordinary fluids of different density

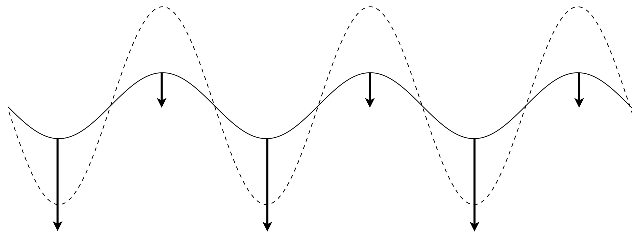


Glass of water turned upside-down.

Air pressure would be enough to hold water up, net force is zero $\Rightarrow$ equilibium.

However (as we know from experience), this equilibrium is unstable. This is the case in general whenever a heavier fluid sits on top a light fluid.

Any ripple/perturbation at the water/air interface will increase (**Rayleigh-Taylor instability**).

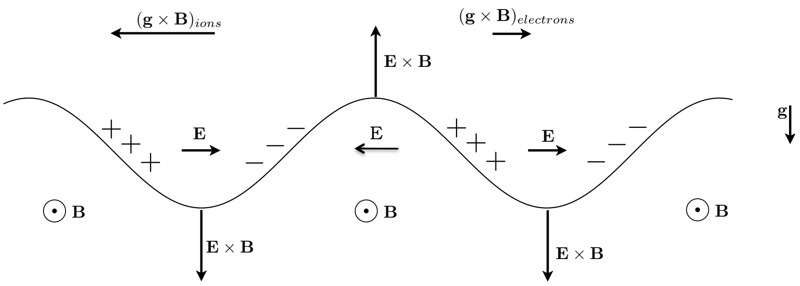


Note. This is an example of interchange instability, as we can think of replacing the two volume elements at the fluid interface.

Plasma

Consider a situation similar to that seen above: plasma on top ("heavier" fluid), vacuum with magnetic field ("lighter" fluid) at the bottom, still in the presence of gravity.

Consider a small perturbation to the interface



Plasma particles are subject to **g**-drift.

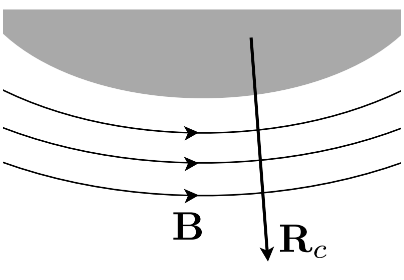
$$\mathbf{v}_g = \frac{m}{q}\frac{\mathbf{g}\times\mathbf{B}}{B^2} \qquad (2.1)$$

(much stronger for ions!)

An electric field will be set-up, giving rise to $\mathbf{E}\times\mathbf{B}$, which will *increase* the perturbation.

One could argue that gravity is, in general, not important in plasma dynamics. In fact, in most cases of interest the effect of gravity in the mechanism of the Rayleigh-Taylor instability is replaced by that of B-field gradient or curvature.

Example: unstable vs. stable



Bad curvature ⇔ unstable

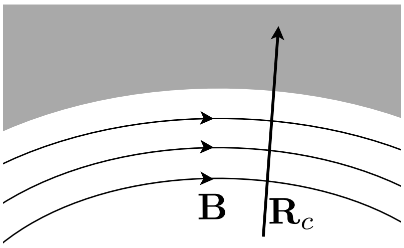
$$\mathbf{v}_{curv} = \frac{m}{q}v_{||}^2\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2B^2} \qquad (2.2)$$

The same mechanism as above applies, if we replace the gravitational force by the centrifugal force

$$m\mathbf{g} \longrightarrow \frac{mv_{||}^2\mathbf{R}_c}{R_c^2}$$

As $\mathbf{R}_c$ point from the "heavier" fluid (plasma) to the "lighter" fluid (vacuum with **B**) we have **instability**. This is a case of "bad curvature": The curvature of the field points away from the region of higher pressure. The plasma is unstable when it is immersed in a magnetic field that is *concave* towards the plasma.

The opposite case is of course **stable**.



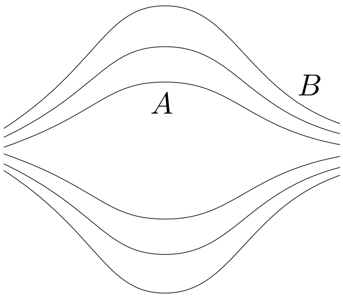
Good curvature ⇔ stable

This corresponds to having **g** pointing from the lighter fluid to the heavier fluid.

The Rayleigh-Taylor instability is a special case of a general class of instabilities, *interchange* instabilities, in plasma. These instabilities arise when by swapping the position of two flux tubes, the energy of the system decreases. They are of fundamental importance at the plasma-vacuum interface.

Note. The criteria for instability must be considered globally. In a real configuration there will be destabilising regions and stabilising regions. The balance between the two will give the global stability properties.

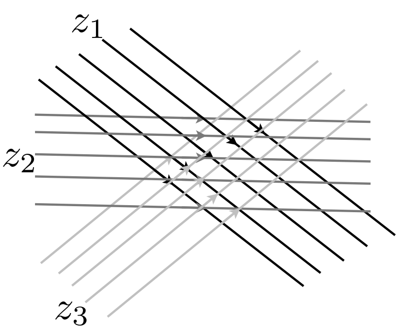
Example: mirror



A bad curvature region
B good curvature region

The balance of the two regions may allow macroscopic stability, although there can be localised instabilities in the bad curvature region.

In addition to field line bending there as another mechanism that can help stabilising interchange instabilities: "*magnetic shear*"



At different depths in the plasma, B-field lines are directed along different directions: interchange becomes impossible unless we bend B-field lines → **stabilising effect**.

This is the case of a tokamak: the "pitch" of the helical field structure depends on the radial position.