Stellar orbits

4th part

Outlines

The third integral in axisymmetric potential

Orbits in planar non-axisymmetric potential

- Surface of sections
    - energy dependency
    - flattening dependency
- Integrals of motions

Orbits in planar non-axisymmetric rotating potential

- The Jacobi integral
- Lagrange points
- Orbits around Lagrange points
- Orbits not confined to Lagrange points

Weak bars

- The Lindblad resonances
- Orbit families in realistic bars

The third integral in axisymmetric potentials

Surfaces of section

Can we visualize the phase phase and check if an
additional integral of motion exists ?

Idea :

We study the orbits in the meridional plane

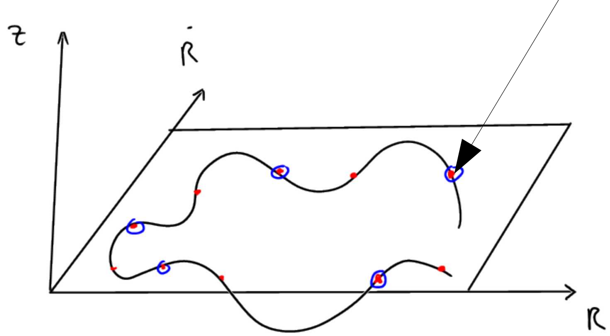
- 4-D — 4 indep. variables $(R, z, \dot{R}, \dot{z})$
- Energy E

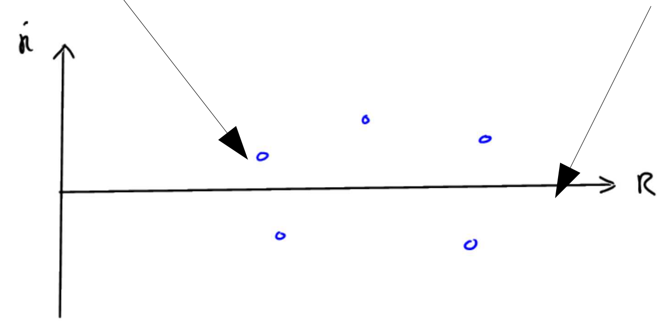
→ 3-D — 3 indep. variables $(R, z, \dot{R})$

- Drawing a 3-D phase space is still not easy. Instead, we draw slices of the phase space. We plot only phase space points that:
  - cross the $z = 0$ plane
  - have $\dot{z} > 0$

consequents

Surface of section
or Poincaré maps



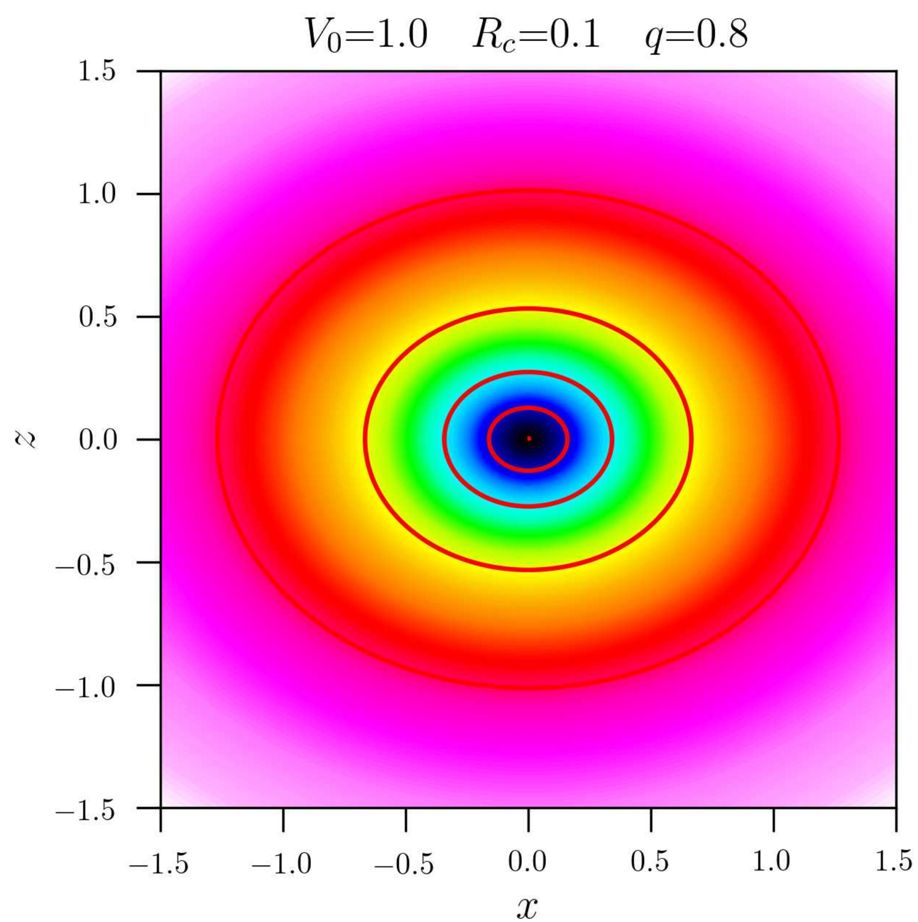


Surfaces of section

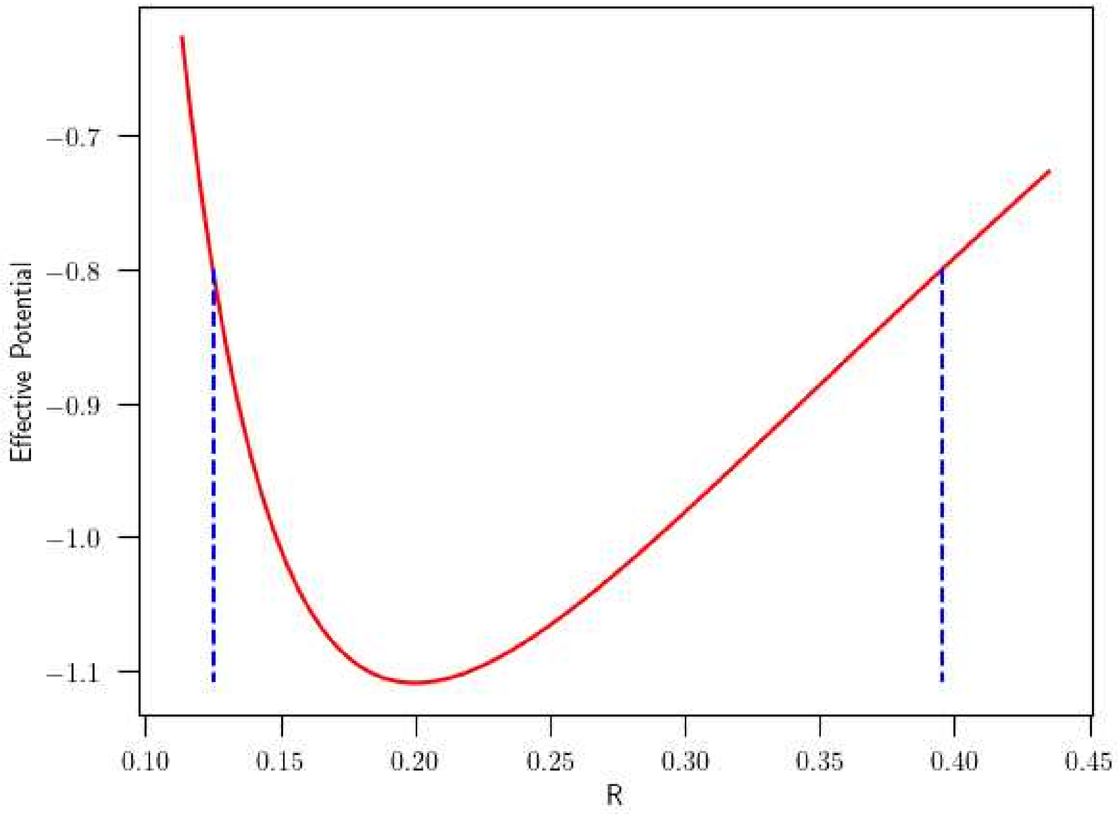
Examples

Logarithmic potential

$$\Phi_{\log}(R,z) = \frac{1}{2}V_0^2 \ln\left(R_c^2 + R^2 + \frac{z^2}{q^2}\right)$$

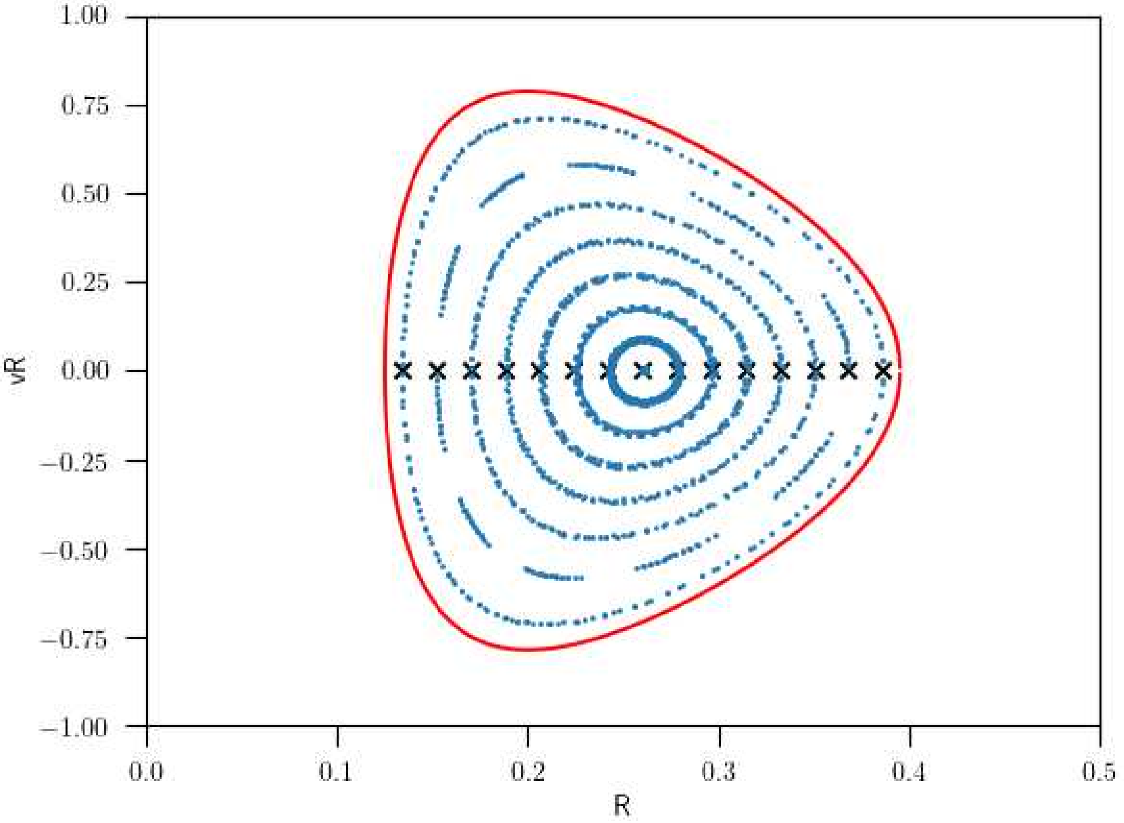


Effective Potential




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --plotpotential
```

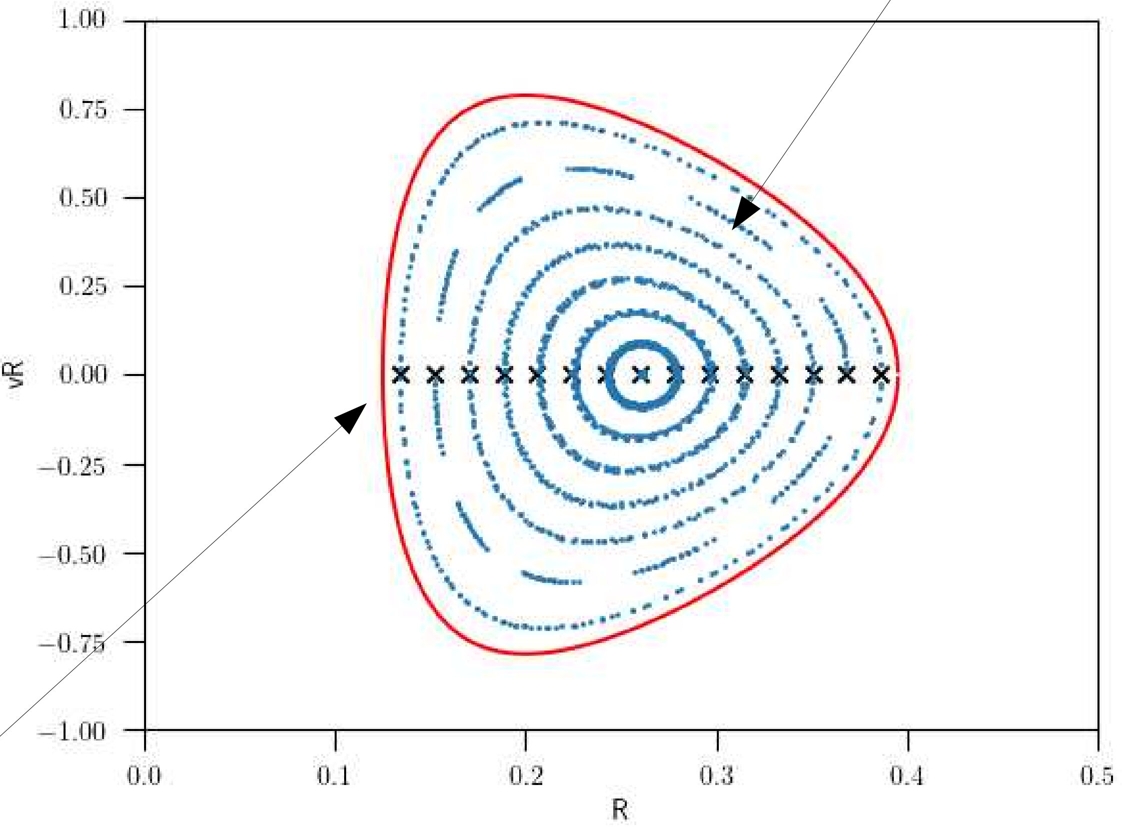
Invariant curves : Third Integral




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 30 --nlaps 1000
```

Invariant curves : Third Integral

Invariant curves




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 30 --nlaps 1000
```

Zero velocity curve : curve defined by $\dot{z} = 0$ $\dot{R}(R) = \pm\sqrt{2\left[E - \Phi_{\rm eff}(R, z=0)\right]}$

The Third Integral I (I is in general non analytical)

Spherical systems : $|\vec{L}| \equiv L$ is conserved

Nearly spherical potential : L is nearly an integral $= I$?

What is the curve in the Poincaré map that satisfies $L = \text{cte}$?

in cylindrical coordinates

$$L^2 = \dot{z}^2 R^2 + L_z^2 \qquad (z=0)$$

$$\dot{z}^2 = \frac{1}{R^2}\left(L^2 - L_z^2\right)$$

Energy conservation

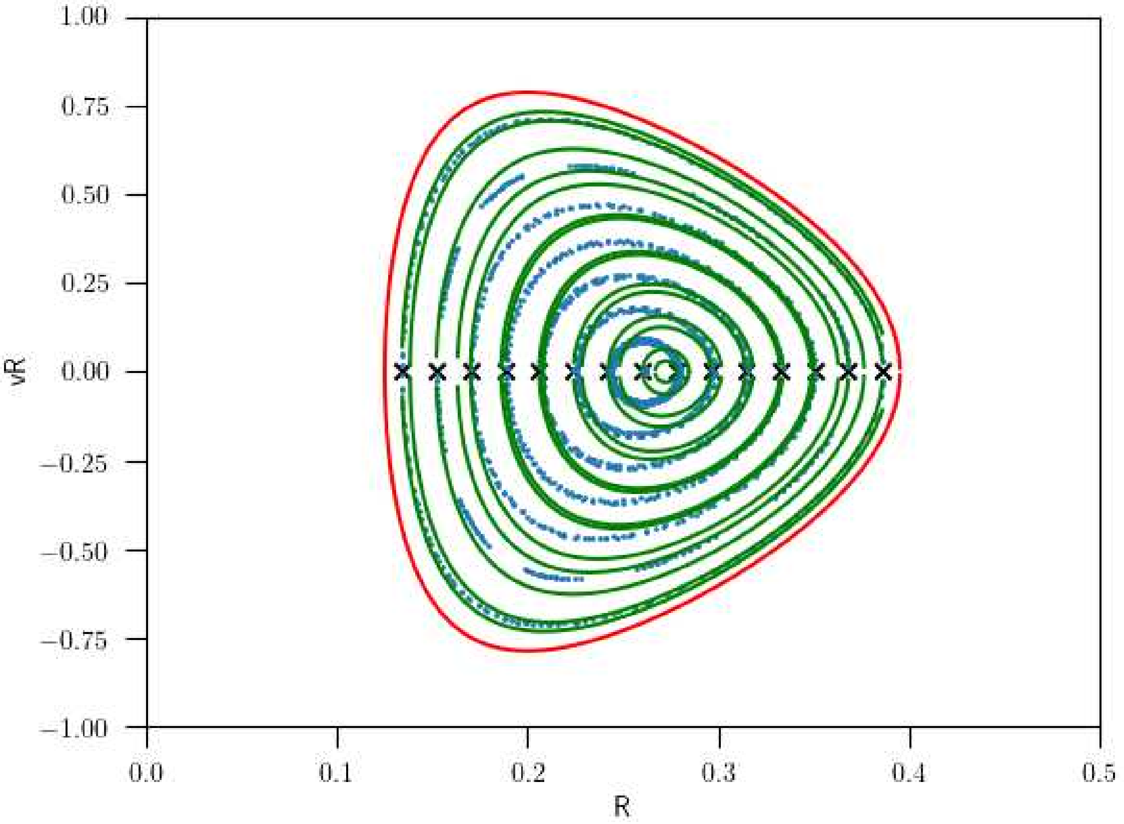
$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{\mathrm{eff}}(R,0)$$

$$= \frac{1}{2}\dot{R}^2 + \frac{1}{2R^2}\left(L^2 - L_z^2\right) + \Phi_{\mathrm{eff}}(R,0)$$

$$\boxed{\dot{R} = \pm\sqrt{2\left(E - \Phi_{\mathrm{eff}}(R,0) - \frac{1}{2R^2}\left(L^2 - L_z^2\right)\right)}}$$

Invariant curves : Third Integral

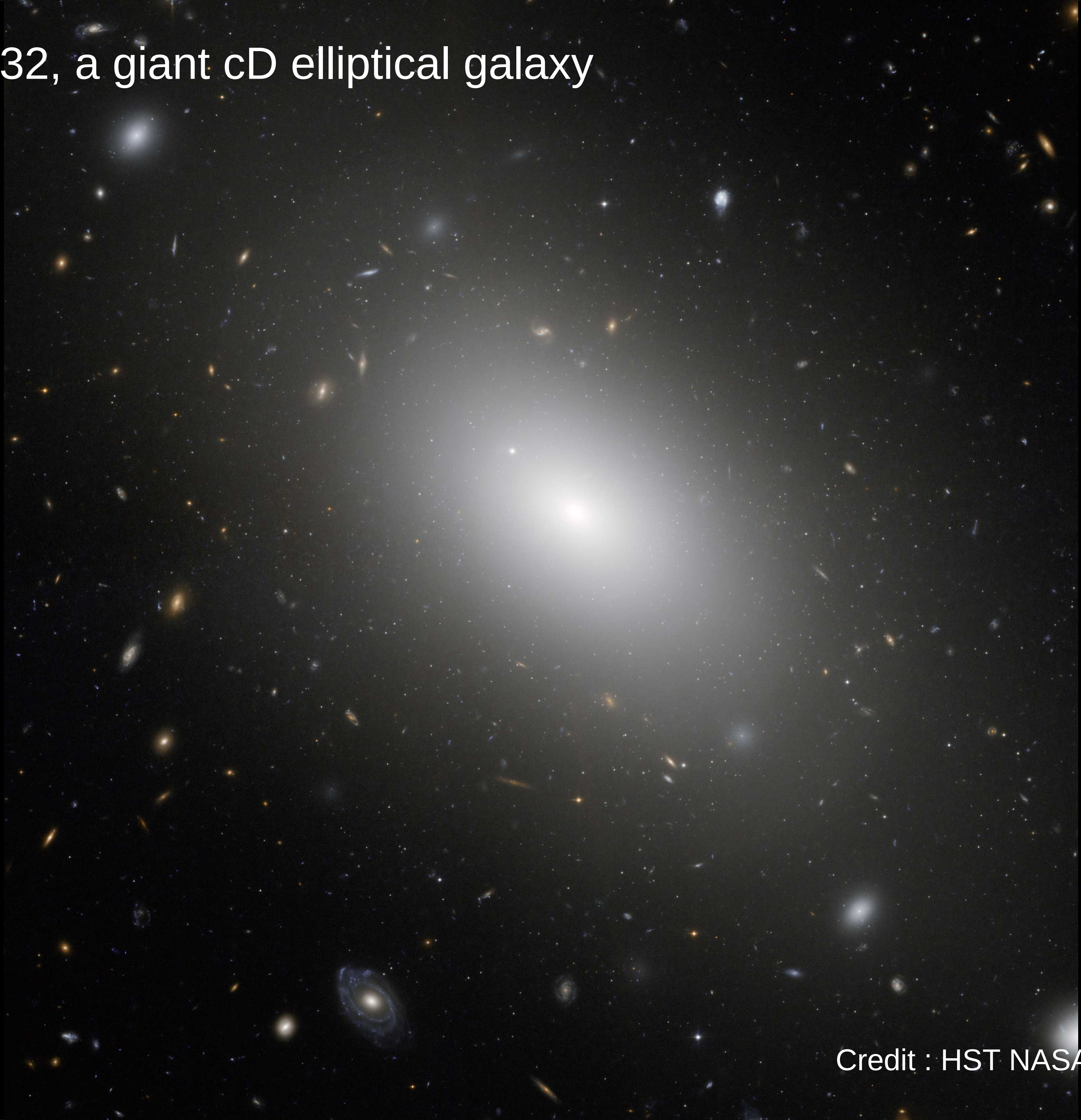
green : contours of constant total
angular momentum




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 15 --nlaps 100 --add_IL
```

Orbits in planar non-axisymmetric potentials

NGC 1132, a giant cD elliptical galaxy



Credit : HST NASA/ESA

NGC 1300 SBb

Surfaces of section (in planar potentials)

Can we visualize the phase phase and check if an additional integral of motion exists ?

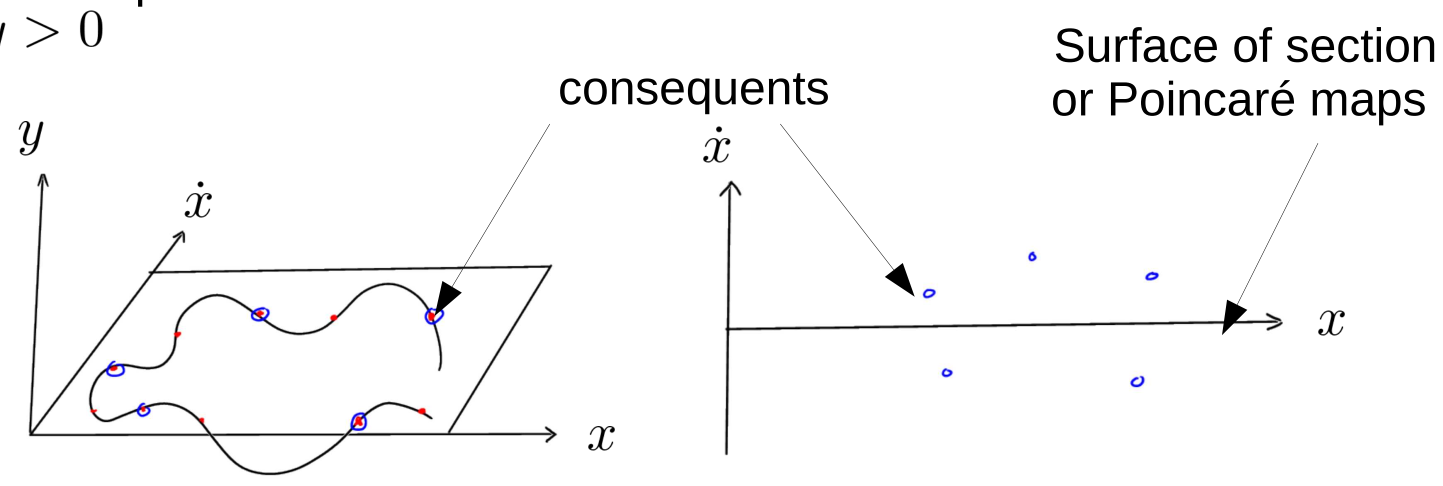
<u>Idea</u> :

We study the orbits in the plane z=0

- 4-D — 4 indep. variables $(x, y, \dot{x}, \dot{y})$
- Energy E

→ 3-D — 3 indep. variables $(x, y, \dot{x})$

- Drawing a 3-D phase space is still not easy. Instead, we draw slices of the phase space. We plot only phase space points that:
  - cross the $y = 0$ plane
  - have $\dot{y} > 0$



Surfaces of section (in planar potentials)

- A point in the surface of section (for a given E) defines an orbit as the three independent variables $(x, \dot{x}, y = 0)$ are defined.

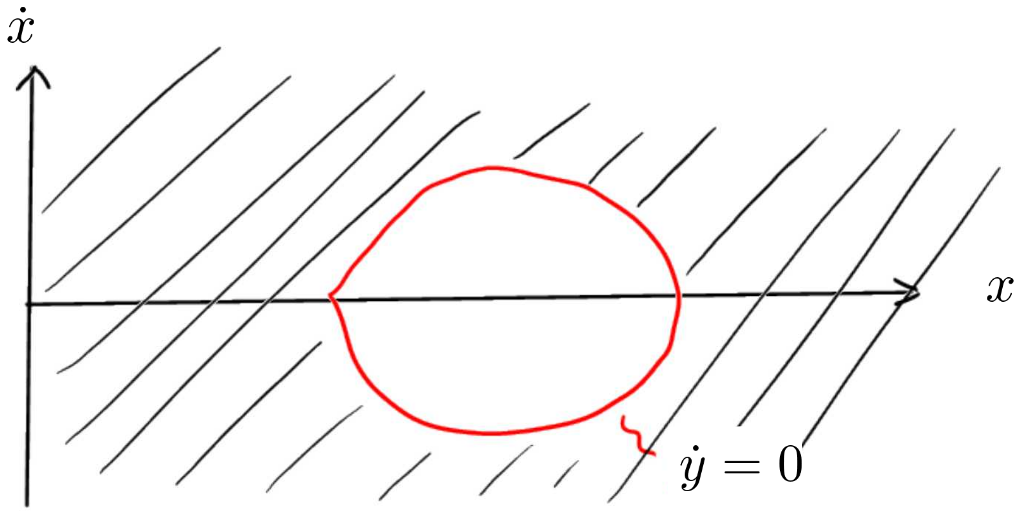
- Even if orbits have the same energy, they will never intersect in the plane.

- Zero velocity curve : curve defined by $\dot{y} = 0$

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \Phi(x, y = 0) \qquad \Rightarrow \qquad \dot{x} \leqslant \pm\sqrt{2\left[E - \Phi(x, y = 0)\right]}$$

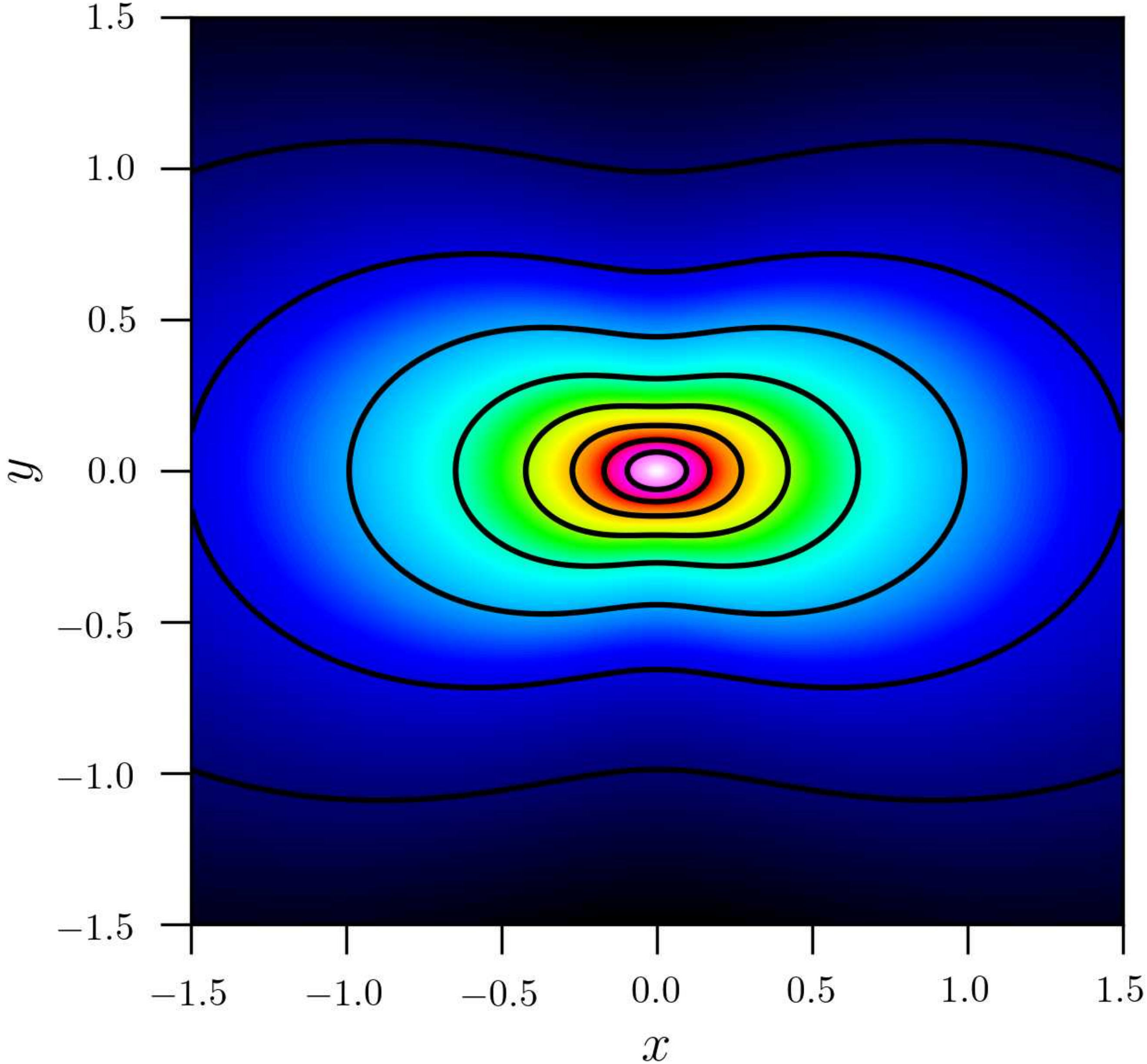
$$\dot{x}(x) = \pm\sqrt{2\left[E - \Phi(x, y = 0)\right]}$$

defines the accessible region of the phase space



Bar model : Logarithmic potential:
Vo=1 Rc=0.13 q=0.8)

$$\Phi_{\log}(x, y) = \frac{1}{2} V_0^2 \ln \left(R_c^2 + x^2 + \left(\frac{y}{q} \right)^2 \right)$$



$$R \ll R_c$$

Orbits in planar non-axisymmetric static potential

Model : logarithmic potential

$$\boxed{\phi(x,y) = \frac{1}{2} V_0^2 \ln\left(R_c + x^2 + \frac{y^2}{q^2}\right)}$$

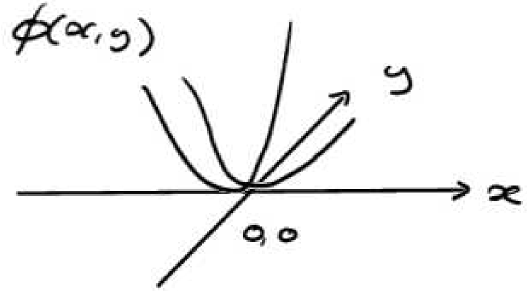
q : flattening parameter
(equipotential axis ratio)

$\boxed{\text{Motions for } R \ll R_c}$

$$\phi(x,y) \simeq \phi(0,0) + \overbrace{\cancel{\frac{\partial\phi}{\partial x}\Big|_{0,0}}}^{=0} x + \overbrace{\cancel{\frac{\partial\phi}{\partial y}\Big|_{0,0}}}^{=0} y + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}\Big|_{0,0} x^2 + \frac{1}{2}\frac{\partial^2\phi}{\partial y^2}\Big|_{0,0} y^2$$

$$\frac{\partial^2\phi}{\partial x^2}\Big|_{0,0} = \frac{V_0^2}{R_c^2}$$

$$\frac{\partial^2\phi}{\partial y^2}\Big|_{0,0} = \frac{V_0^2}{R_c^2}\,\frac{1}{q^2}$$



Equations of motion

$$\boxed{\begin{aligned} \ddot{x} &= -\frac{\partial \phi}{\partial x} \\ \ddot{y} &= -\frac{\partial \phi}{\partial x} \quad \text{y !} \end{aligned}} \quad \longrightarrow \quad \boxed{\begin{aligned} \ddot{x} &= -\frac{V_0^2}{R_c^2}\, x \\ \ddot{y} &= -\frac{V_0^2}{q^2 R_c^2}\, y \end{aligned}} \qquad \begin{aligned} \omega_x &= \frac{V_0}{R_c} \\ \omega_y &= \frac{V_0}{q R_c} \end{aligned}$$

2 decoupled harmonic oscillators with different frequencies

$$\omega_y = \frac{1}{q}\,\omega_x \qquad (q < 1)$$

if $q = \frac{n}{m}$ $\quad n, m \in \mathbb{N}$

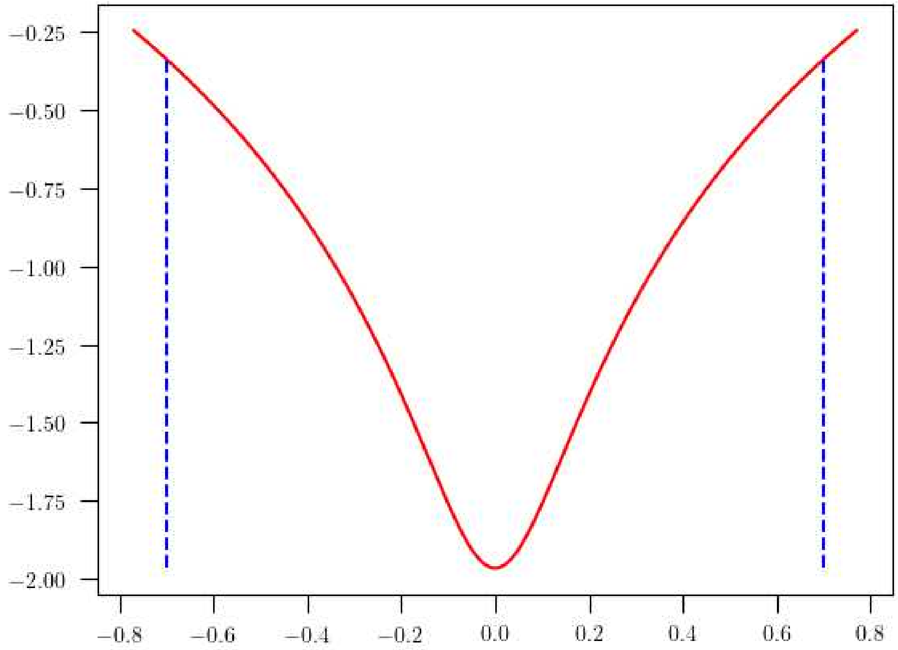
$\Rightarrow$ closed orbit

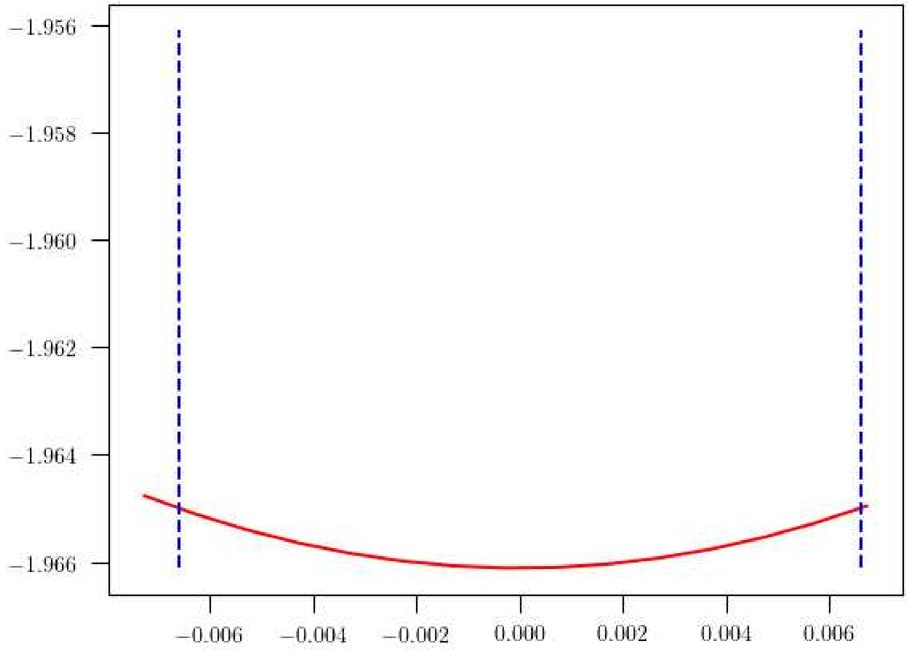
Integrals of motions (Hamiltonians)

$$H_x = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_x^2 x^2$$

$$H_y = \frac{1}{2}\dot{y}^2 + \frac{1}{2}\omega_y^2 y^2$$

Potential and energy



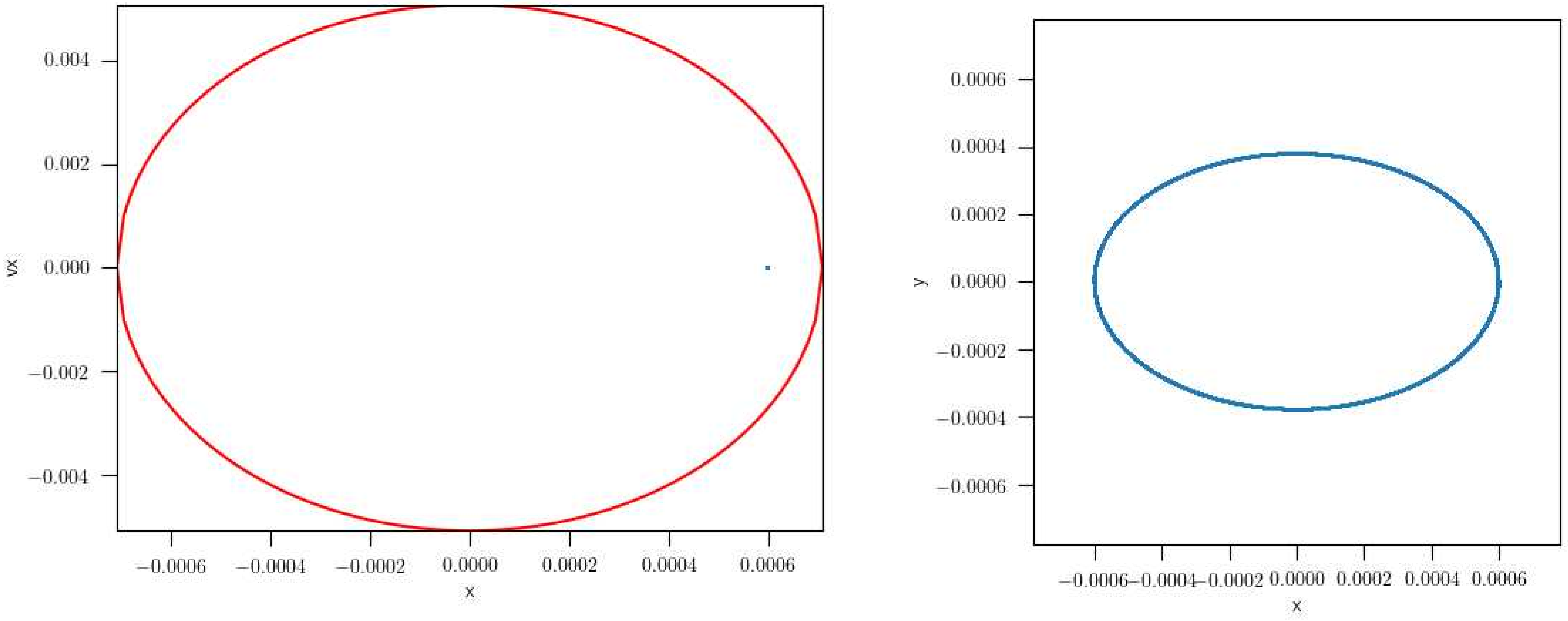


$R << R_c$

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --plotpotential
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.965  --plotpotential
```

The flattening – frequency dependency

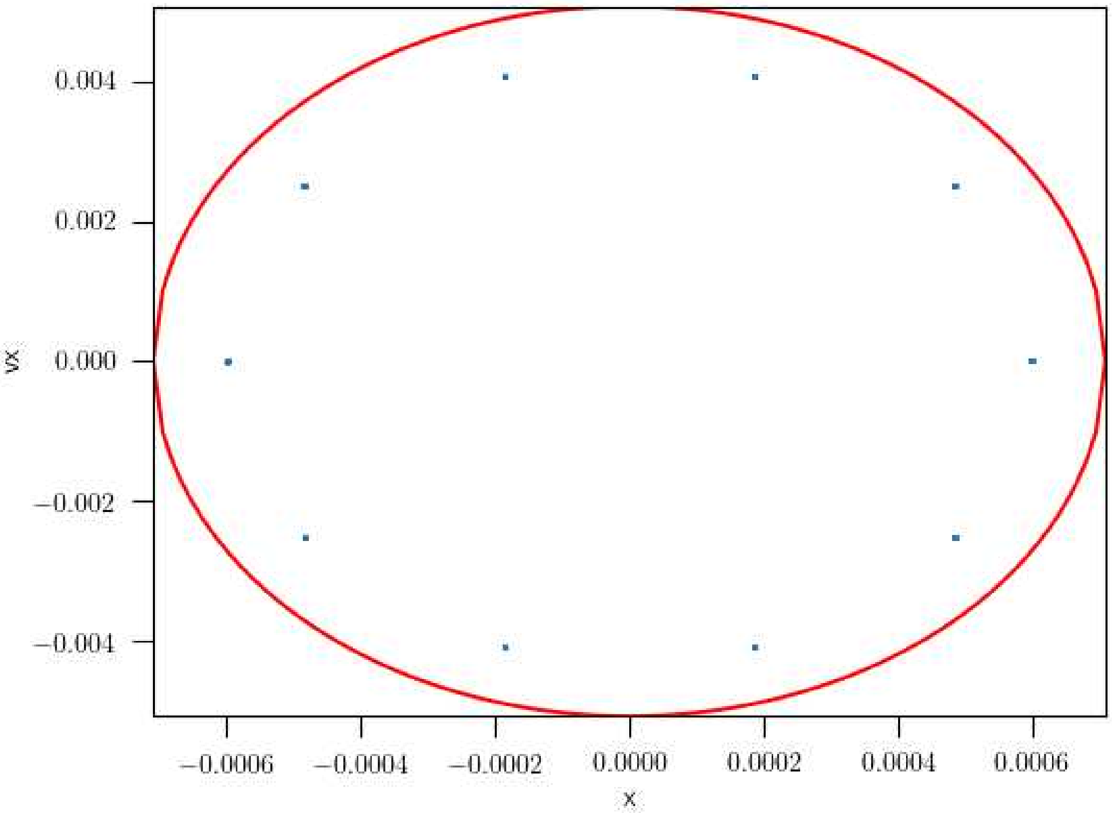
$$q = 1$$

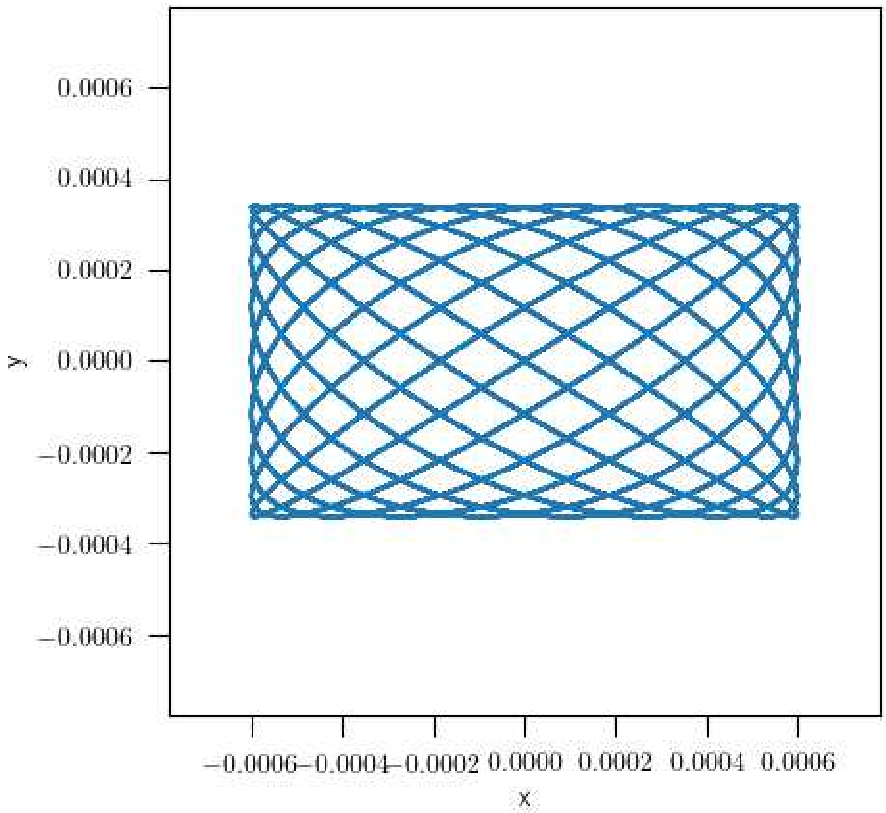



```
./mapping.py --V0 1. --Rc 0.14 --q 1.0   -E -1.9661   --x 0.0006
```

The flattening – frequency dependency

$$q = 0.9$$

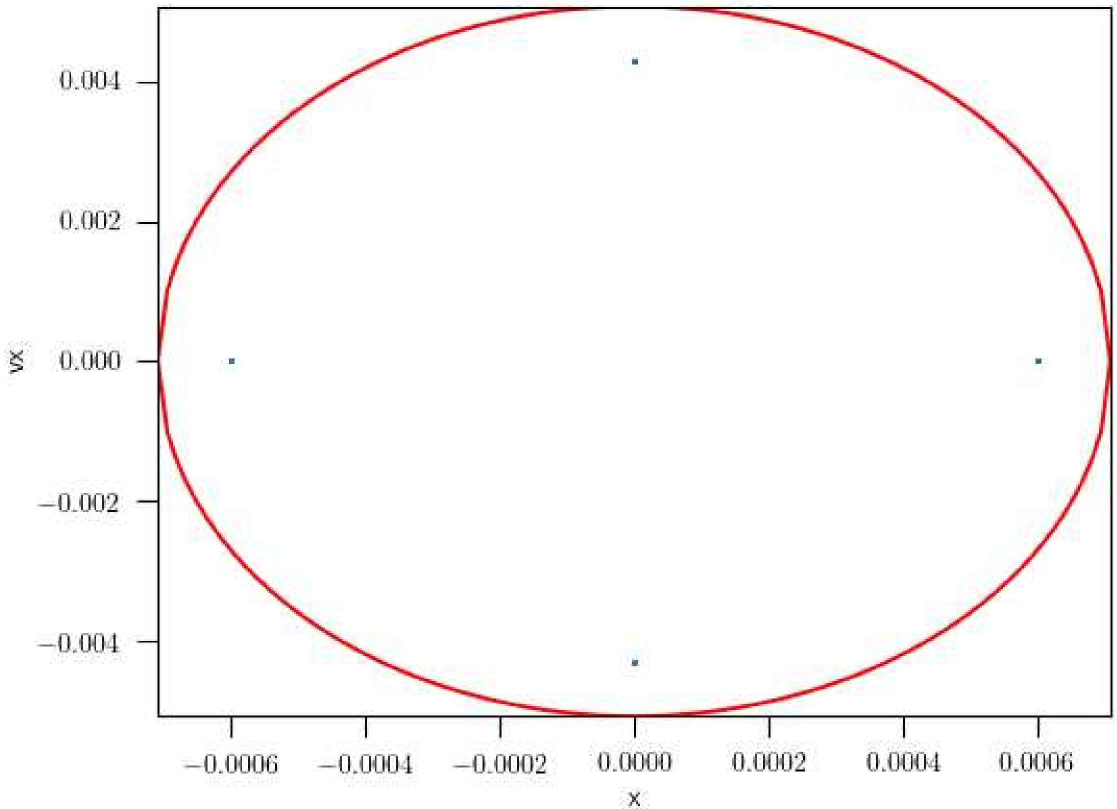


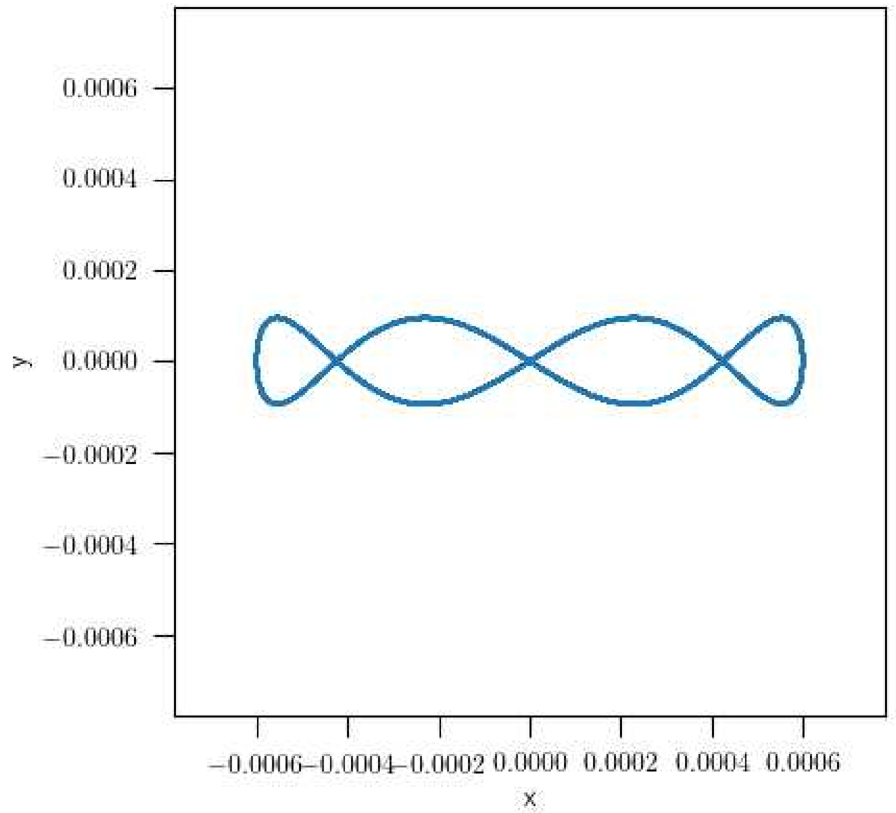



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9   -E -1.9661   --x 0.0006
```

The flattening – frequency dependency

$$q = 0.25$$

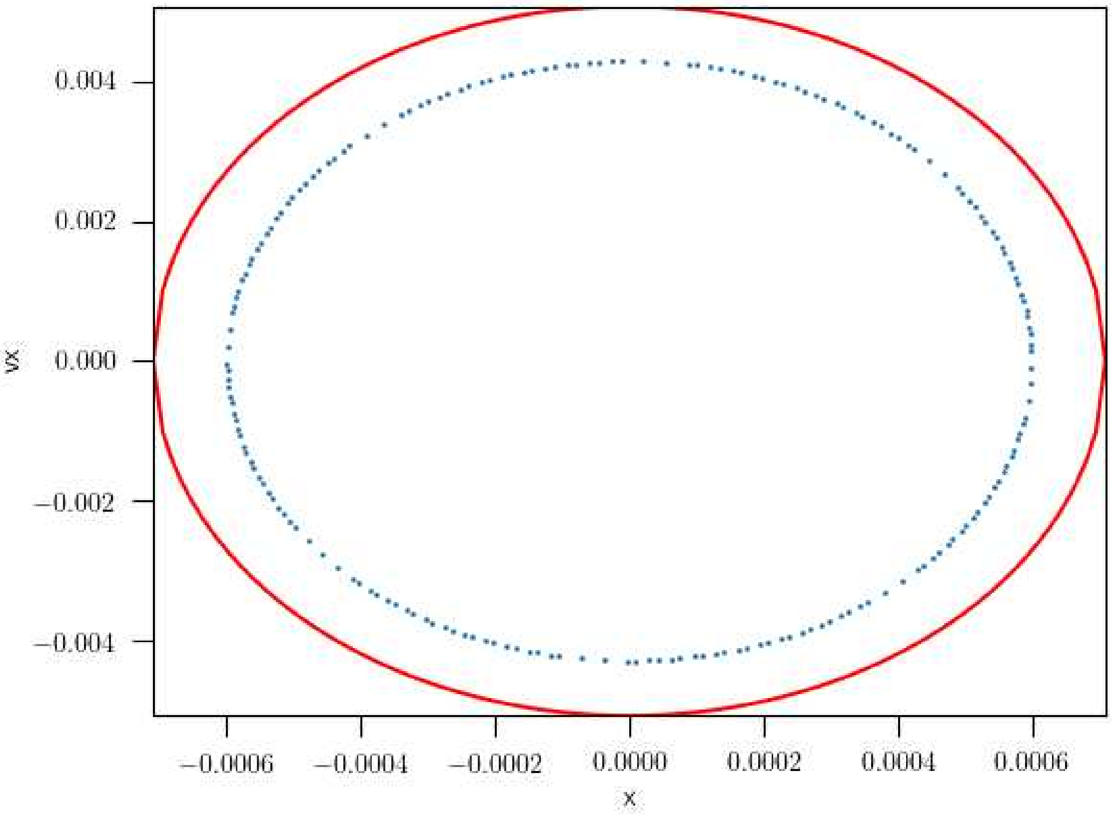


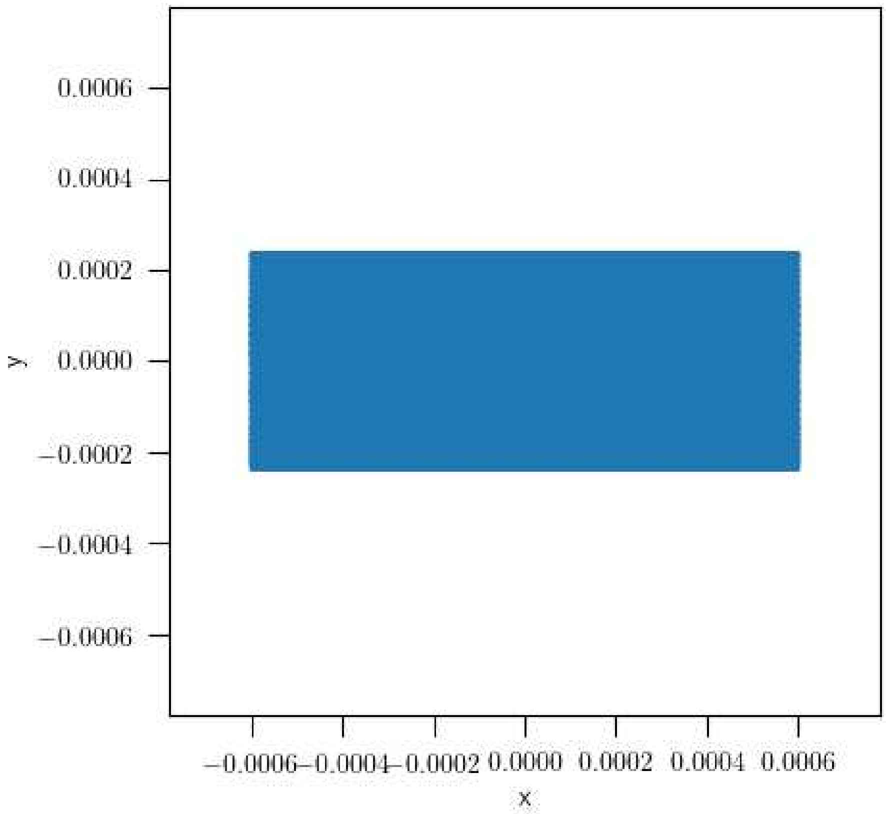



```
./mapping.py --V0 1. --Rc 0.14 --q 0.25   -E -1.9661   --x 0.0006
```

The flattening – frequency dependency

$$q = 0.62388462341$$

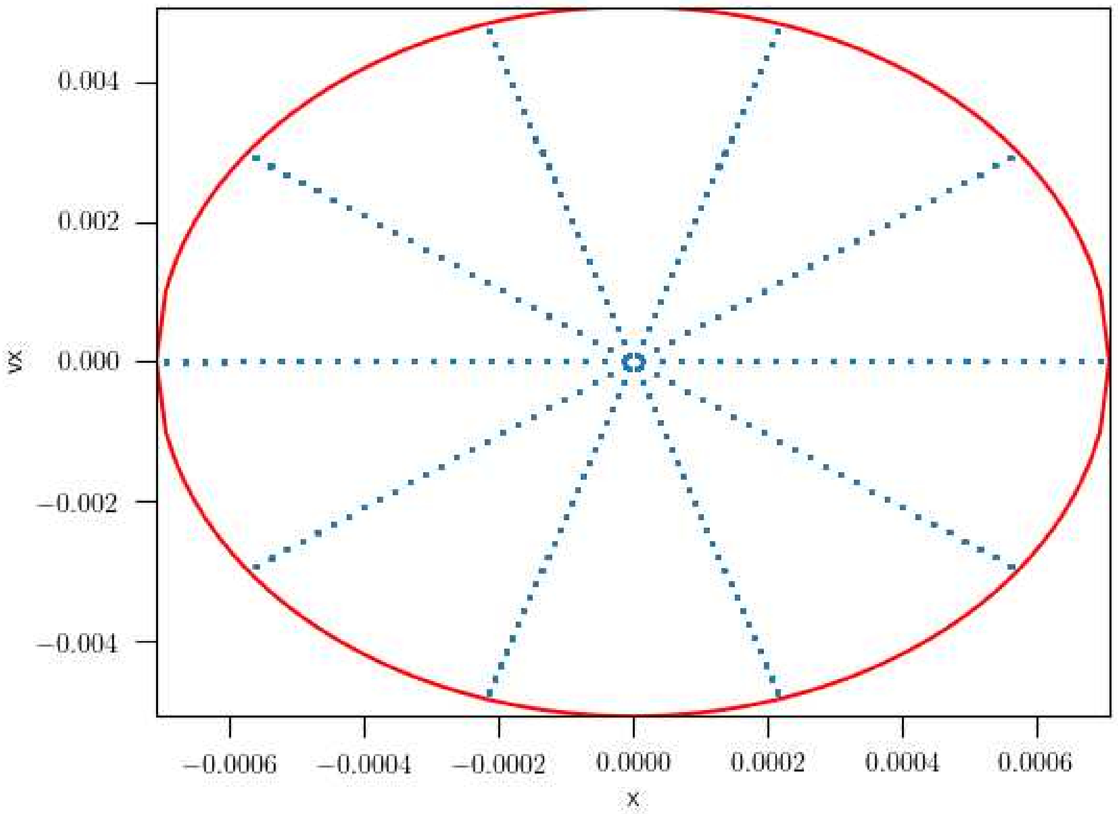





```
./mapping.py --V0 1. --Rc 0.14 --q 0.62388462341 -E -1.9661 --x 0.0006 --nlaps 200
```

Complete phase space

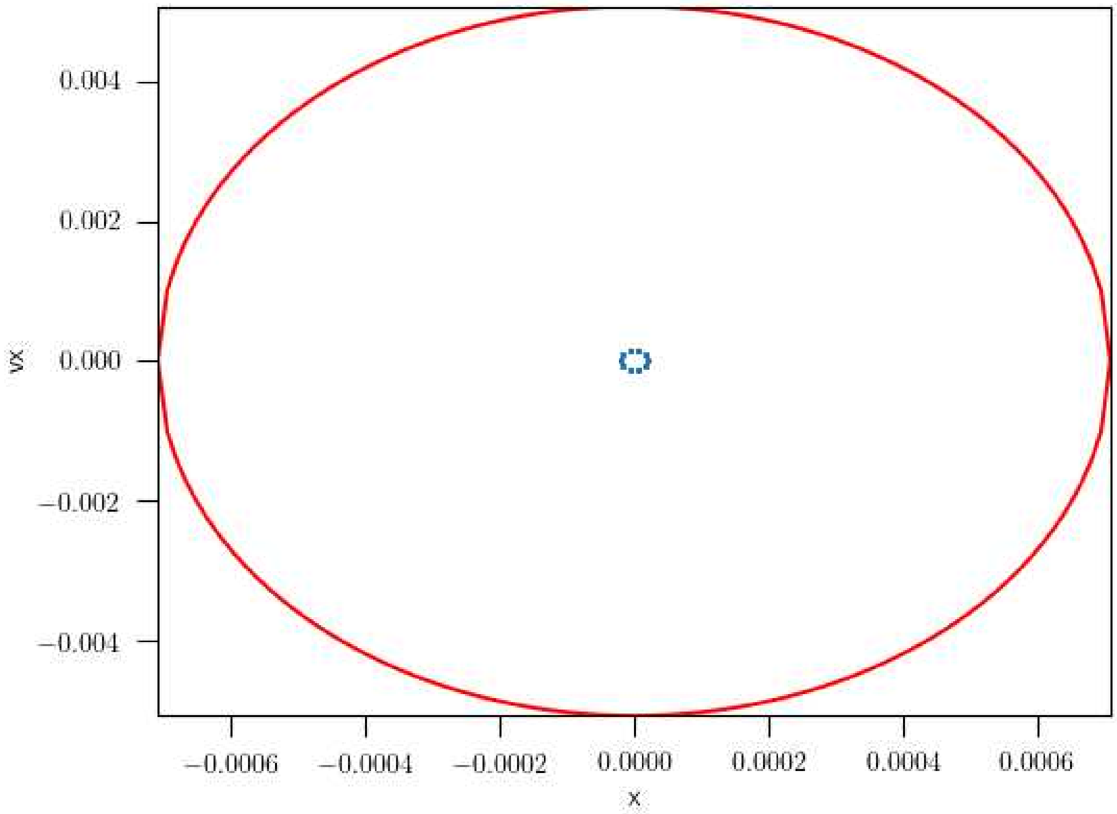
$$q = 0.9$$

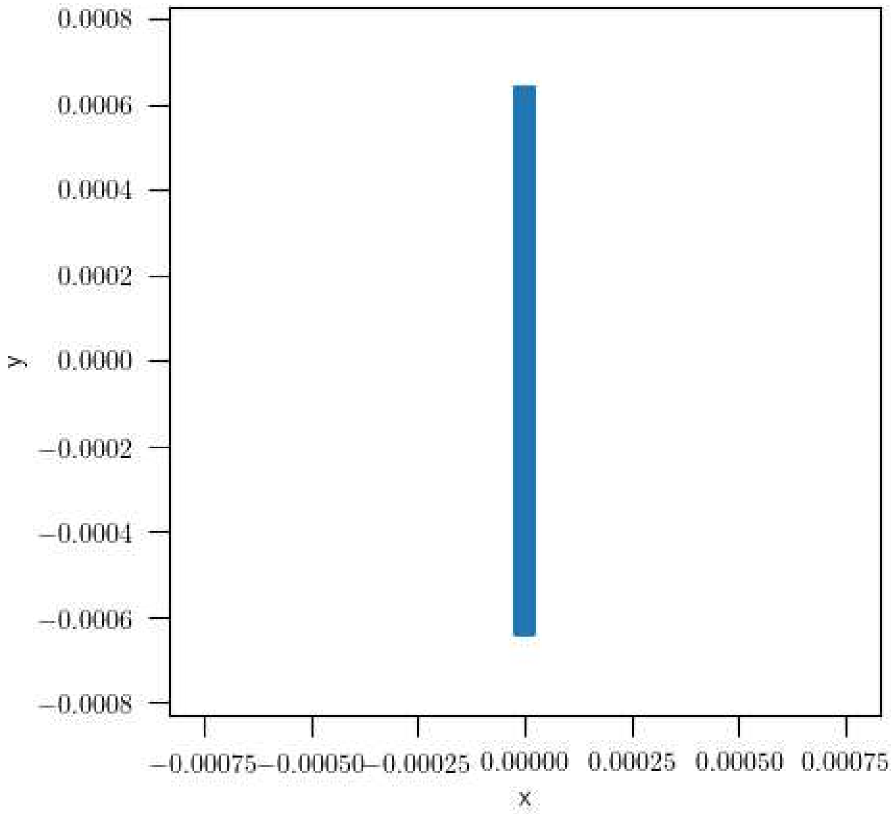



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.9661   --norbits 50
```

small x, Y-elongated orbits (box orbit)

$$q = 0.9$$

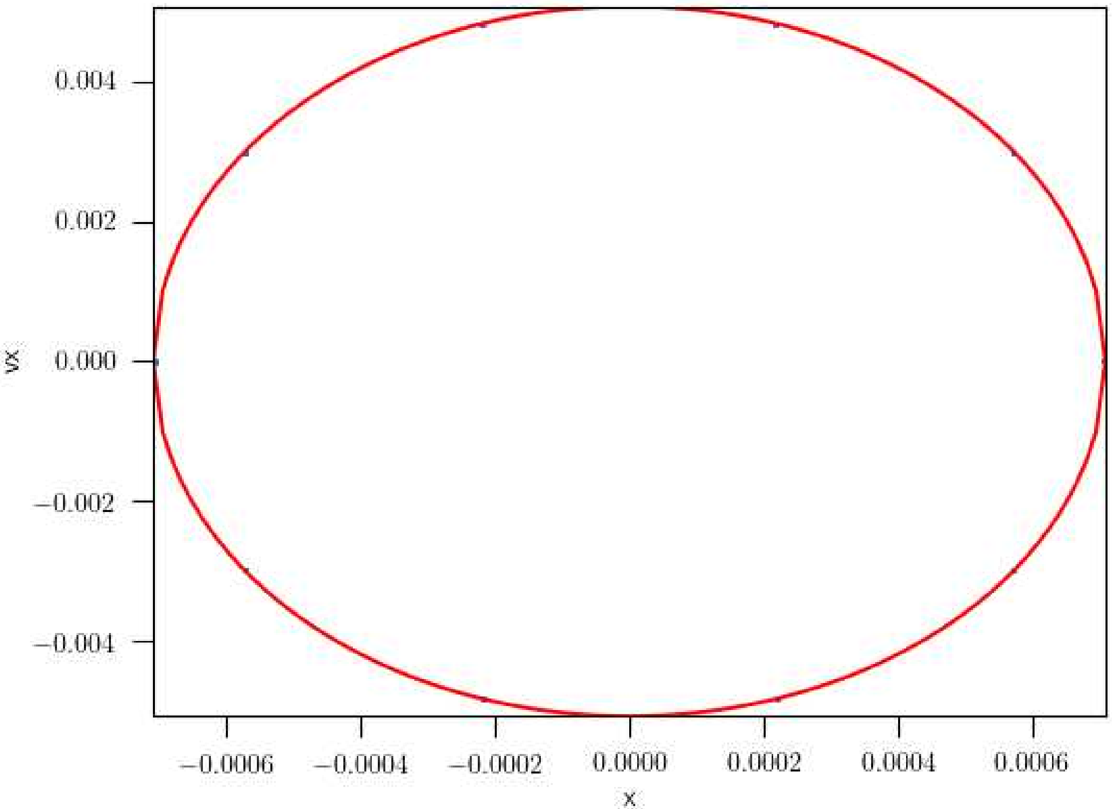


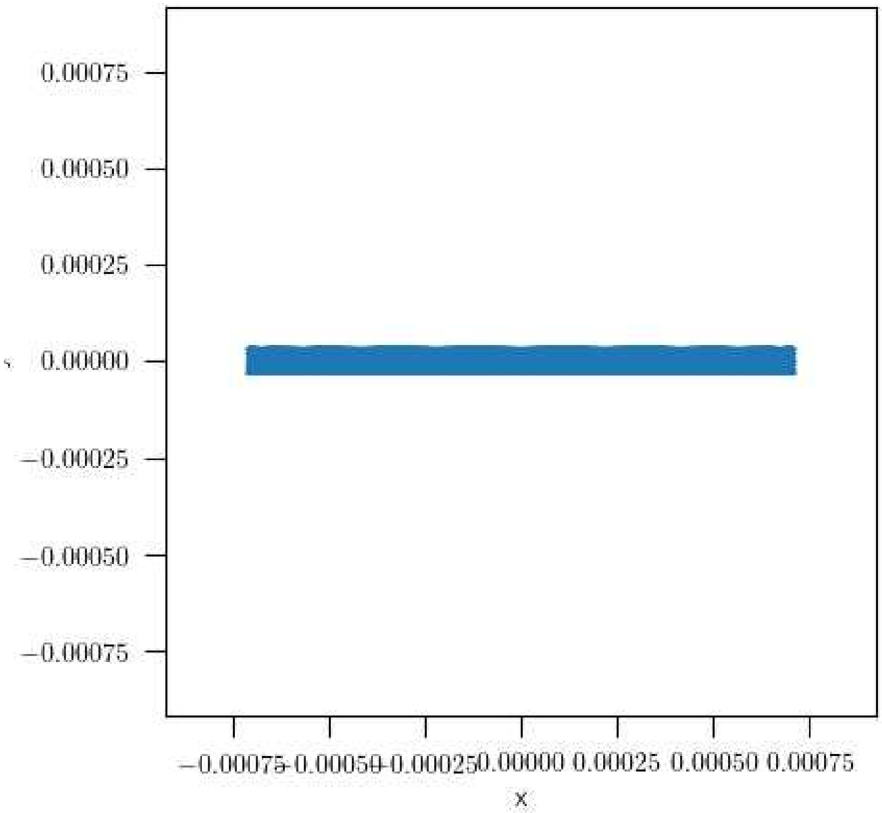



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.9661   --x 0.00002
```

large x, X-elongated orbits (box orbit)

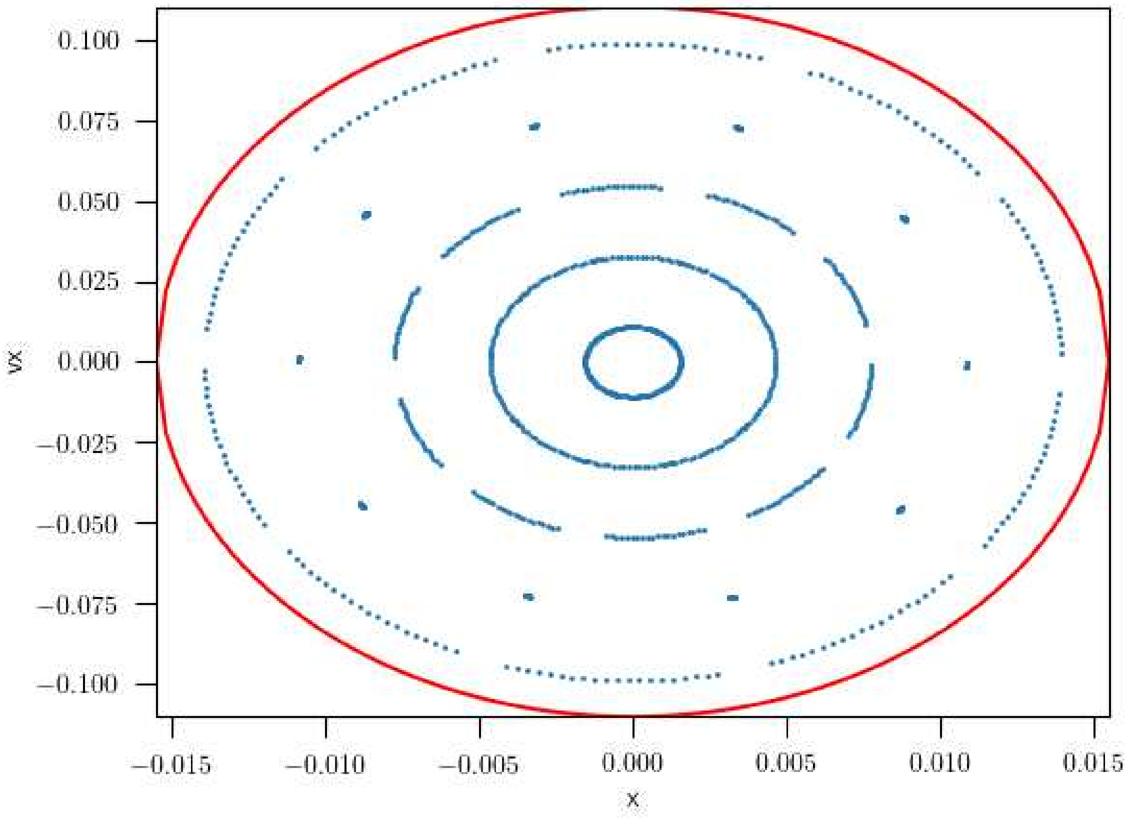
$$q = 0.9$$





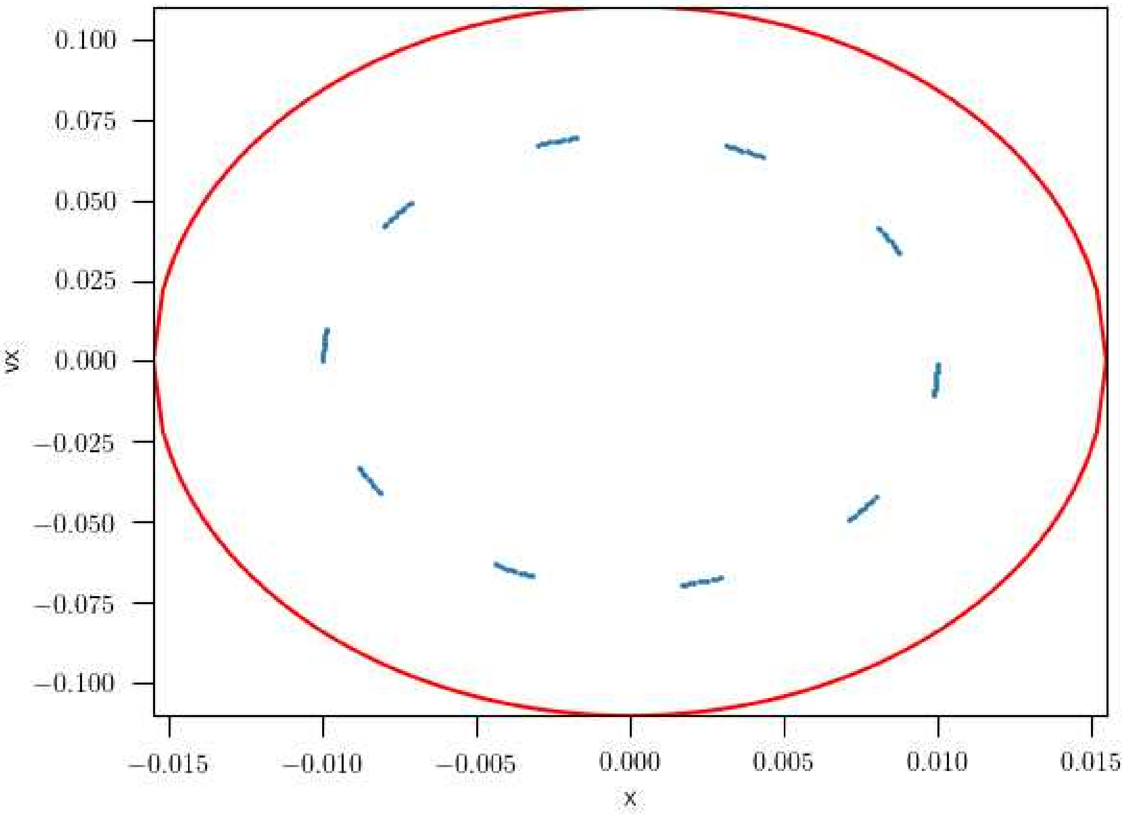

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.9661   --x 000709
```

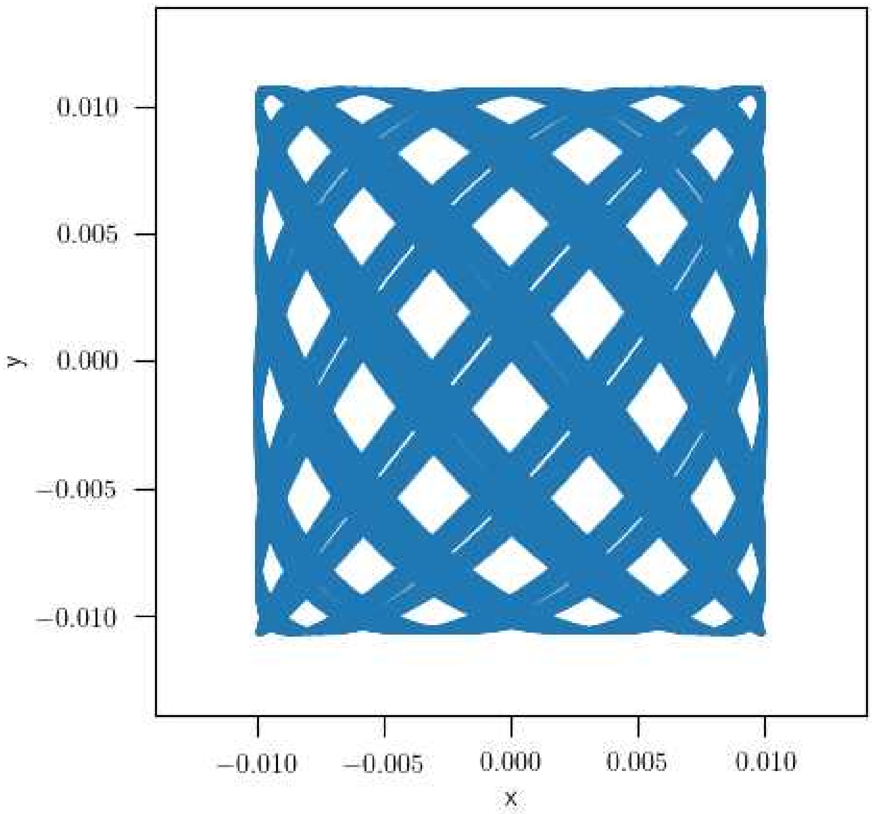
Increasing energy : perturbed harmonic oscillator (coupling terms)




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96
```

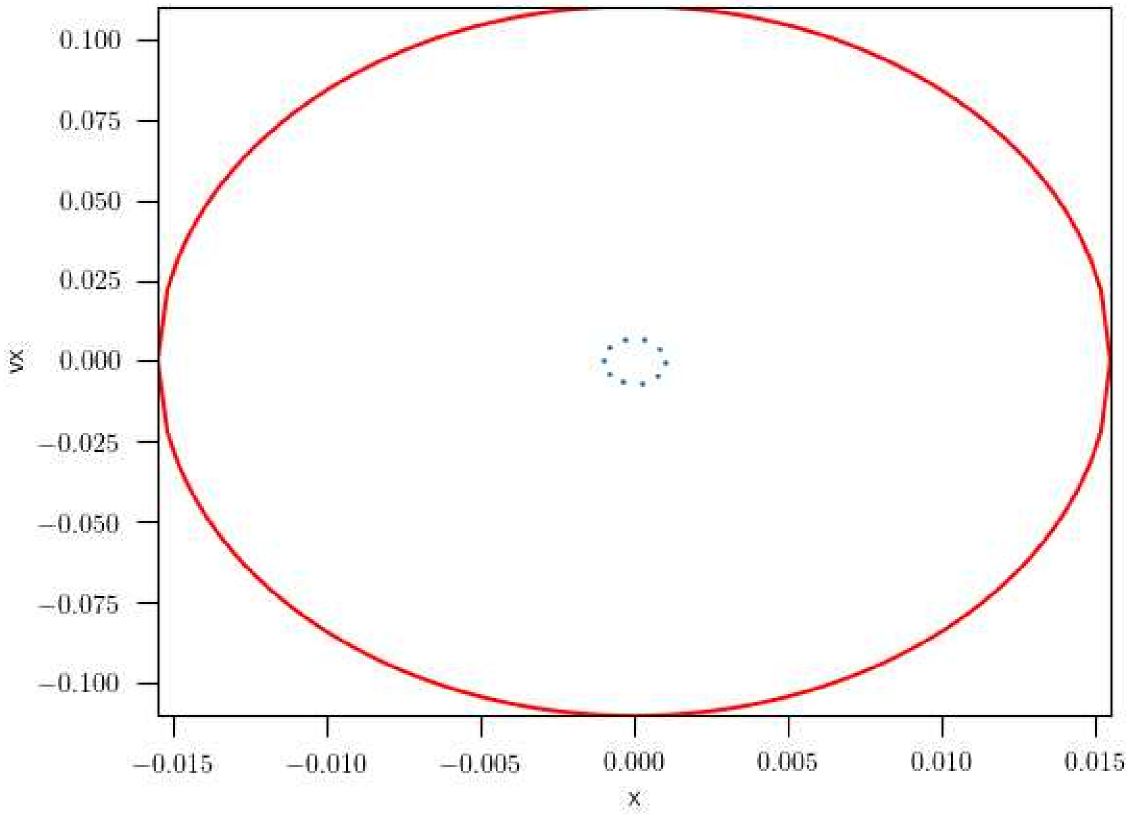
Increasing energy : perturbed harmonic oscillator

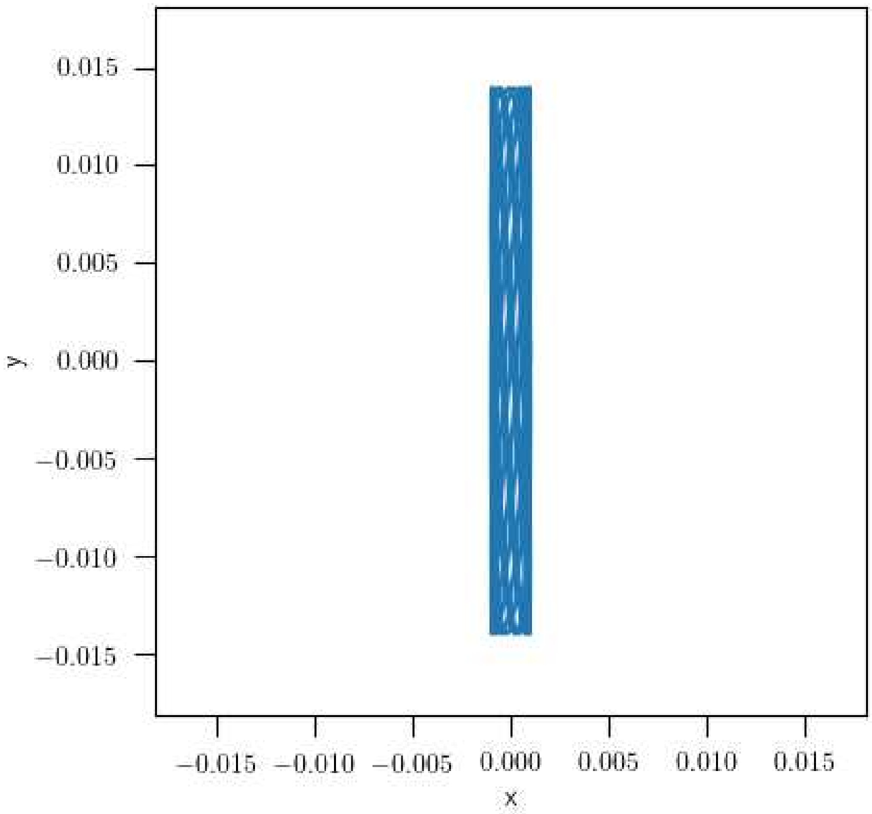





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96   --x 0.01
```

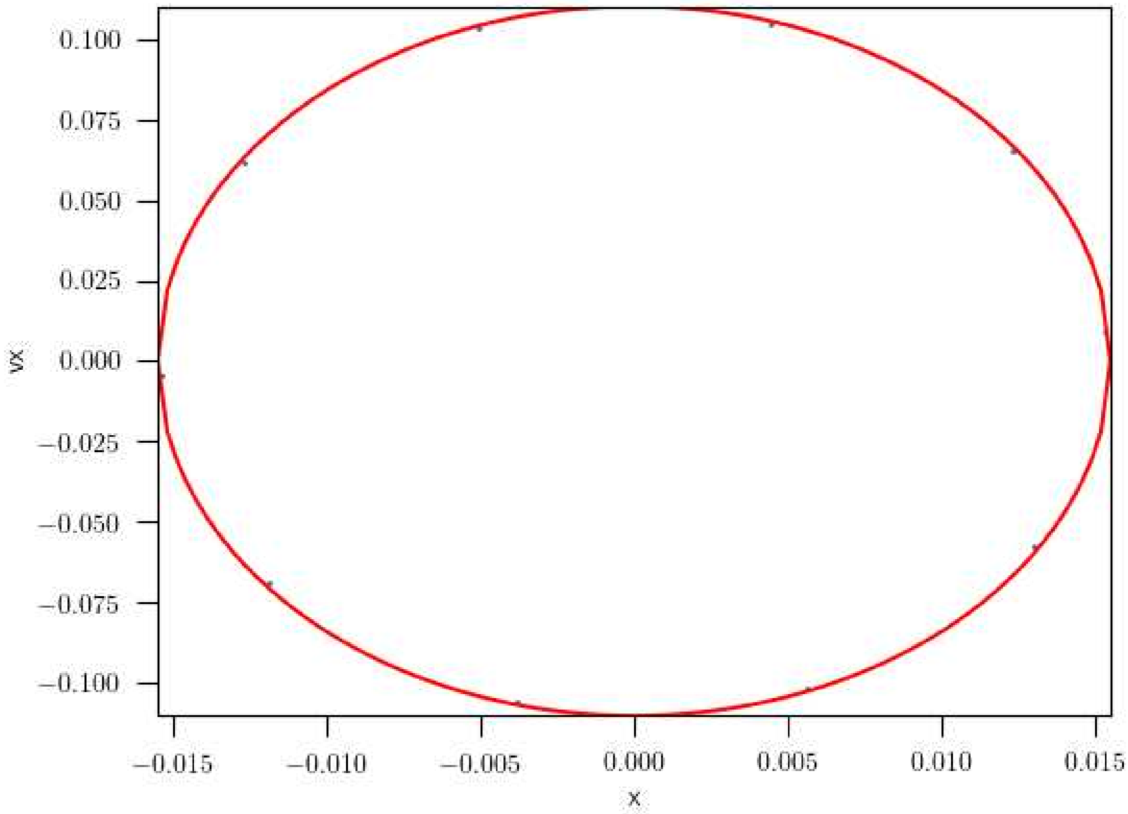
small x, Y-elongated orbits (box orbit)

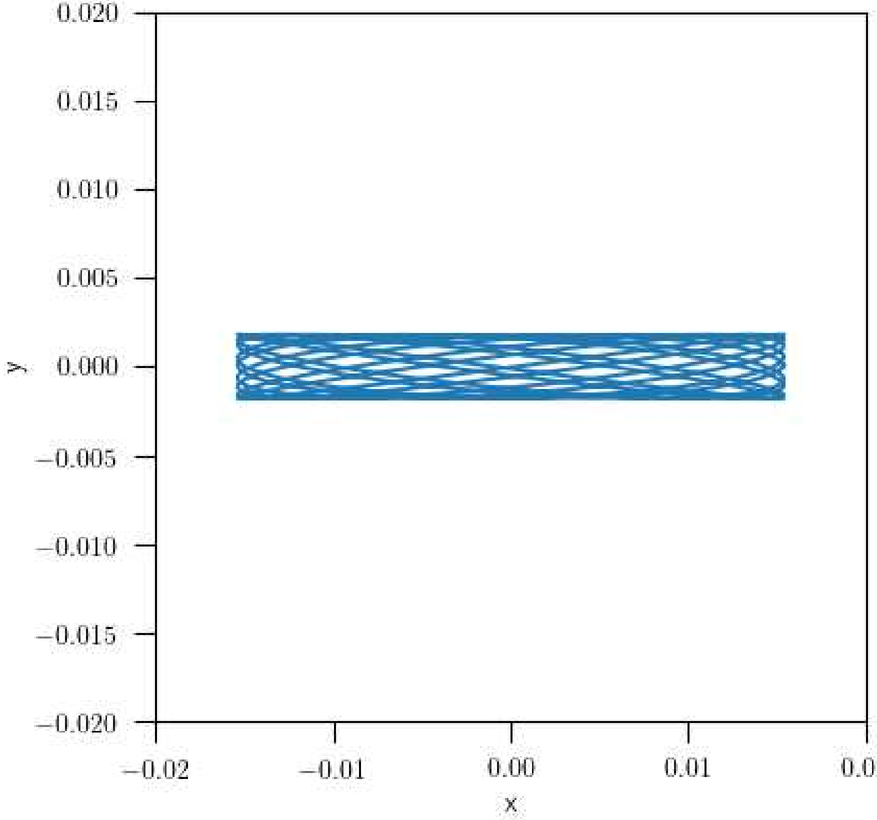





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96   --x 0.001  --nlaps 10
```

large x, X-elongated orbits (box orbit)






```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96   --x 0.0154 --nlaps 10
```

$$R \gg R_c$$

Motions for $R \gg R_c$

$$\phi(x,y) = \frac{1}{2} V_0^2 \ln\left(R_c + x^2 + \frac{y^2}{q^2}\right)$$

$$\simeq \frac{1}{2} V_0^2 \ln\left(x^2 + \frac{y^2}{q^2}\right) \underset{q \simeq 1}{\sim} \frac{1}{2} V_0^2 \ln\left(R^2\right)$$

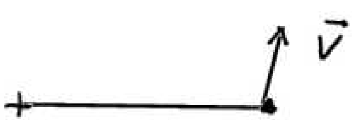
Orbit families

① **box orbits** (disturbed 2D harmonic oscillator)

$V_{//}$: $V_y \simeq 0$

if $V_y = 0$: radial orbit ($L_z = 0$)

② **loop orbits**

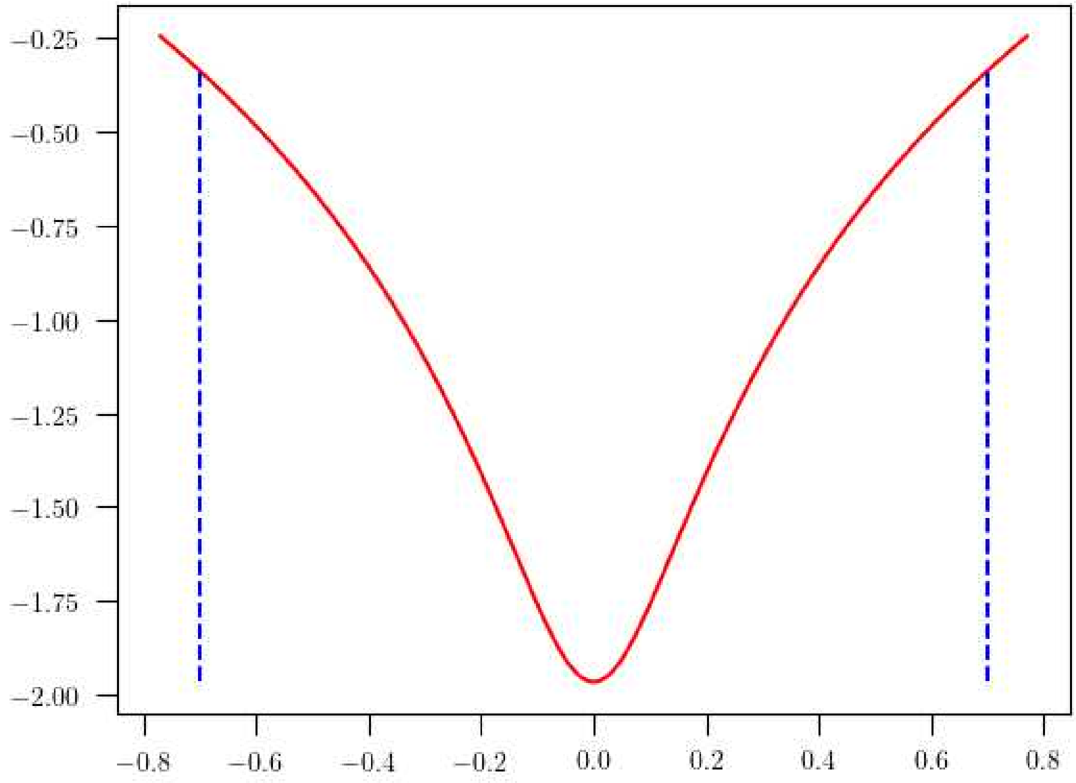


$V_\perp$: $V_y \simeq V_0$

if $V_y = V_0$: circular orbit

$q = 1$

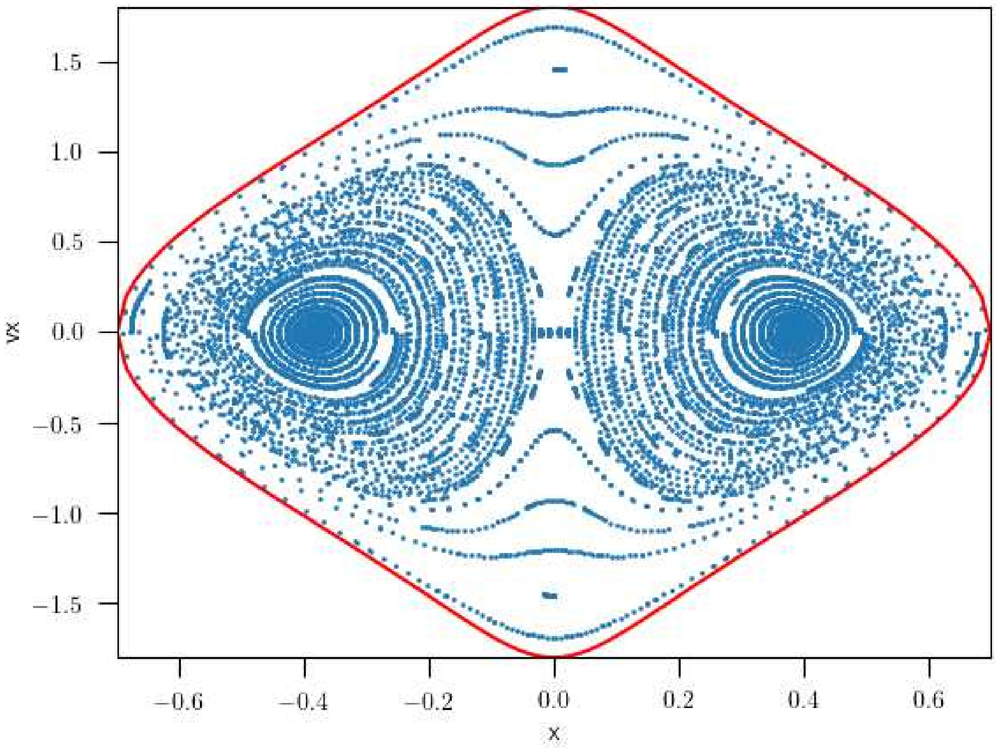
Potential and energy



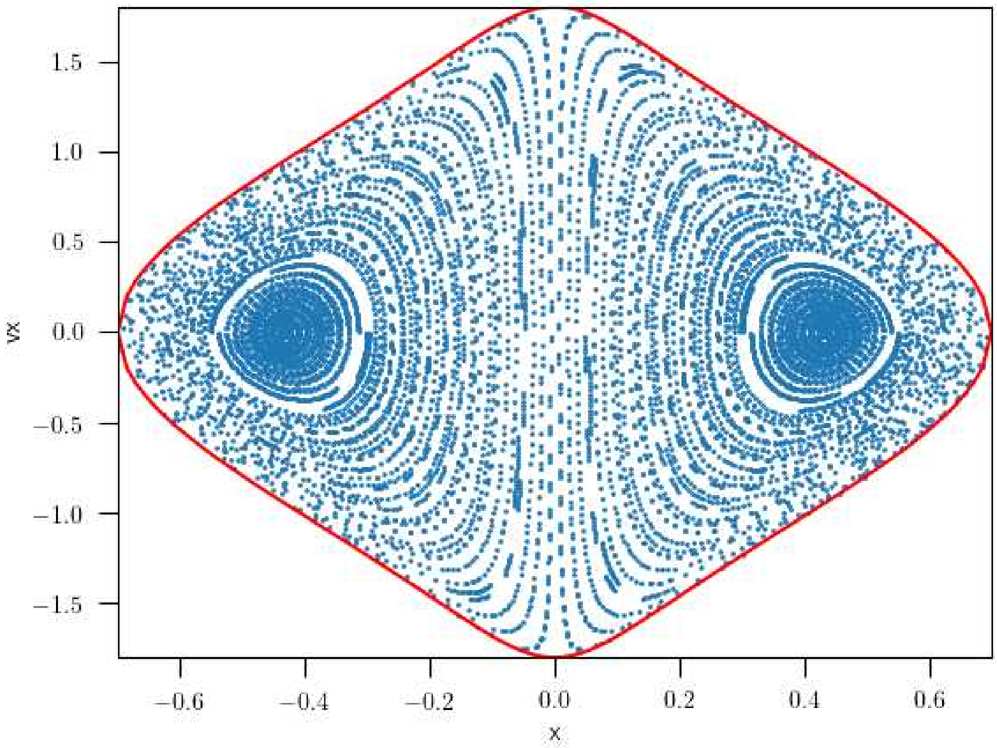

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --plotpotential
```

Phase space

$q = 0.9$

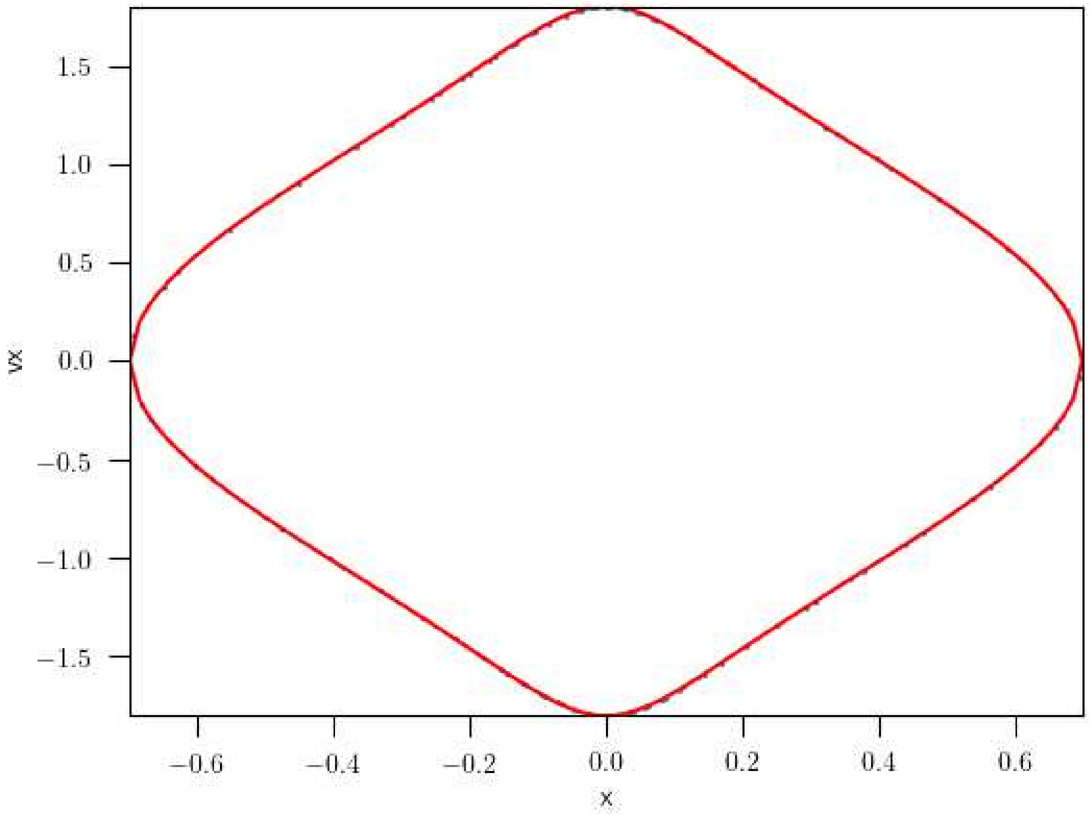


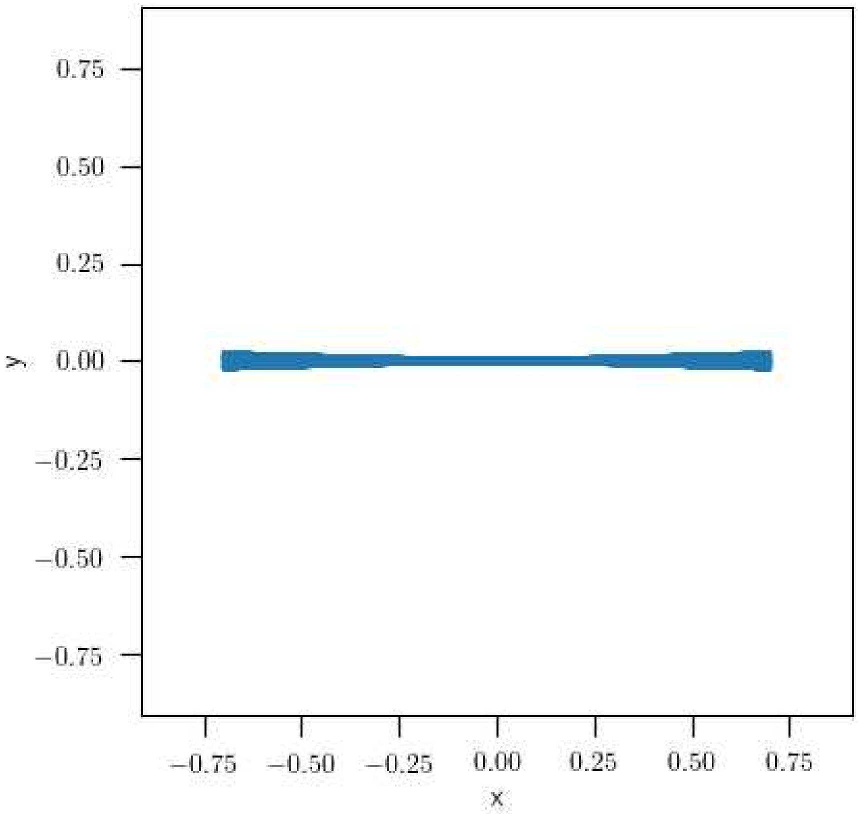
$q = 1.0$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337 --norbits 100
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337 --norbits 100
```

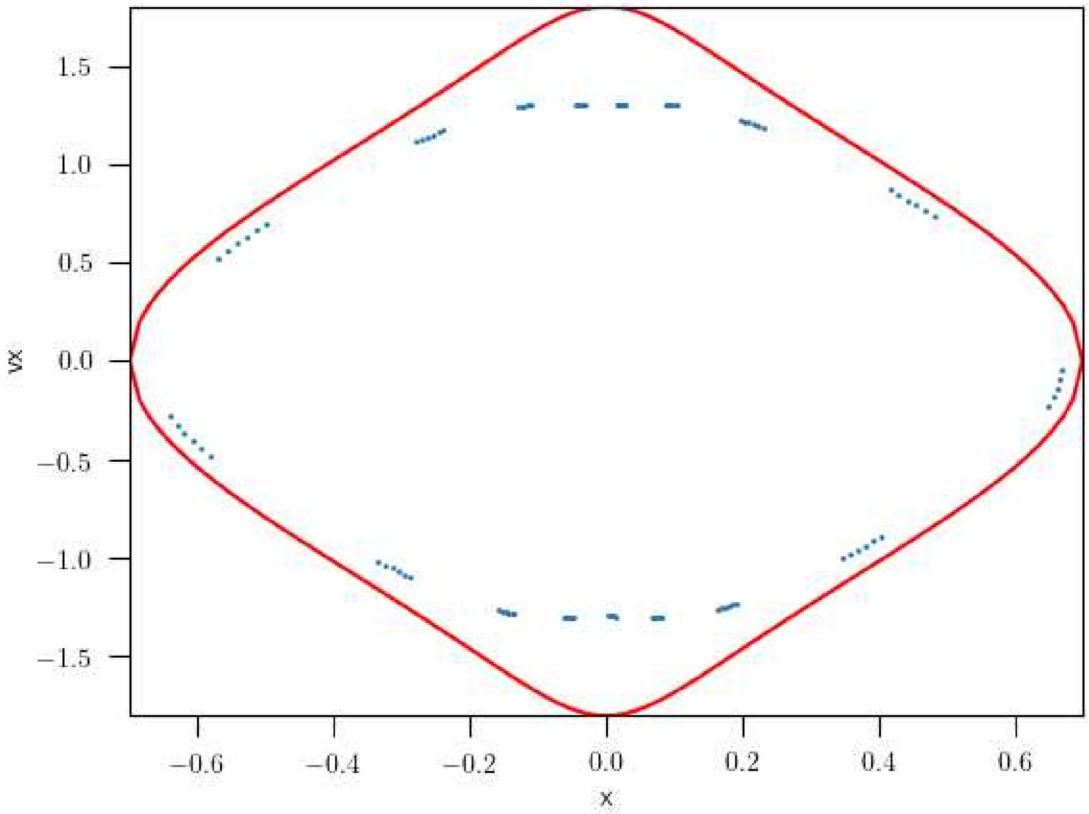
Box orbits

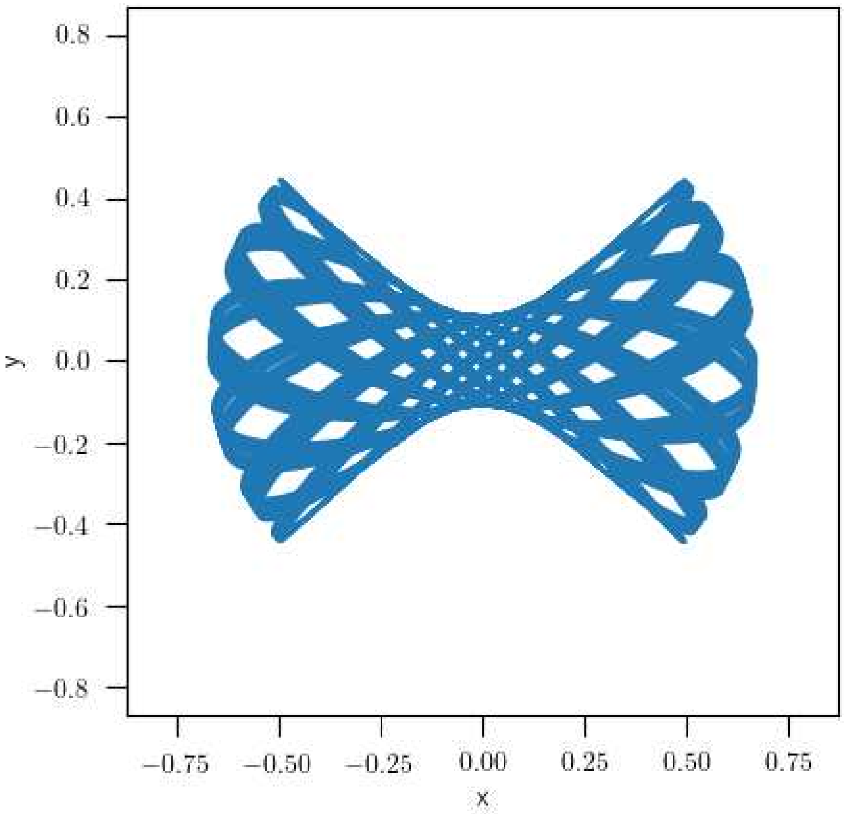





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.7
```

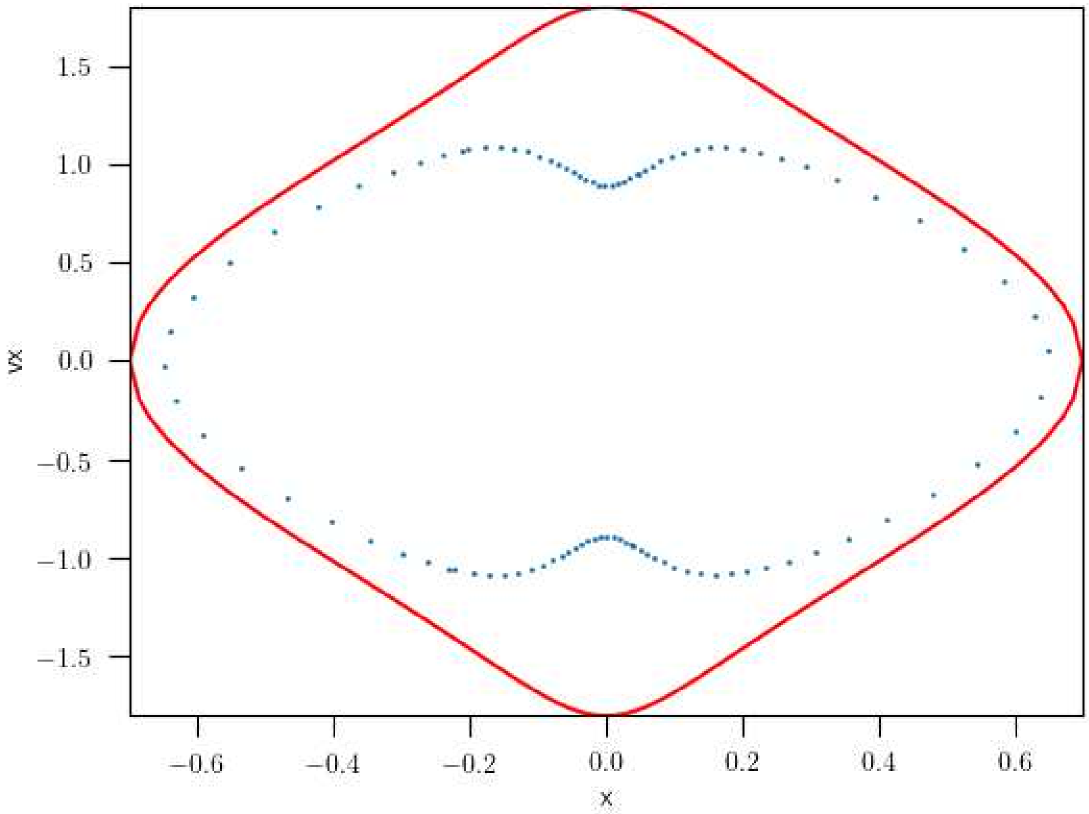
Box orbits

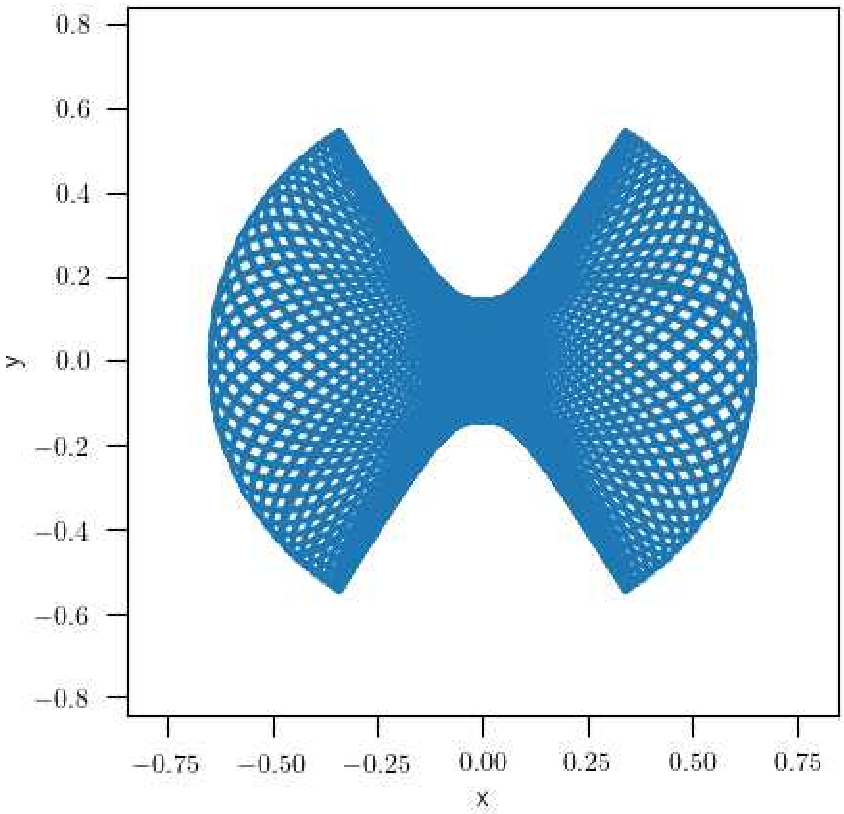





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.67
```

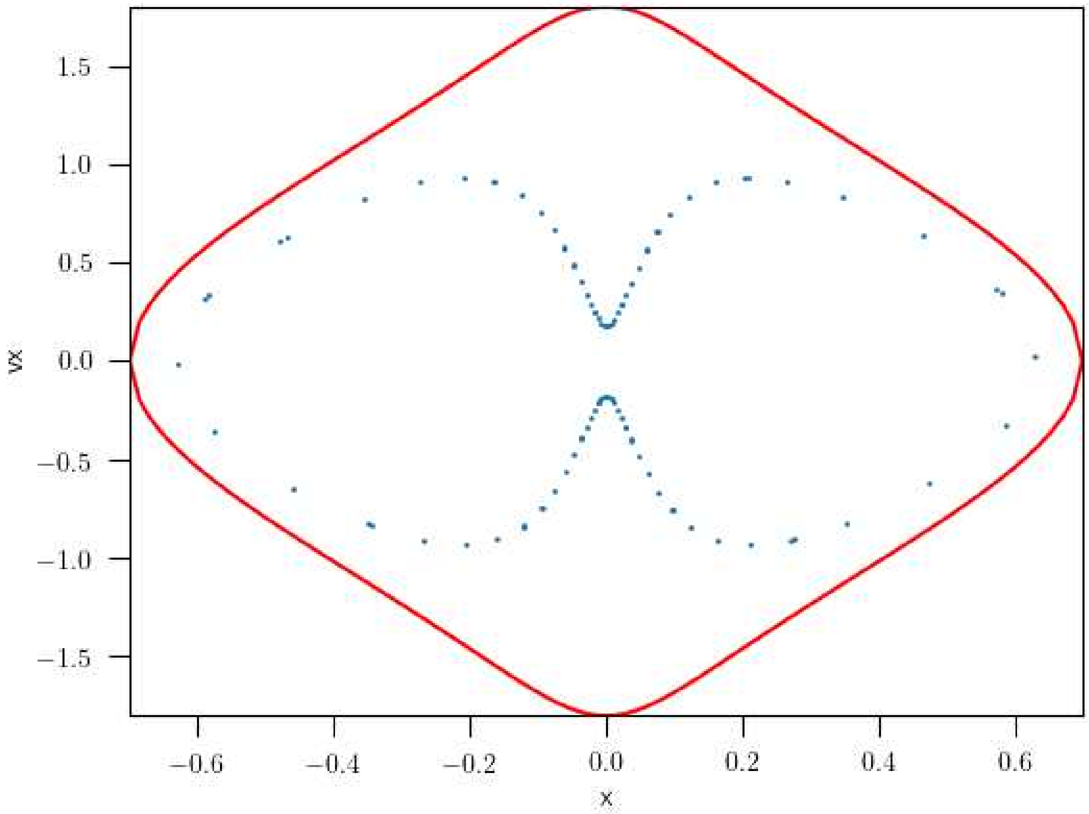
Box orbits

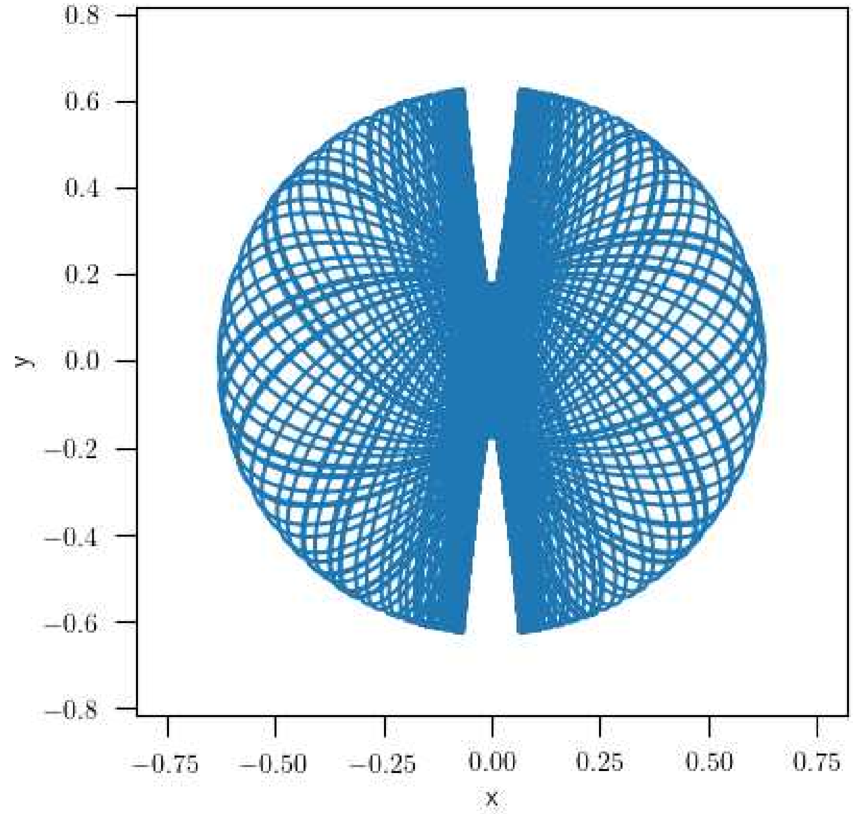





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.65
```

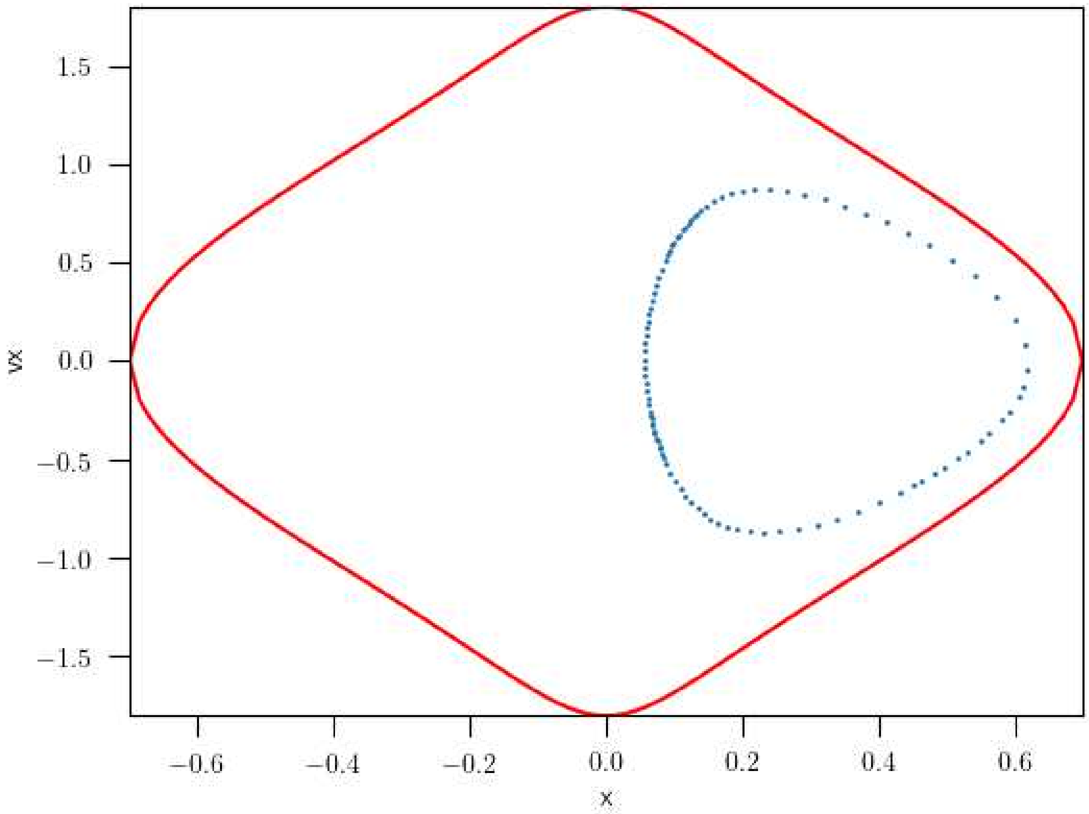
Box orbits

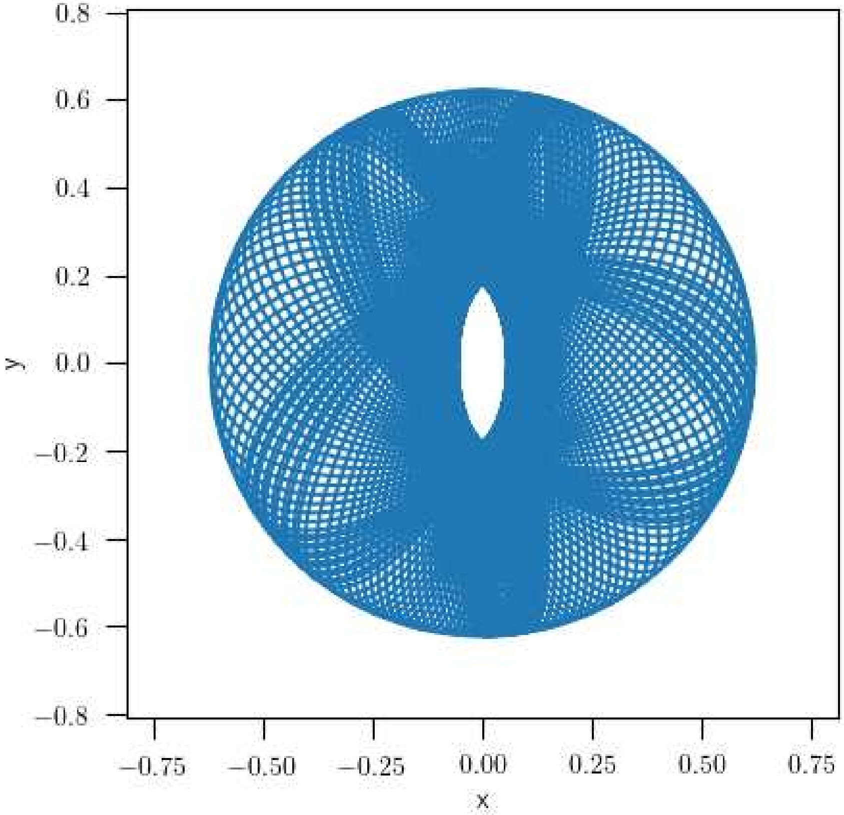





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.63
```

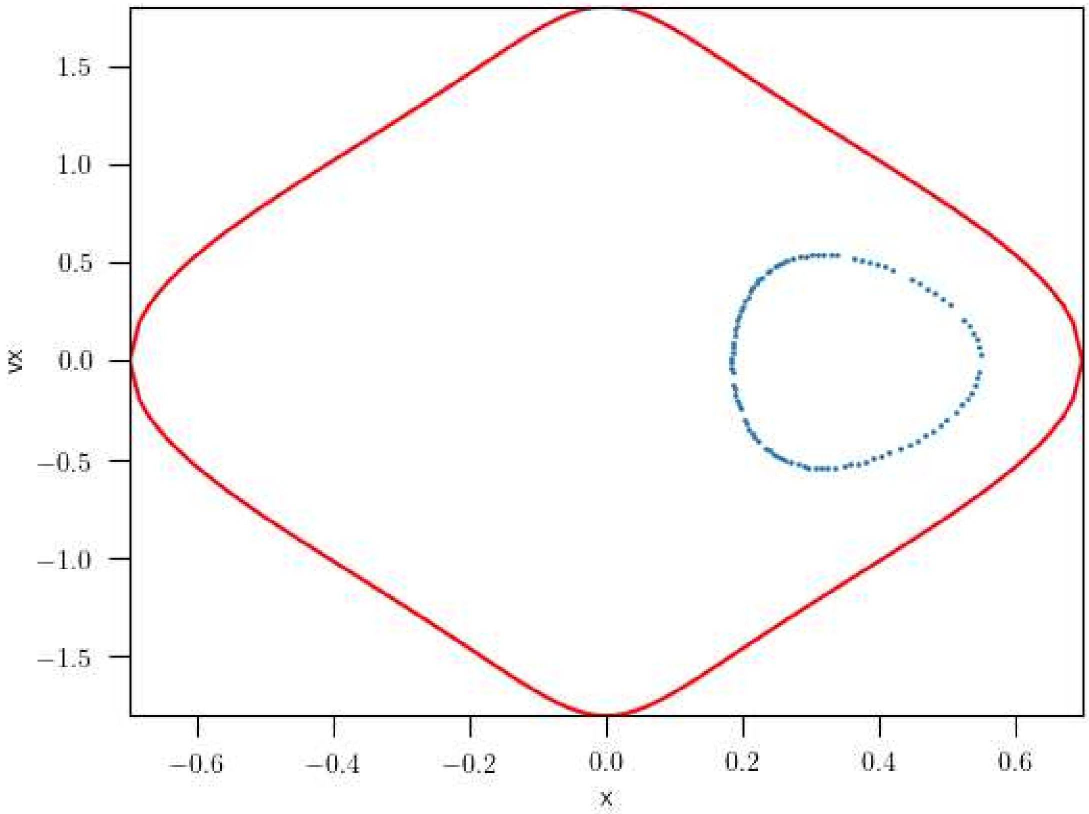
Loop orbits

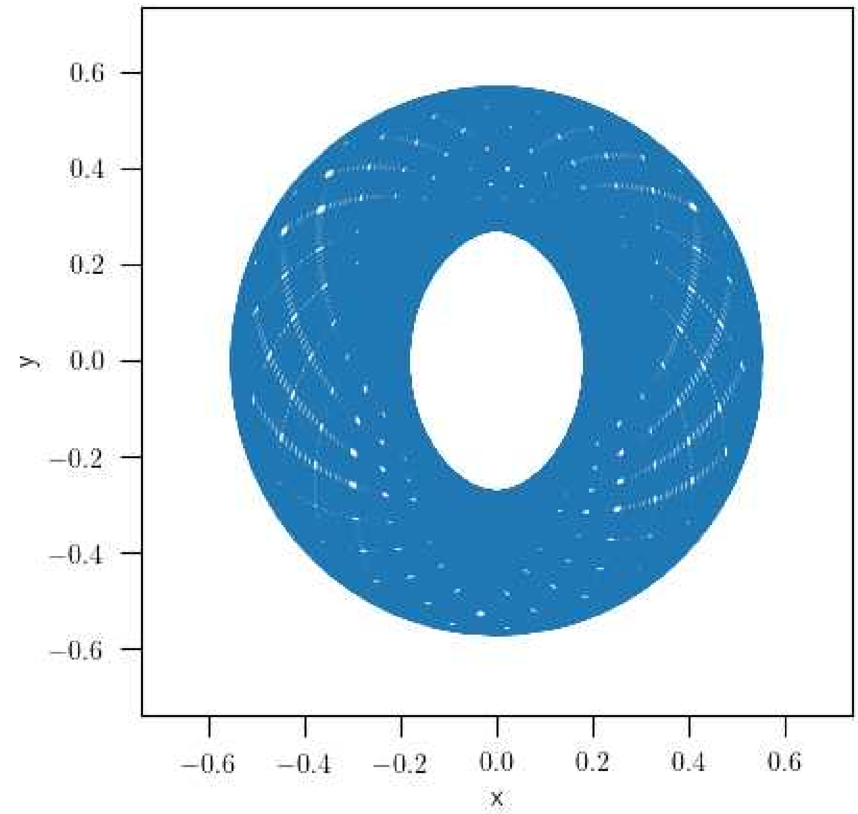





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.62
```

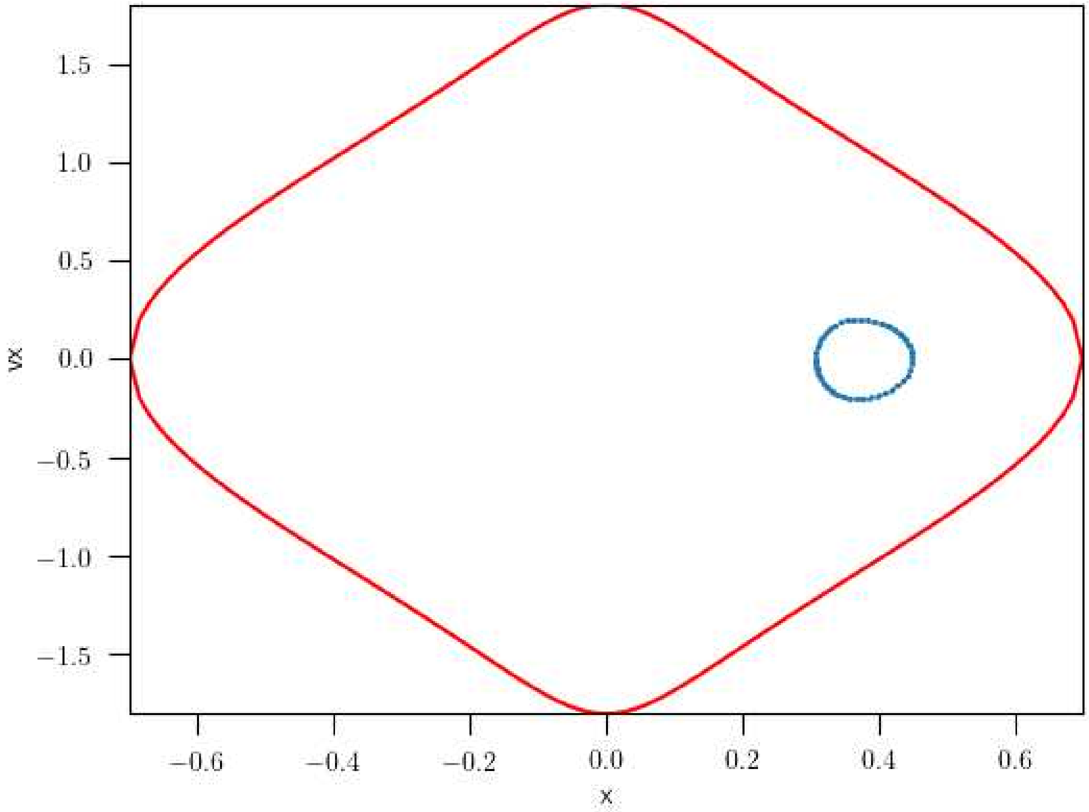
Loop orbits

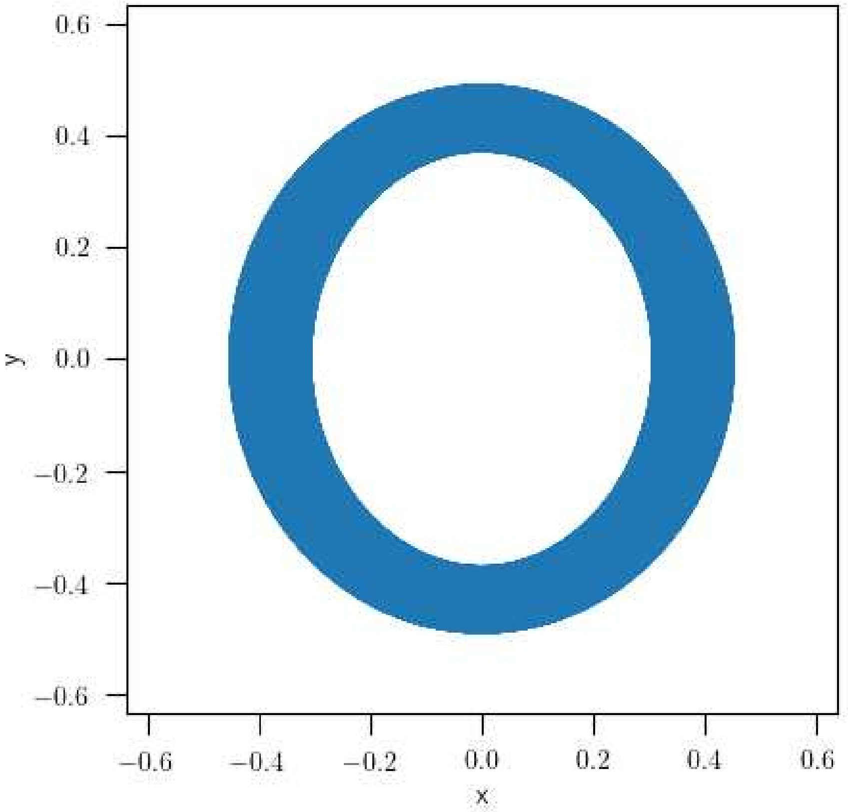





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.55
```

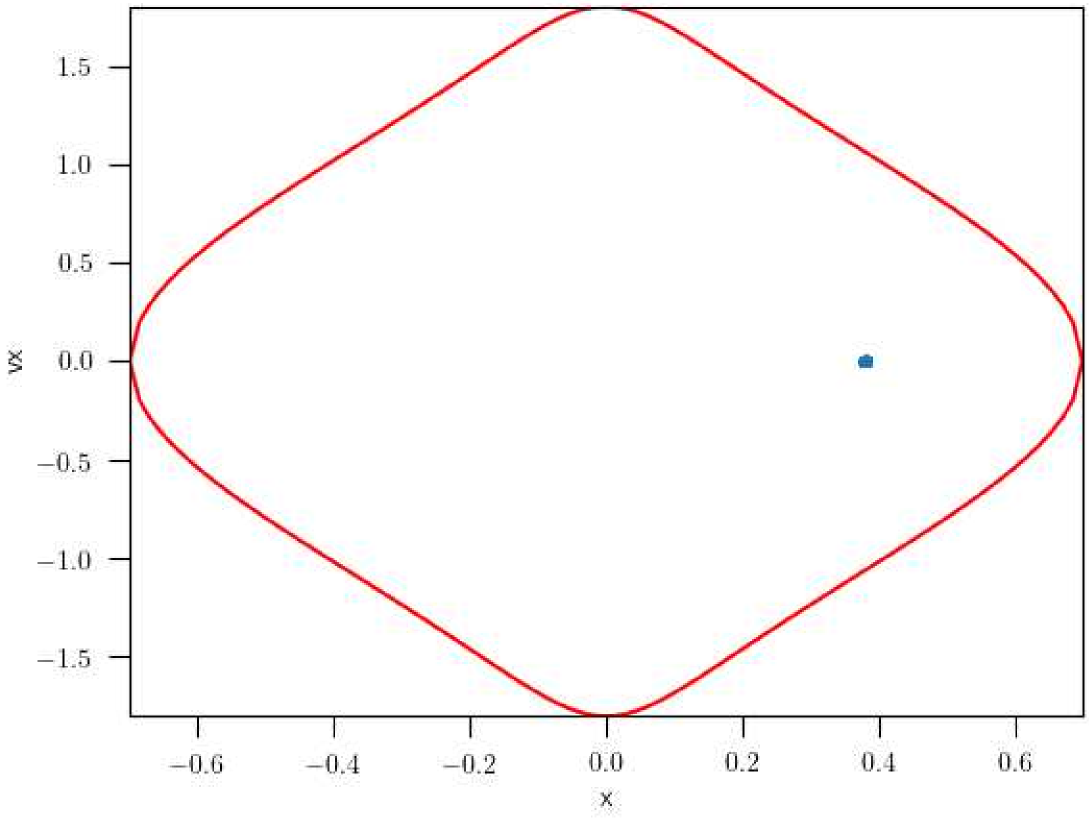
Loop orbits

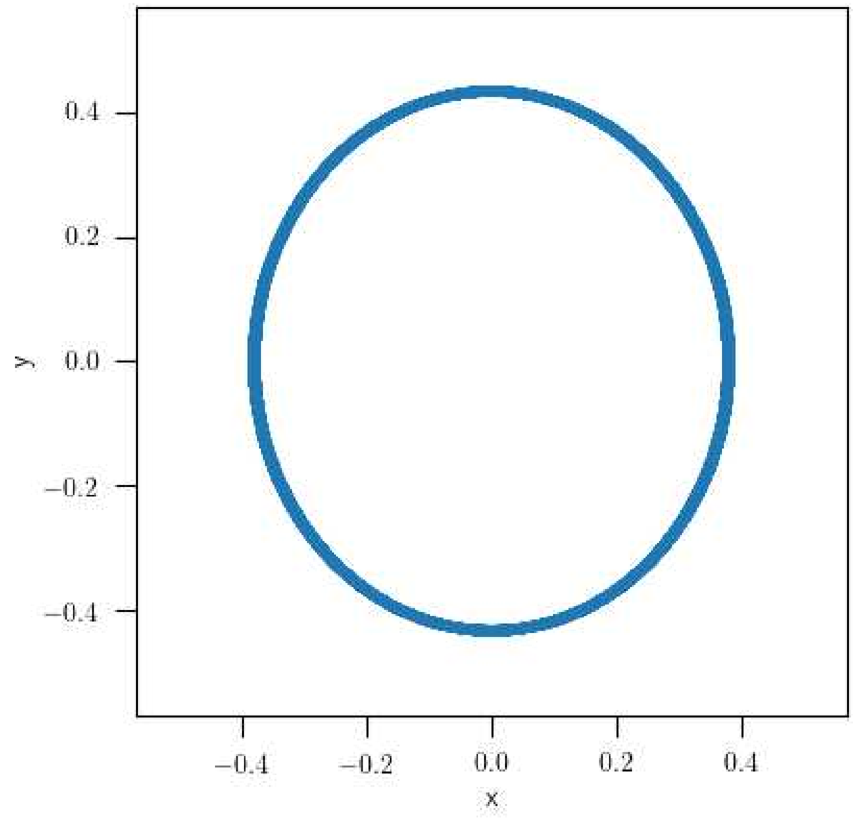





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.45
```

Loop orbits



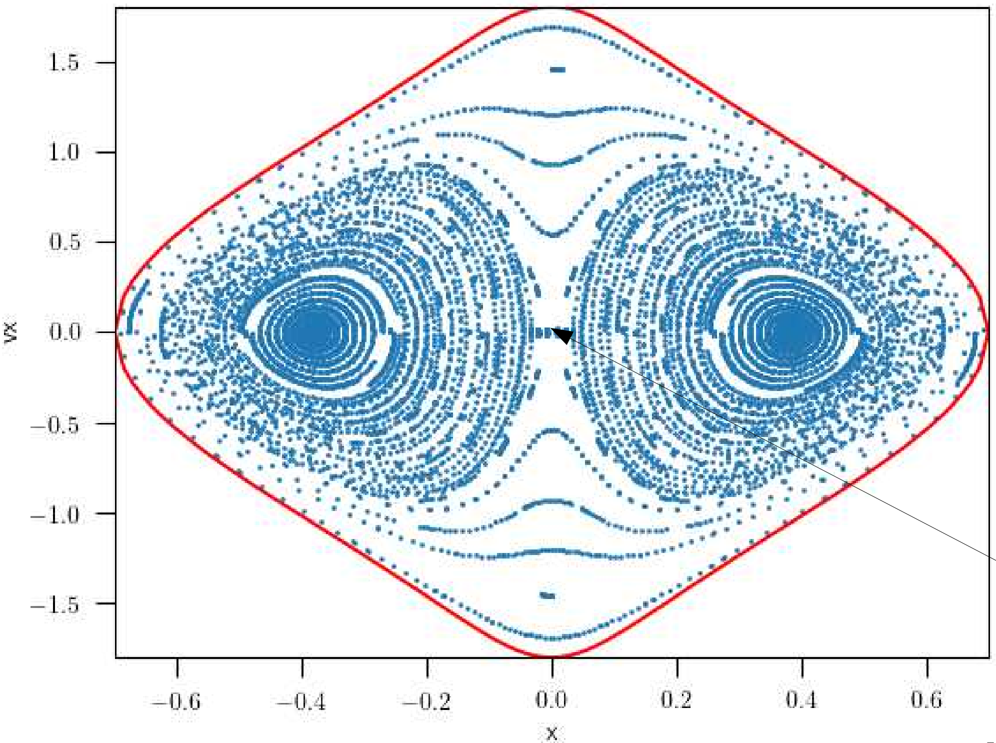


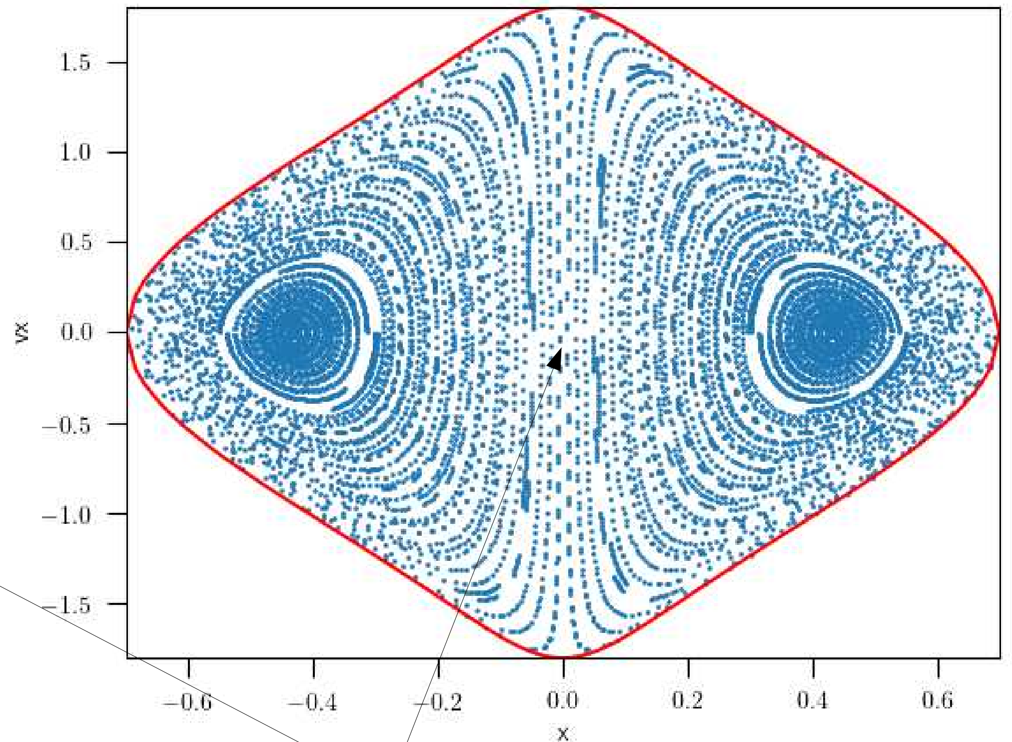

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.374
```

Box orbits elongated towards the y axis

$q = 0.9$

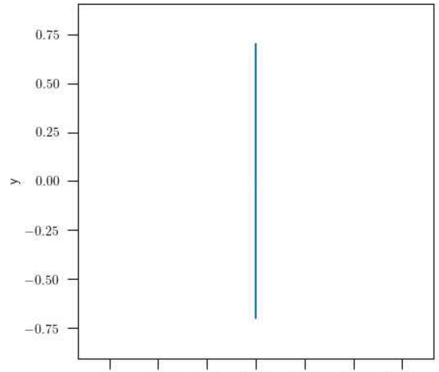
$q = 1.0$





unstable

stable

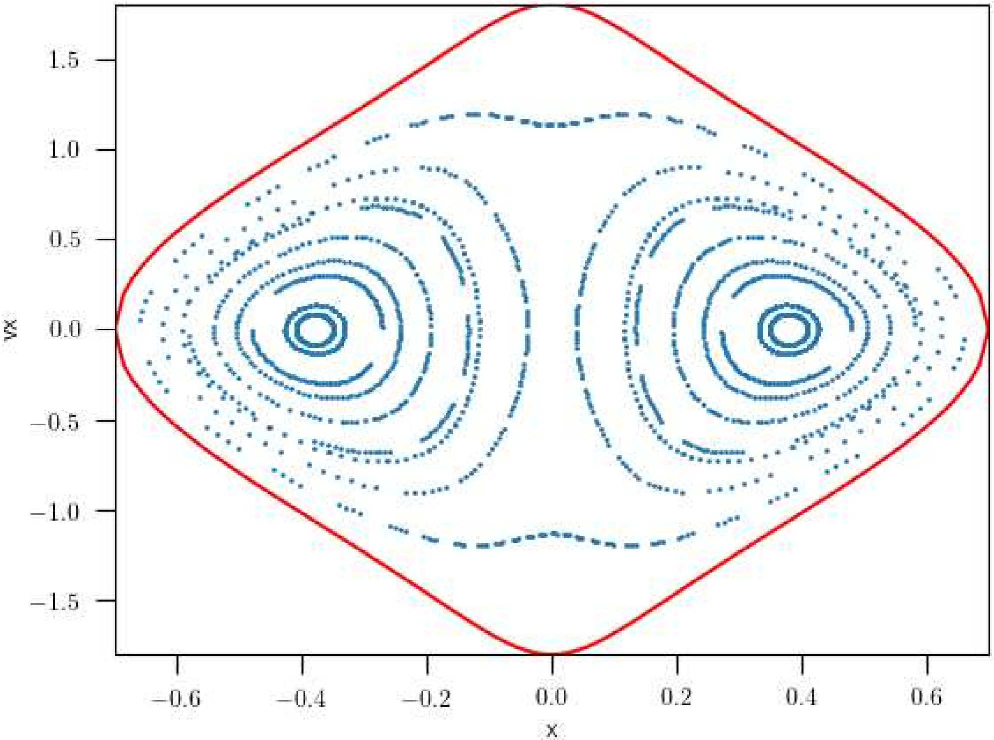


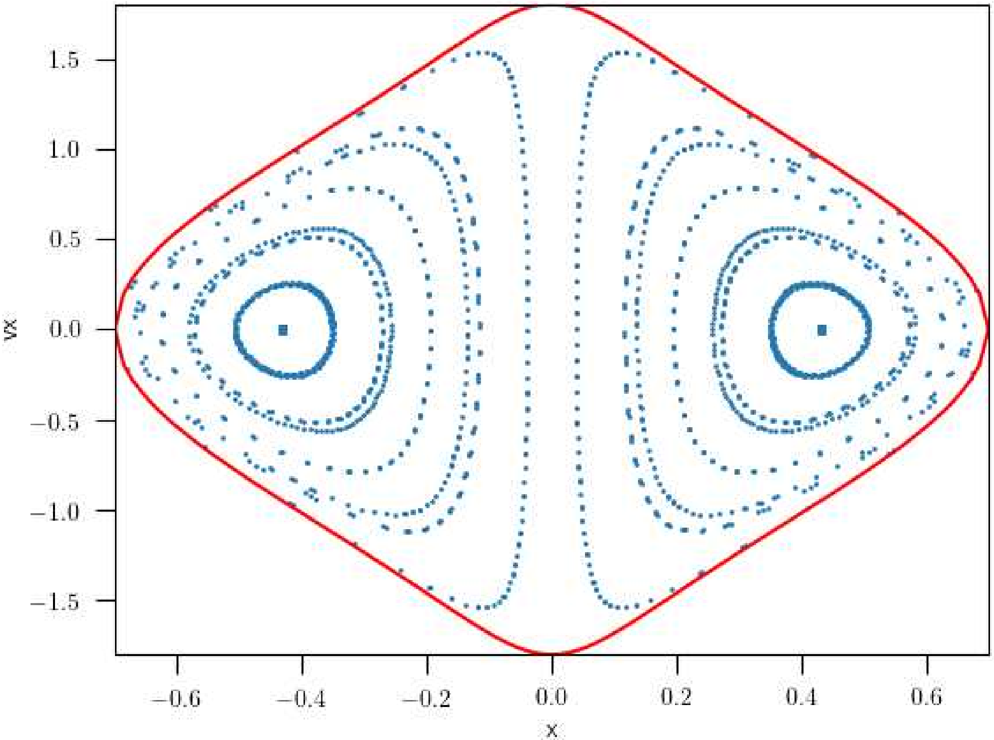

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337 --norbits 100
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337 --norbits 100
```

Integral of motions ?

Integral of motions ?

$q = 0.9$ $q = 1.0$






```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --norbits 18
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --norbits 18
```

Integrals of motions

① "nearly circular orbits"

Angular momentum conservation $\quad L_z = x\dot{y} - y\dot{x}$

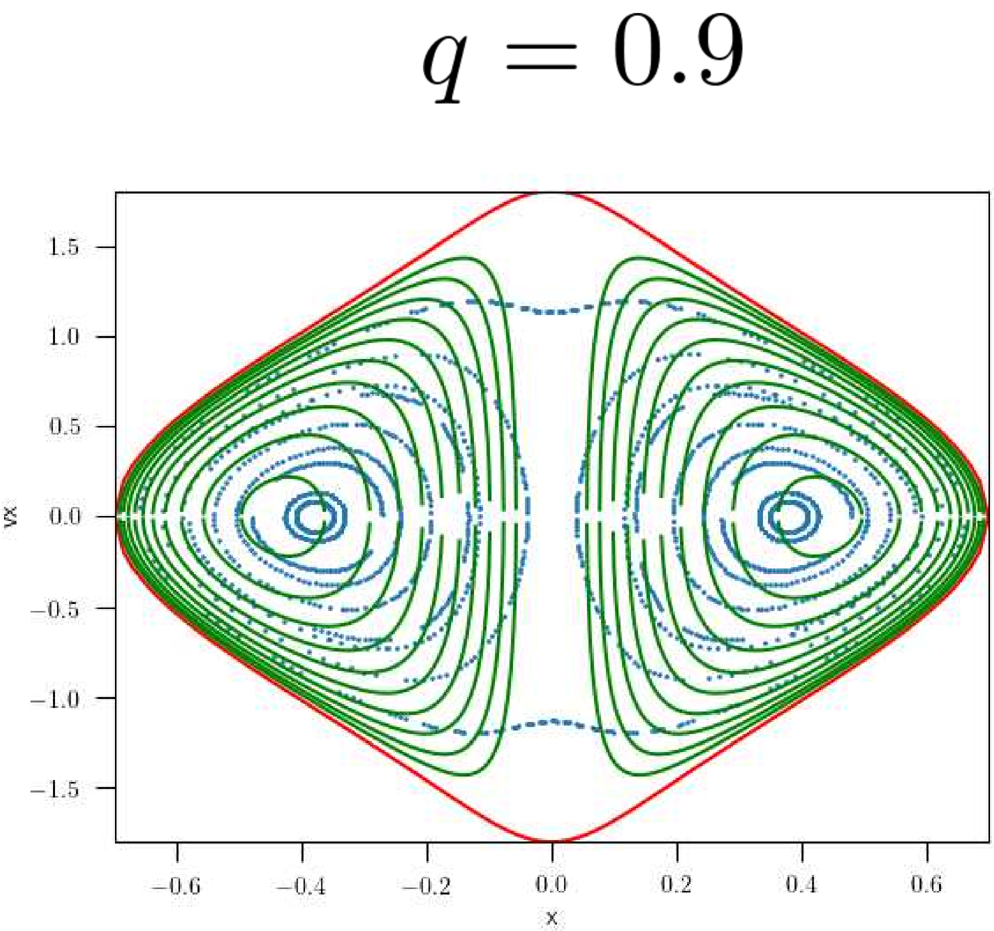
can we compute $x = x(\dot{x})$ in the plane $y = 0$?

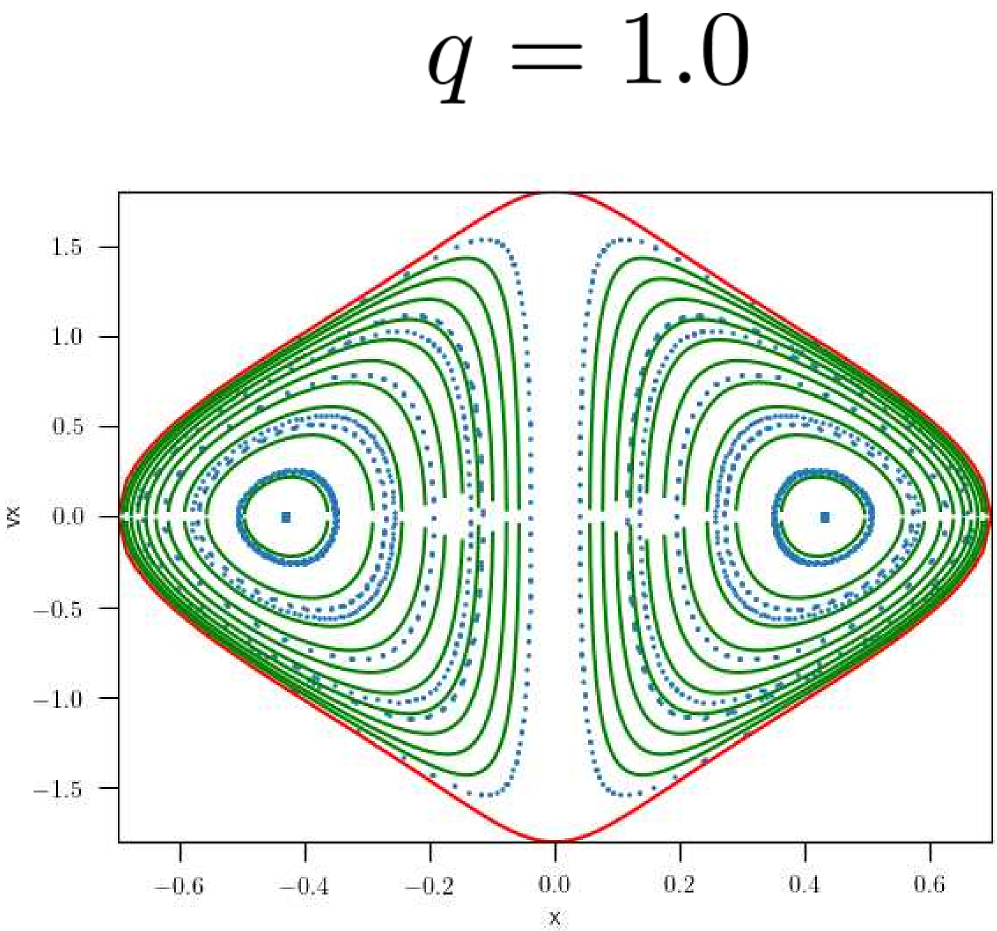
$$L_z = x\dot{y}$$

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \phi(x, y=0)$$

$$\boxed{\dot{x} = \sqrt{2(E-\phi) - \dot{y}^2} = \sqrt{2\left(E - \phi - \frac{L_z^2}{x^2}\right)}}$$

L_z






```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --norbits 18  --add_ILz
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --norbits 18  --add_ILz
```

Integrals of motions

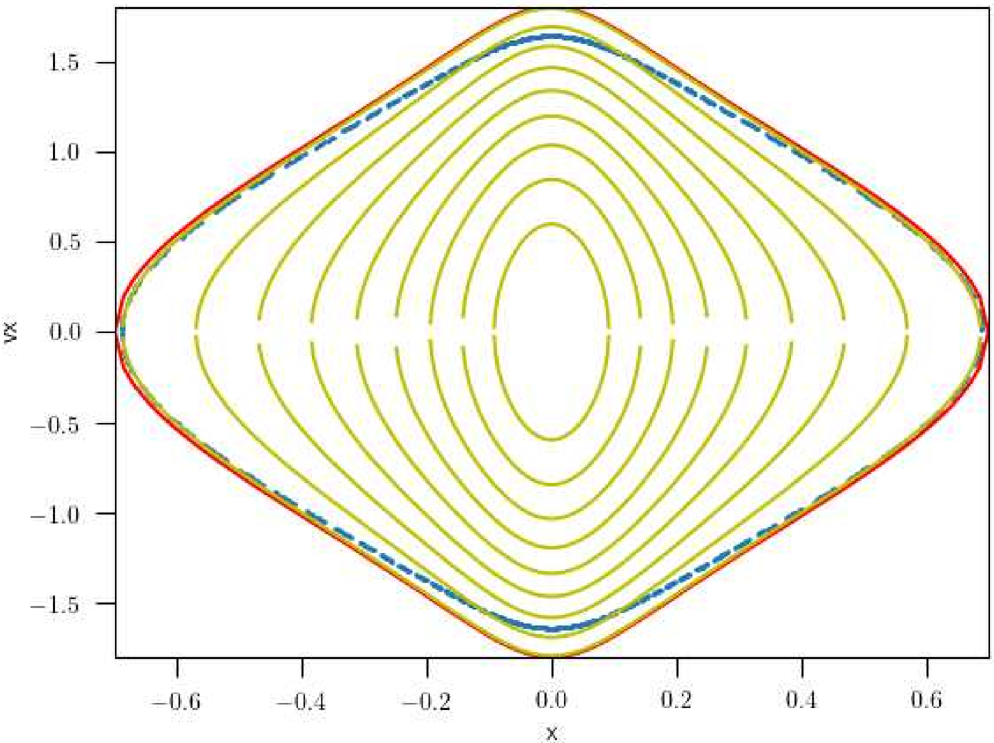
② Motion parallel to the long axis $\quad (y = \dot{y} = 0)$

$$H_x = \frac{1}{2}\dot{x}^2 + \phi(x, y=0) = E_x \quad \text{(harmonic oscillator)}$$

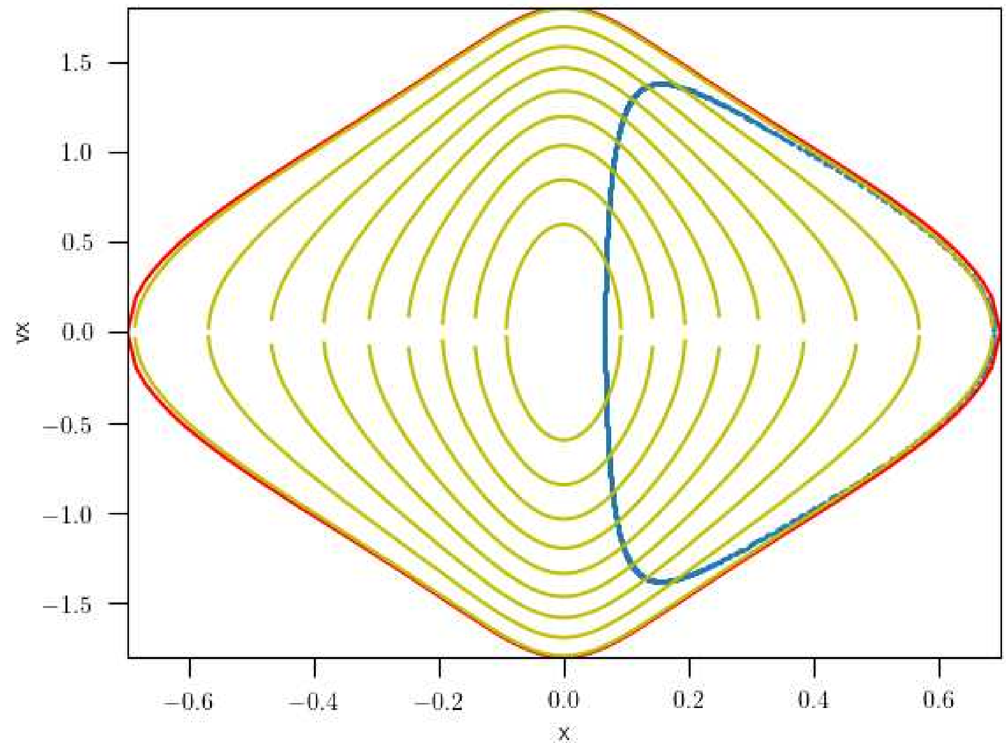
$$\boxed{\dot{x} = \sqrt{2\left(E_x - \phi(x, y=0)\right)}}$$

H_x

$q = 0.9$



$q = 1.0$



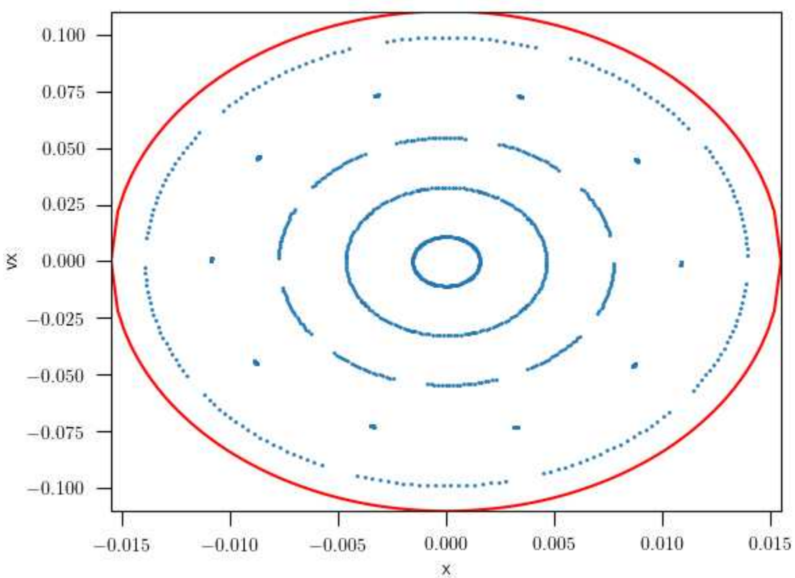

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --x 0.69  --nlaps 1000    --add_Ix
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --x 0.69  --nlaps 1000    --add_Ix
```

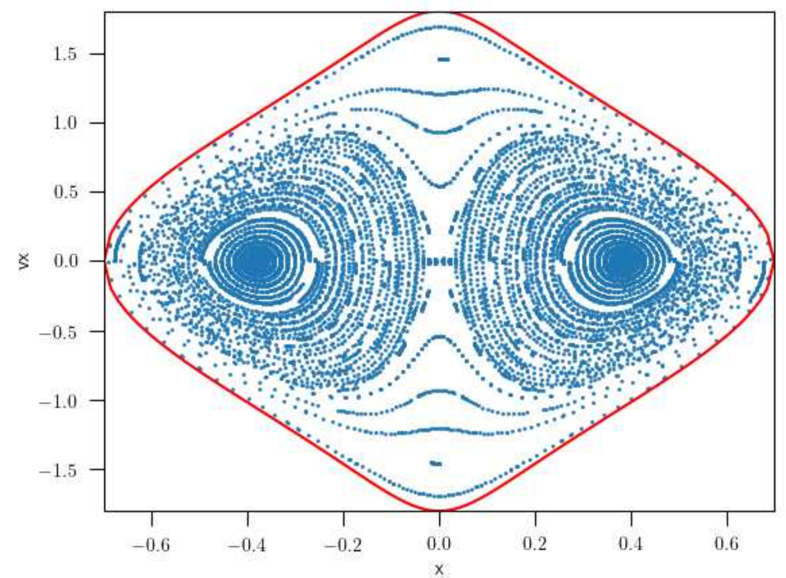
$$R \sim R_c$$

Family decoupling

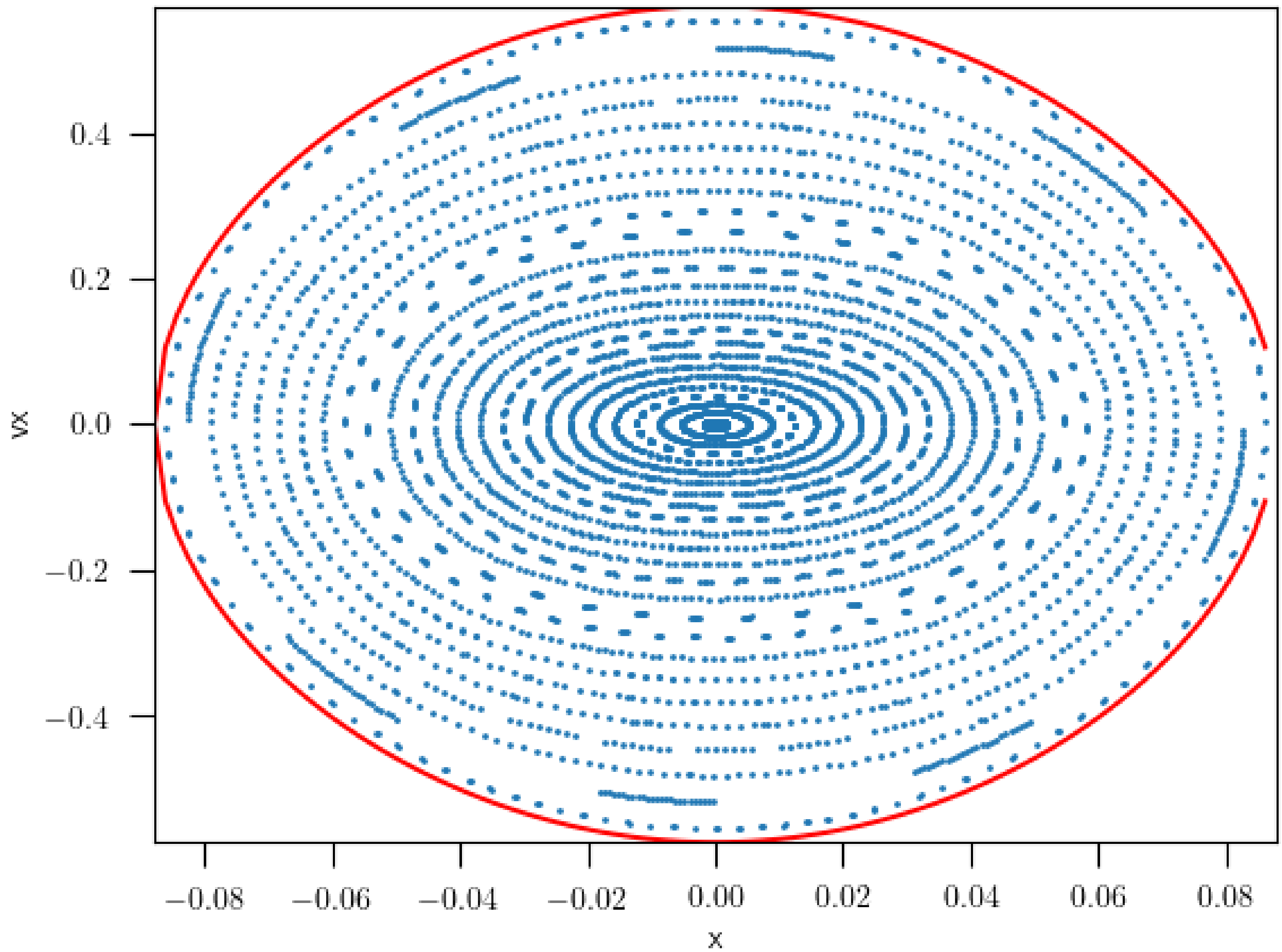
from low energy   to   high energy

1 family   2 famillies



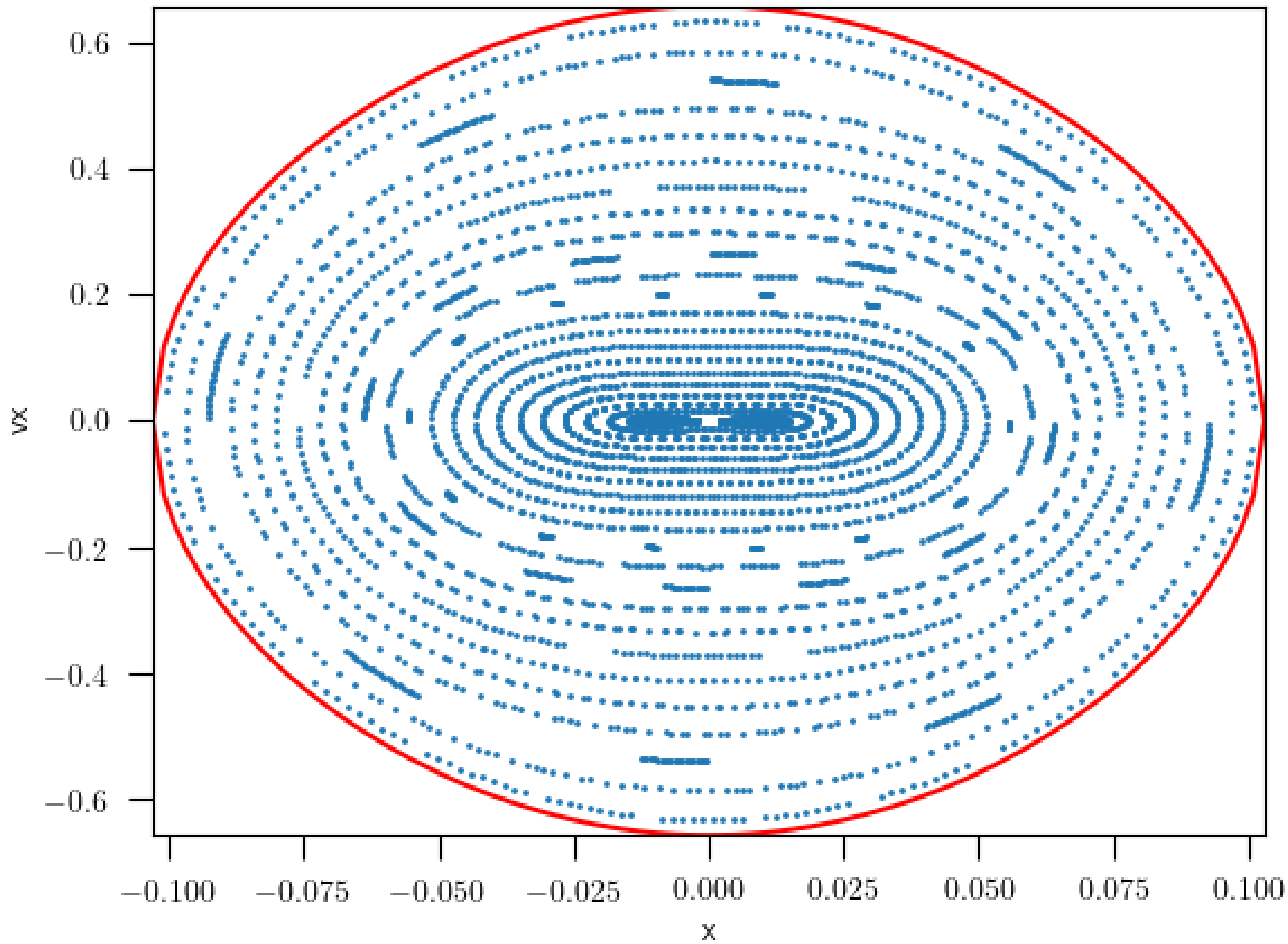


$$E = -1.8$$



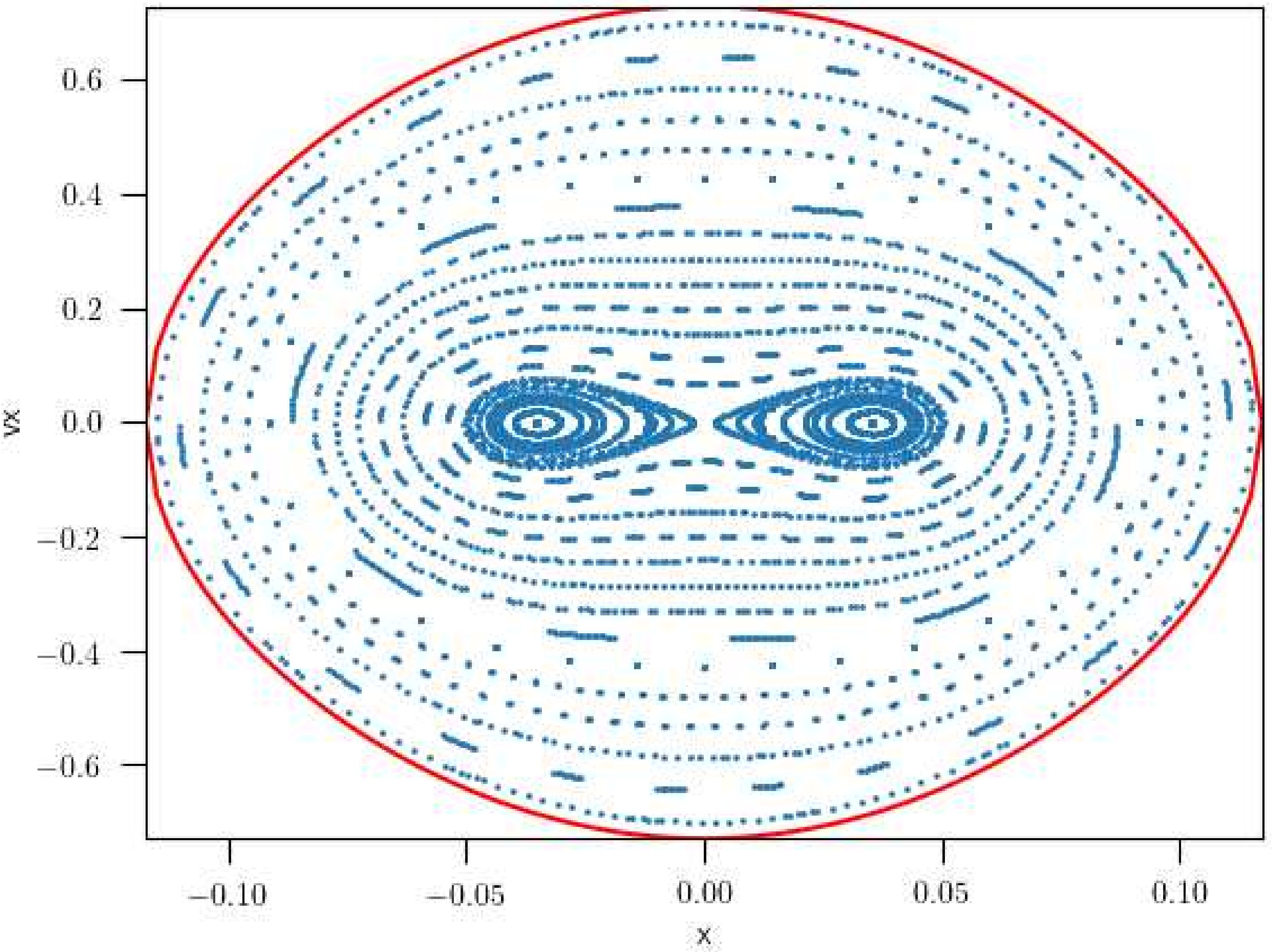

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.80  --norbits 50
```

$E = -1.75$



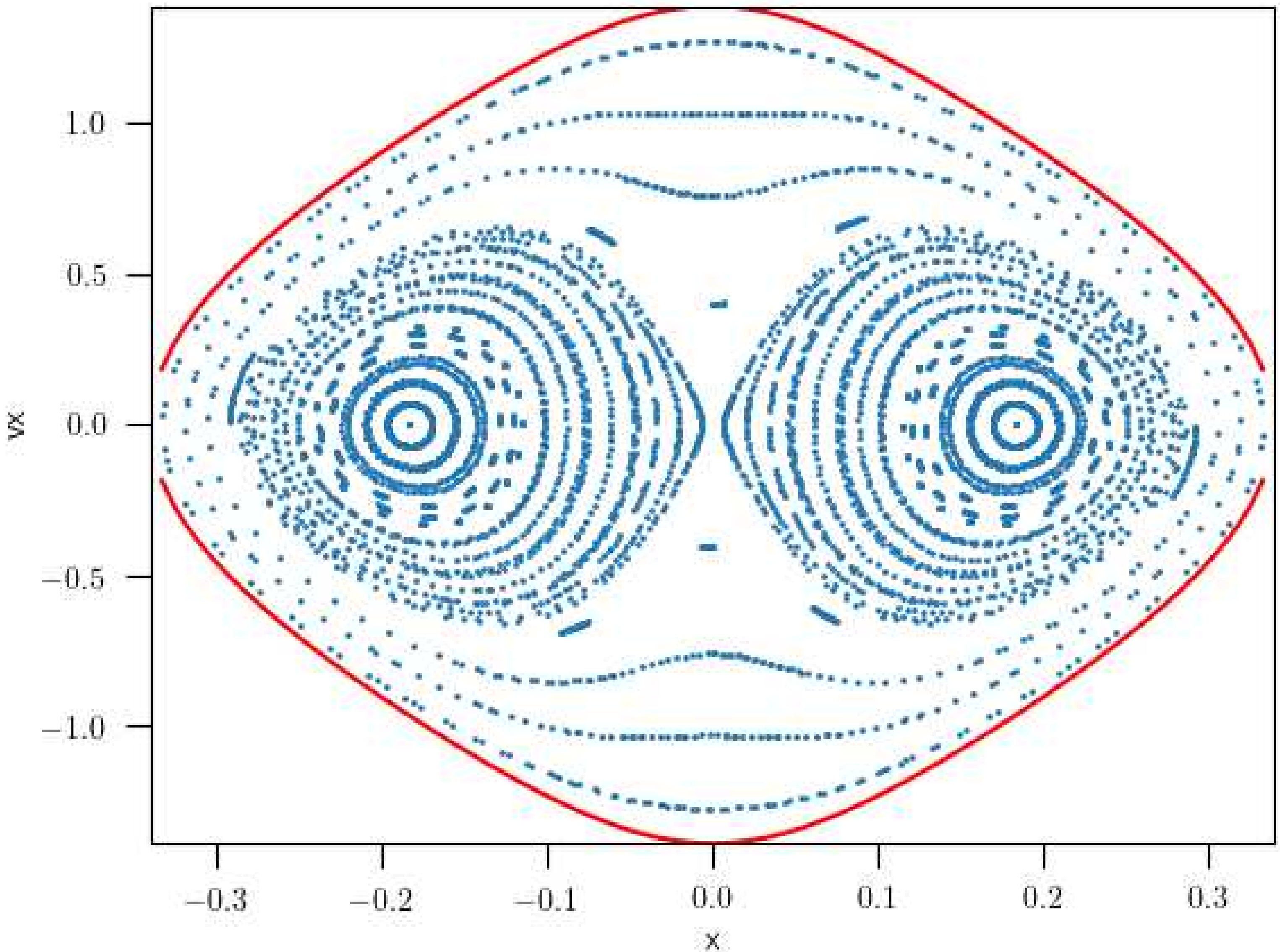

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.75  --norbits 50
```

bifurcation $\qquad E = -1.70$



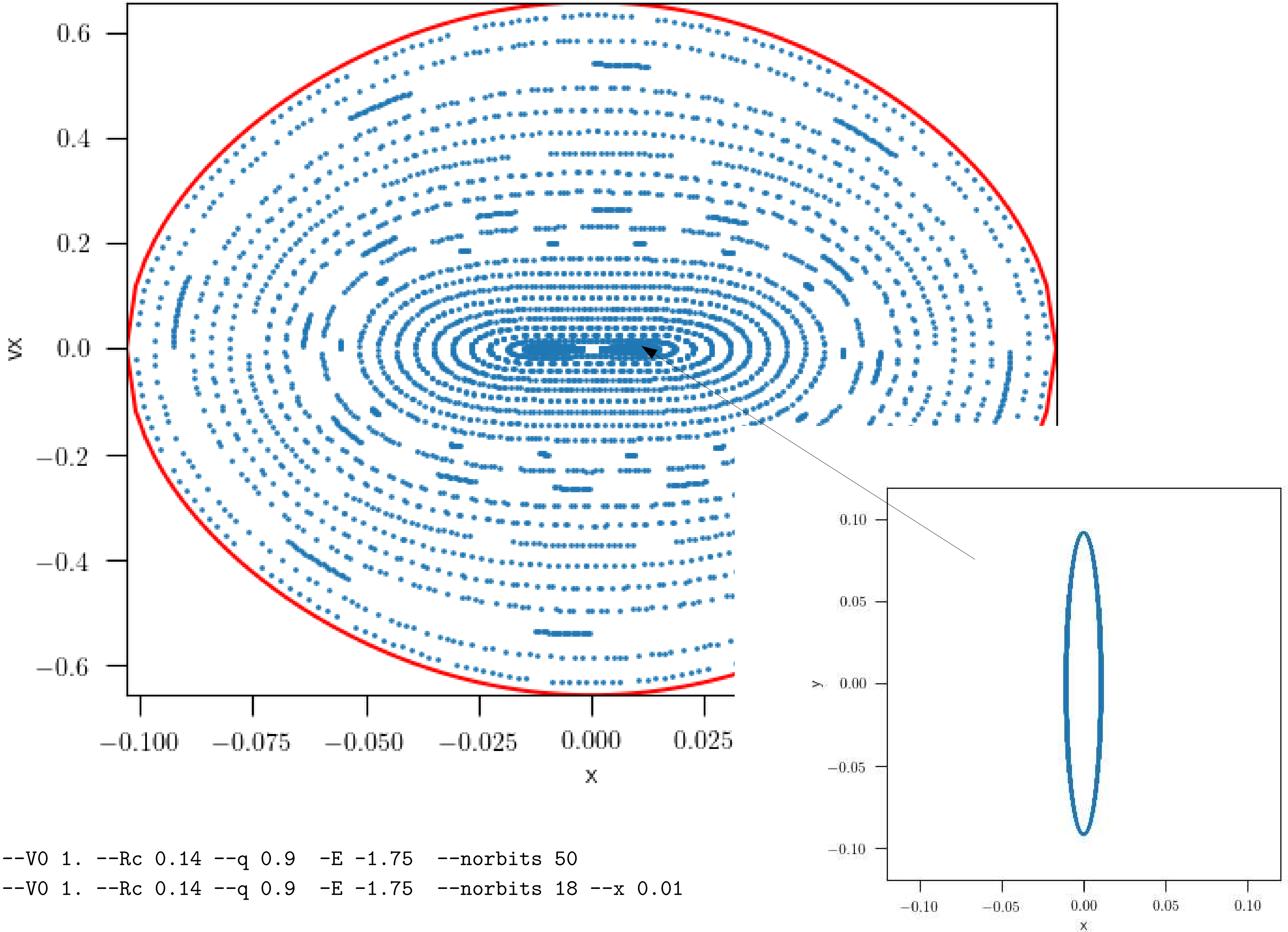

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.70  --norbits 50
```

$$E = -1.$$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1     --norbits 50
```

$$E = -1.75$$

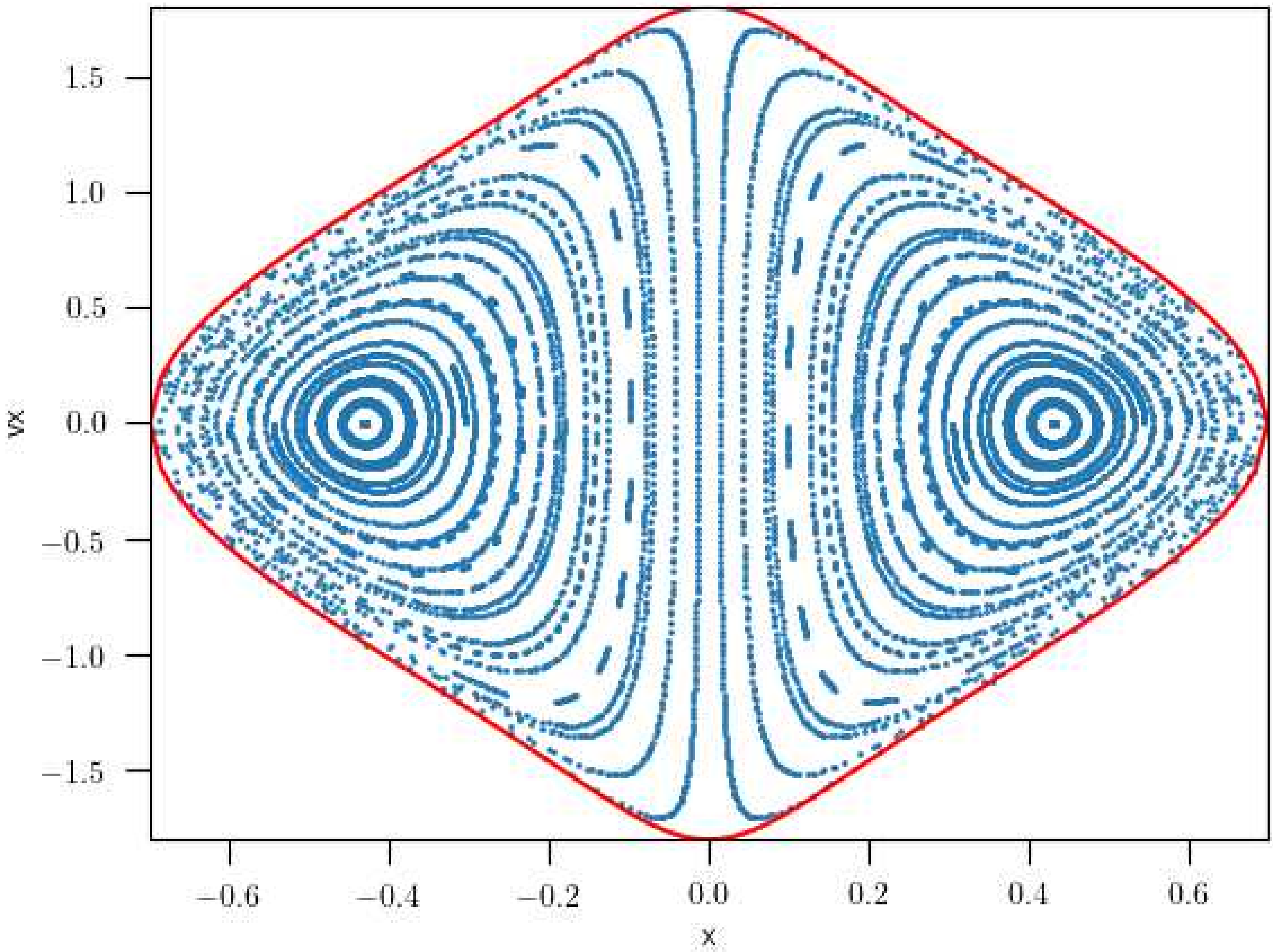



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.75  --norbits 50
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.75  --norbits 18 --x 0.01
```

Evolution with the flattening

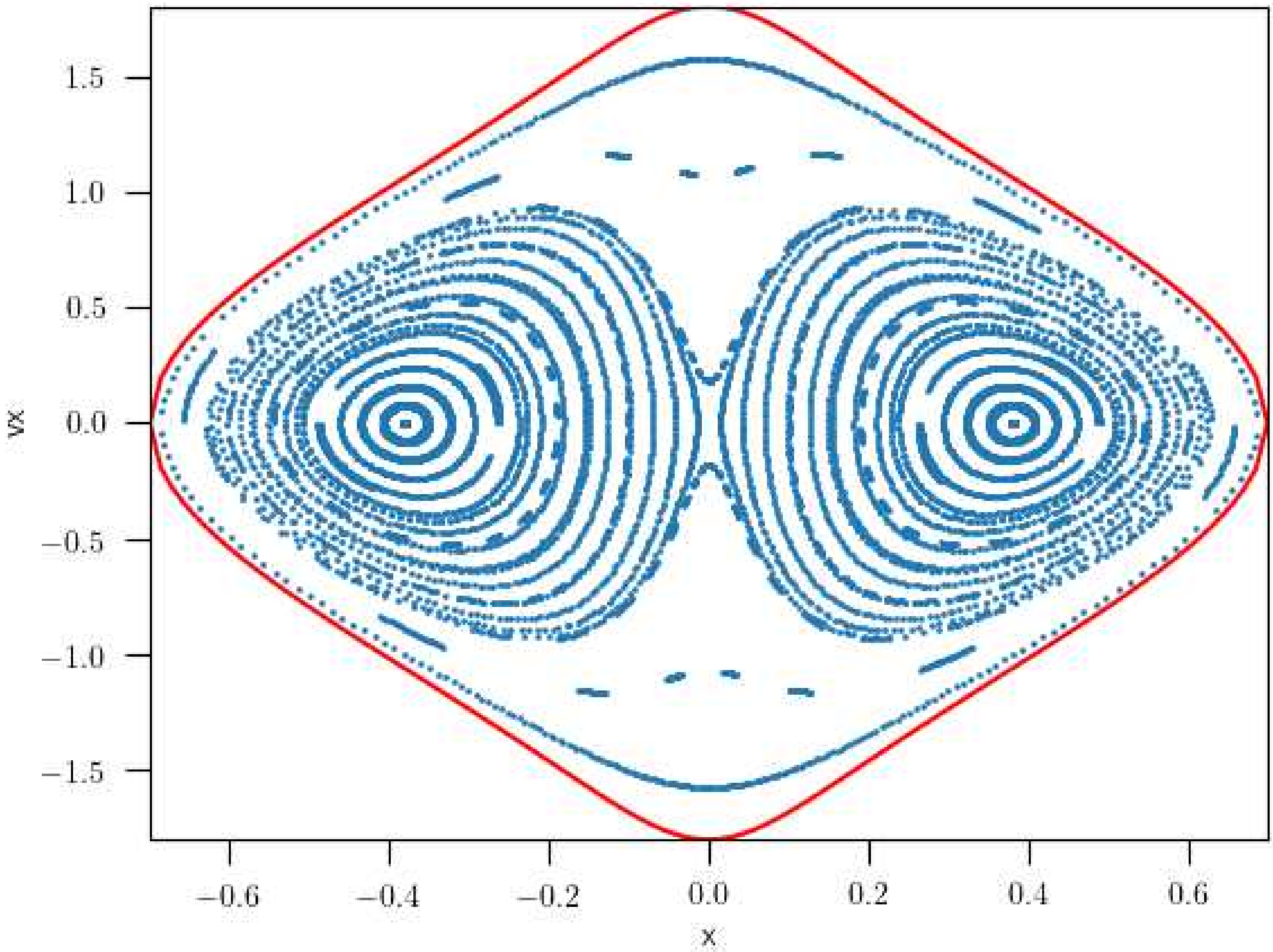
keeping the nergy fixed

$$q = 1.0$$




```
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --norbits 50  --nlaps 200
```

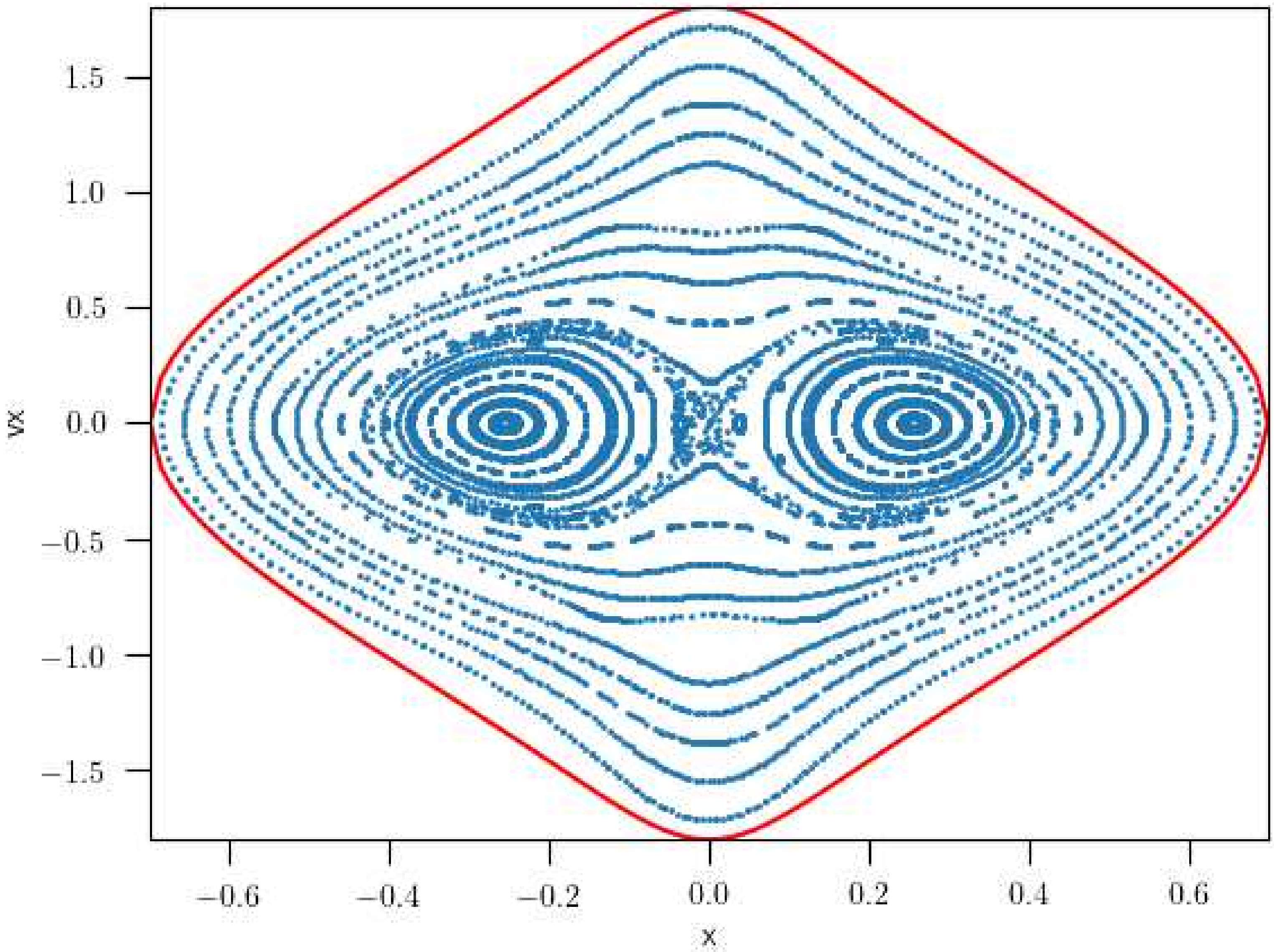
$$q = 0.9$$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --norbits 50  --nlaps 200
```

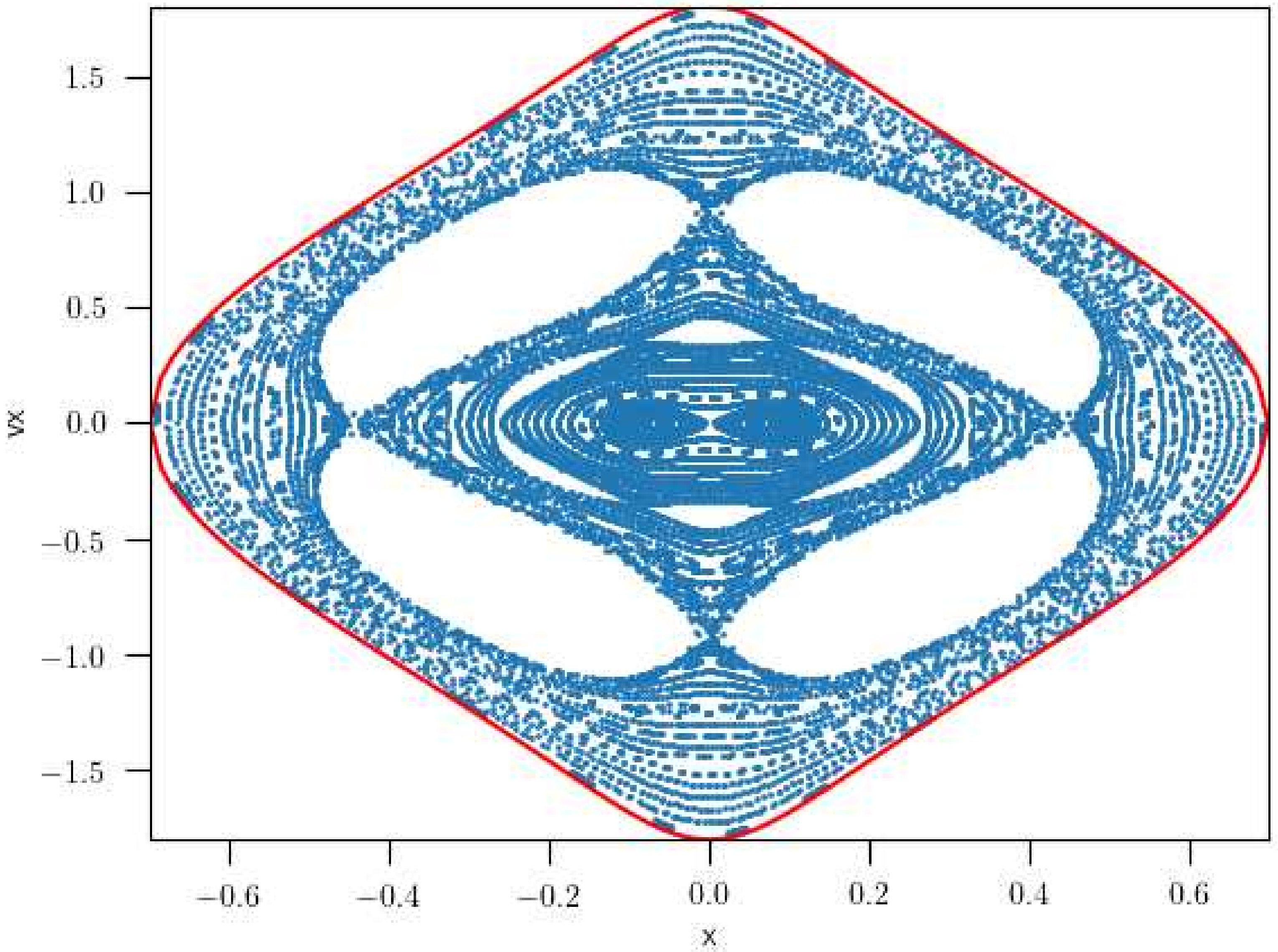
$$q = 0.7$$

Box orbits dominate the phase space !



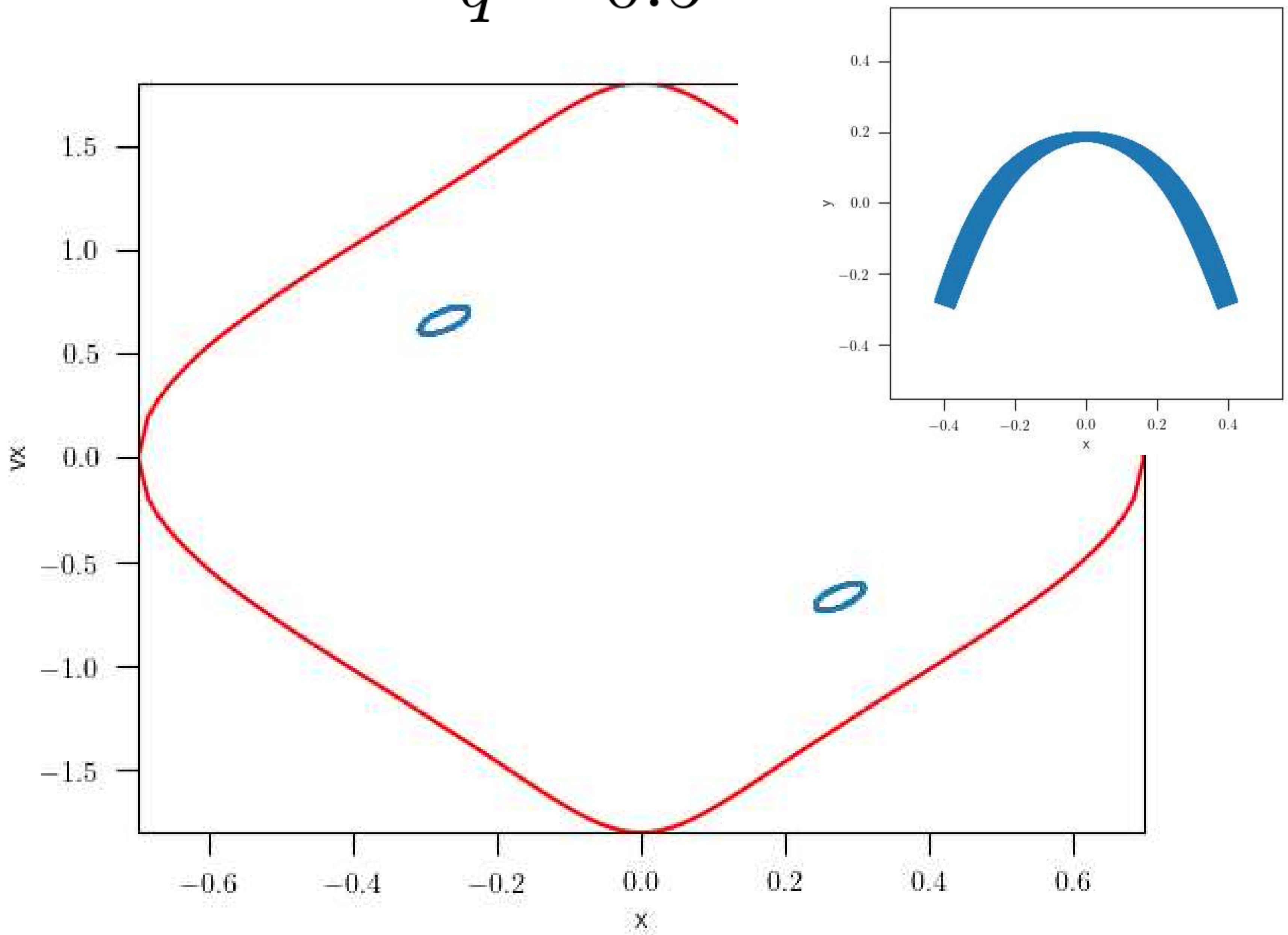

```
./mapping.py --V0 1. --Rc 0.14 --q 0.7  -E -0.337  --norbits 50  --nlaps 200
```

$$q = 0.5$$



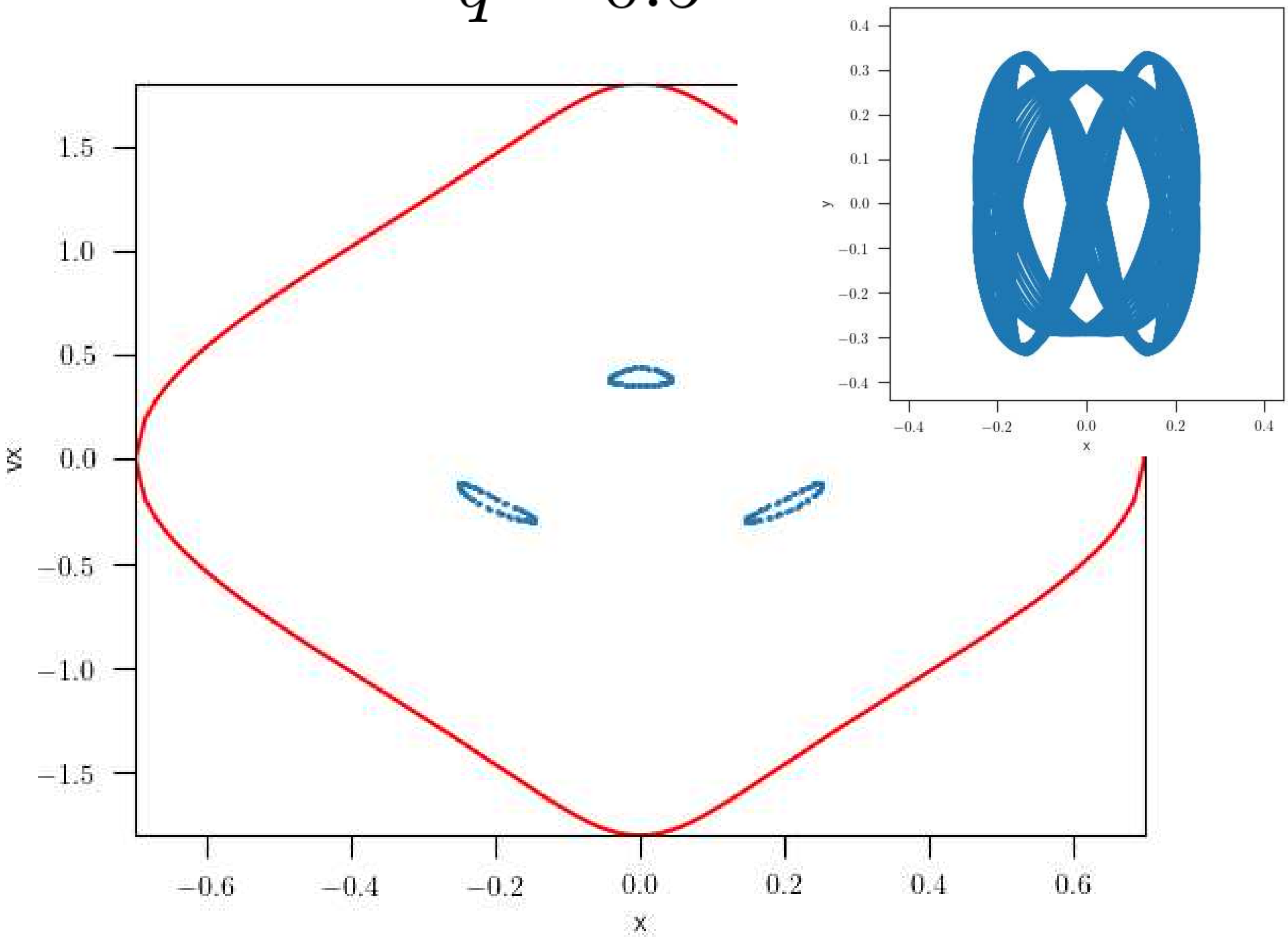

```
./mapping.py --V0 1. --Rc 0.14 --q 0.5  -E -0.337  --norbits 100  --nlaps 200
```

$q = 0.5$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.5  -E -0.337  --norbits 50  --nlaps 200 --x 0.3 --vx -0.6
```

$q = 0.5$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.5  -E -0.337  --norbits 50  --nlaps 200 --x 0.3 --vx -0.6
```

Conclusions

Many 2D bared potential have orbital structures like the logarithmic potential:

- Most orbits respect a 2nd integral (L_z or H_x)
- 2 types of orbits:
  - Loop : - fixed sense of rotation
    - never reach the centre
  - Box : - no fixed sense of rotation
    - many reach the centre

Loop orbits dominate when the axis ratio of of the potential is nearly unity.
Box orbits dominate instead.

Orbits in planar non-axisymmetric rotating potentials

Two dimentional rotating potential

$$\boxed{\phi(\theta, t) \quad \begin{cases} \theta \rightarrow L_z \neq \text{cte} \\ t \rightarrow E \neq \text{cte} \end{cases}}$$

Assume a static rotation of the bar at constant angular frequency Ω_b

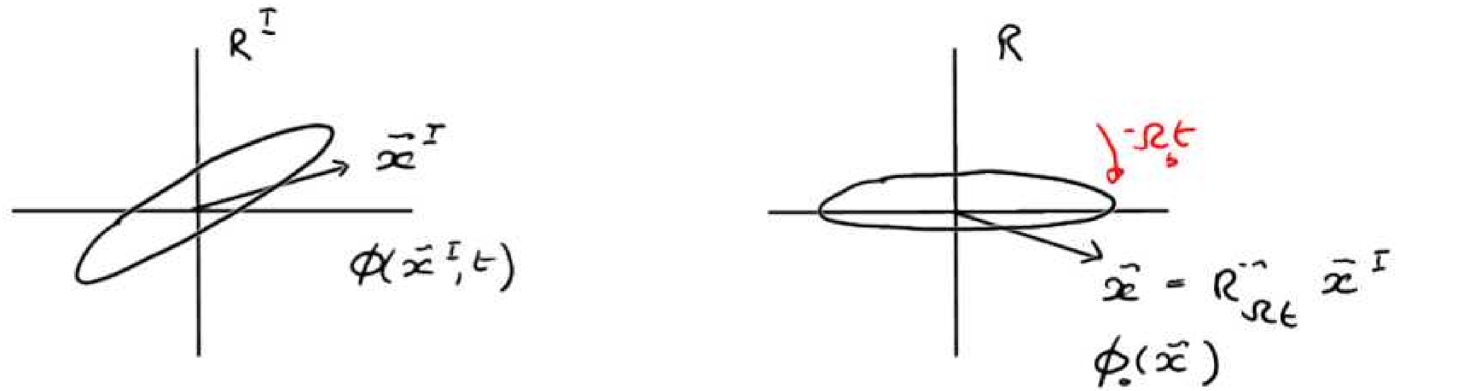
<u>Idea</u> : Describe the motion from the rotating frame where the bar is static

$$\underset{\boxed{\text{inertial frame } R^I}}{(\vec{x}^I, \dot{\vec{x}}^I)} \rightarrow \underset{\boxed{\text{rotating frame } R}}{(\vec{x}, \dot{\vec{x}})}$$

Positions

$$\boxed{\vec{x} = R^{-1}_{\Omega_b t}\, \vec{x}^{\,I}}$$

$R^{-1}_{\Omega_b t}$: brings the bar to its original position $(t=0)$



Potential

$$\phi(\vec{x}^{\,I}, t) \equiv \phi\left(\vec{x} = R^{-1}_{\Omega t}\, \vec{x}^{\,I}, t=0\right) = \phi_0(\vec{x})$$

$$\boxed{\phi(\vec{x}^{\,I}, t) = \phi_0(\vec{x})}$$

Velocities

$$\boxed{\dot{\vec{x}}^{\,I} = \dot{\vec{x}} + \vec{\Omega}_b \times \vec{x}}$$

Lagrangian

In the inertial frame $\mathcal{R}^I$

$$\boxed{\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} \dot{\vec{x}}^{I\,2} - \phi^I(\vec{x}^I, t)}$$

In the rotating frame $\mathcal{R}$

- $\frac{1}{2} \dot{\vec{x}}^{I\,2} \longrightarrow \frac{1}{2}\left(\dot{\vec{x}} + \vec{\Omega}_b \times \vec{x}\right)^2$
- $\phi^I(\vec{x}^I, t) \longrightarrow \phi(R_{\Omega t}^{-1}\,\vec{x}, t=0) \equiv \phi_0(\vec{x})$

$$\boxed{\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}\left(\dot{\vec{x}} + \vec{\Omega}_b \times \vec{x}\right)^2 - \phi_0(\vec{x})}$$

Momentum

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} = \dot{\vec{x}} + \vec{\Omega}_b \times \vec{x}$$

Hamiltonian

$$H_J = \vec{p} \cdot \dot{\vec{x}} - \mathcal{L}(\vec{x}, \dot{\vec{x}})$$

$$\boxed{H_J(\vec{x}, \vec{p}) = \frac{1}{2} \vec{p}^{\,2} - \vec{\Omega} \cdot (\vec{p} \times \vec{x}) + \phi(\vec{x})}$$

H_J has no explicit time dependency

$\Rightarrow$ $H_J = E_J = \text{cte}$

Jacobi integral

Equations of motion from Hamilton's equations

$$H_J = \frac{1}{2}\vec{p}^{\,2} - \vec{\Omega}\cdot(\vec{p}\times\vec{x}) + \phi(\vec{x})$$

$$\boxed{\begin{aligned} \dot{\vec{x}} &= \frac{\partial H_J}{\partial \vec{p}} = \vec{p} - \vec{\Omega}\times\vec{x} \\ \dot{\vec{p}} &= -\frac{\partial H_J}{\partial \vec{x}} = -\vec{\nabla}\phi - \vec{\Omega}\times\vec{p} \end{aligned}}$$

Effective potential

split the kinetic term in the Lagrangian

$$\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}\left(\dot{\vec{x}} + \vec{\Omega}_s \times \vec{x}\right)^2 - \phi_0(\vec{x})$$

$$\equiv \frac{1}{2}\dot{\vec{x}}^2 + \dot{\vec{x}}\left(\vec{\Omega}_s \times \vec{x}\right) \underbrace{- \phi_0(\vec{x}) + \frac{1}{2}\left(\vec{\Omega}_s \times \vec{x}\right)^2}_{\text{depends only on } \vec{x}}$$

$$\boxed{\begin{aligned} \phi_{\text{eff}}(\vec{x}) &:= \phi(\vec{x}) - \frac{1}{2}\left(\vec{\Omega} \times \vec{x}\right)^2 \\ &= \phi(\vec{x}) \underbrace{- \frac{1}{2}\vec{\Omega}^2\vec{x}^2 + \frac{1}{2}\left(\vec{\Omega}\cdot\vec{x}\right)^2}_{\phi_{\text{centr}}(\vec{x})\,:\ \text{repulsive centrifugal potential}} \end{aligned}}$$

Note : $\phi_{\text{cent}}(\vec{x}) = -\frac{1}{2}\vec{\Omega}^2\vec{x}^2 + \frac{1}{2}\left(\vec{\Omega}\cdot\vec{x}\right)^2$

Equations of motion from the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} - \frac{\partial \mathcal{L}}{\partial \vec{x}} = 0$$

$$\ddot{\vec{x}} = -\vec{\nabla}\phi_{eff}(\vec{x}) - 2\left(\Omega \times \dot{\vec{x}}\right)$$

$$\boxed{\ddot{\vec{x}} = -\vec{\nabla}\phi(\vec{x}) \underbrace{+\,\Omega^2\vec{x} - \vec{\Omega}(\vec{\Omega}\cdot\vec{x})}_{\text{centrifugal force}} \underbrace{-\,2\left(\Omega \times \dot{\vec{x}}\right)}_{\text{Coriolis force}}}$$

$$= \Omega^2\vec{x} \quad \text{if } \vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$$

Stationary points

$$\dot{\vec{x}} = \ddot{\vec{x}} = 0$$

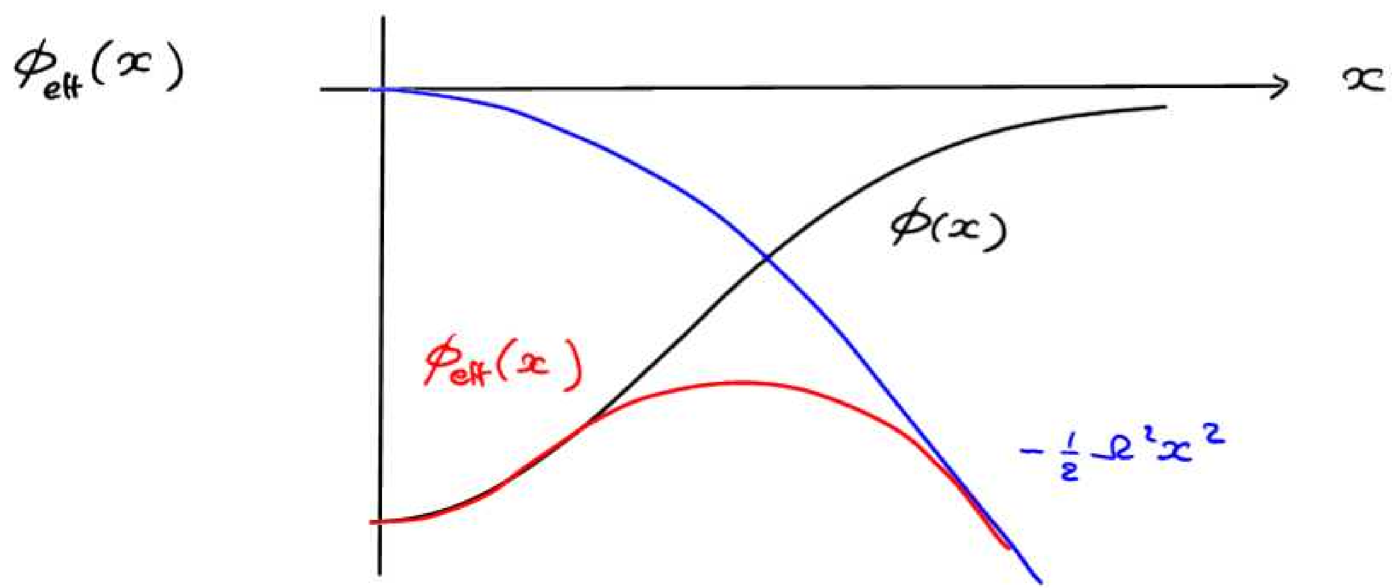
with $$\ddot{\vec{x}} = -\vec{\nabla}\phi_{\text{eff}} - 2\vec{\Omega}\times\dot{\vec{x}}$$

$$\boxed{\vec{\nabla}\phi_{\text{eff}} = 0}$$

Shape of the effective potential

$\boxed{y = 0}$ $\qquad \vec{\Omega}_b = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$

$$\phi_{\text{eff}}(\vec{x}) = \phi(\vec{x}) - \frac{1}{2}\Omega^2 R^2$$



Stationary points

$$\dot{\vec{x}} = \ddot{\vec{x}} = 0$$

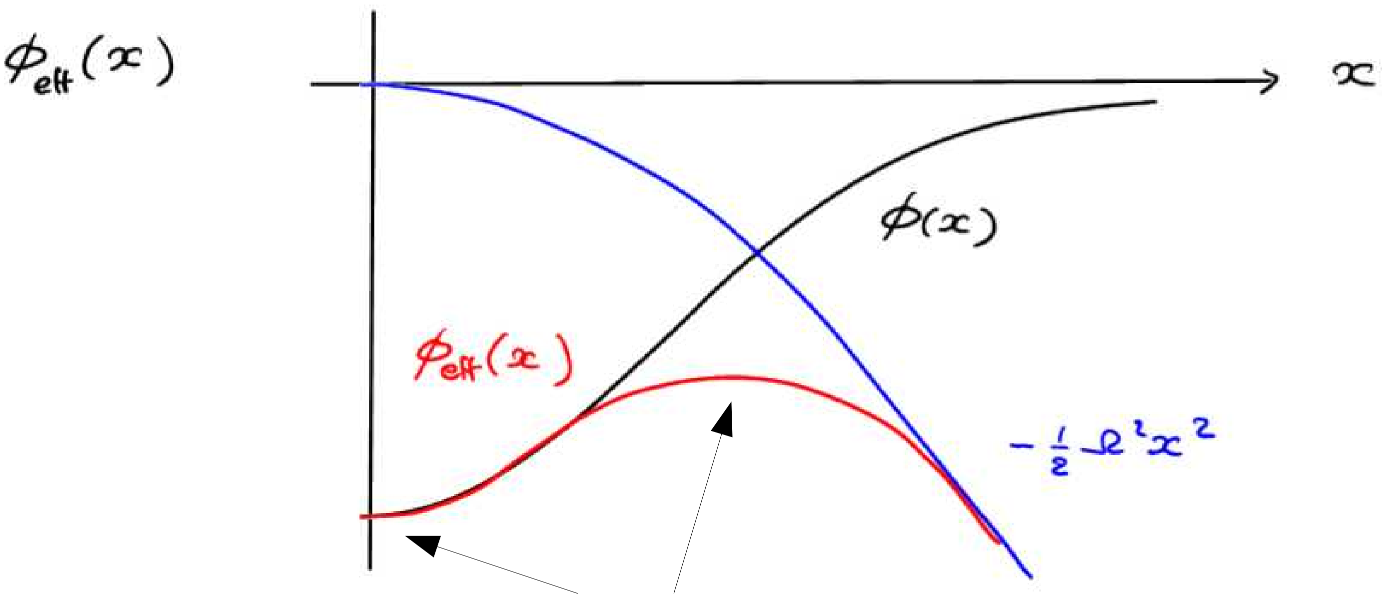
with $$\ddot{\vec{x}} = -\vec{\nabla}\phi_{eff} - 2\vec{\Omega}\times\dot{\vec{x}}$$

$$\boxed{\vec{\nabla}\phi_{eff} = 0}$$

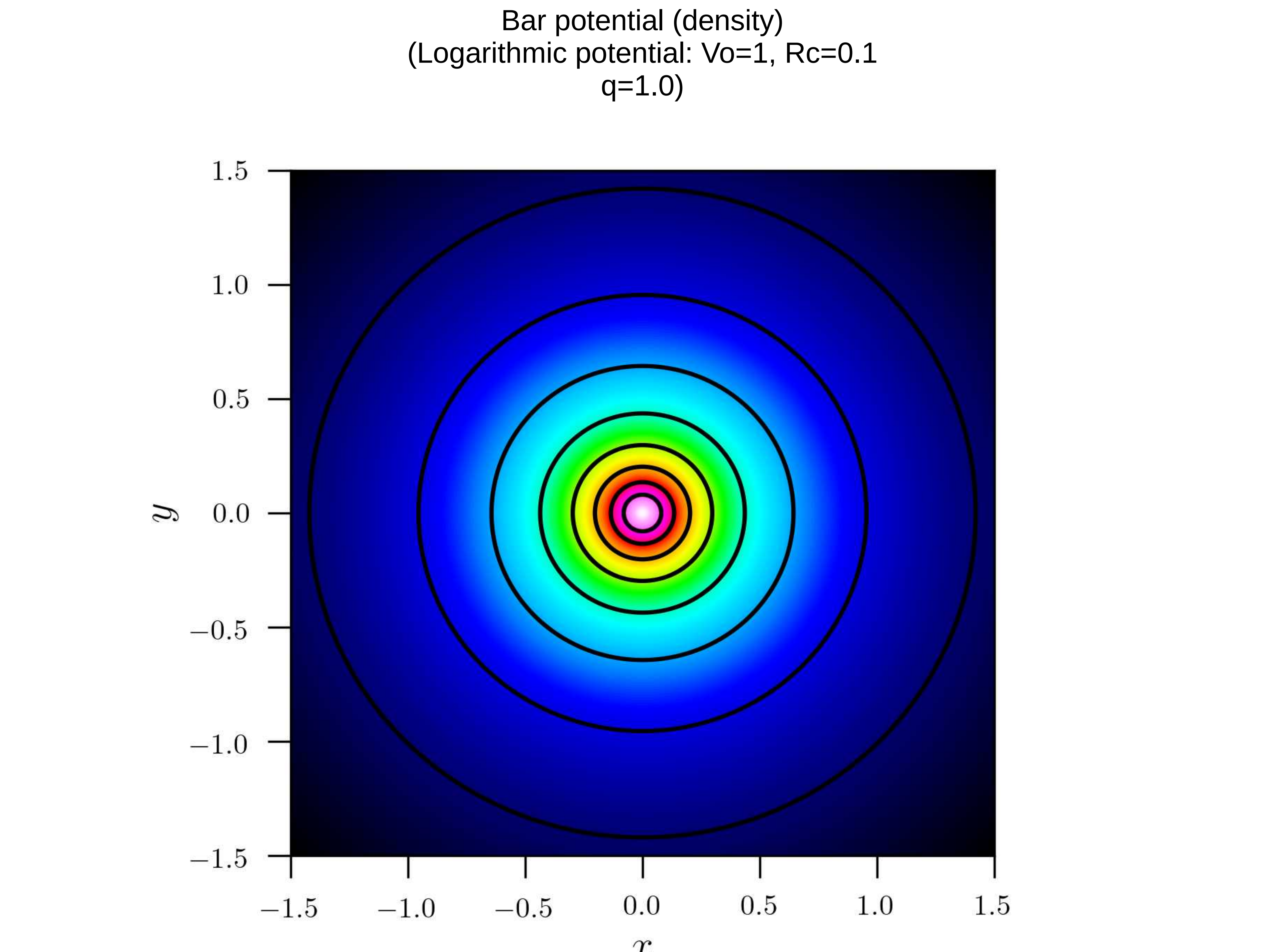
Shape of the effective potential

$\boxed{y = 0}$ $\quad \vec{\Omega}_b = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$

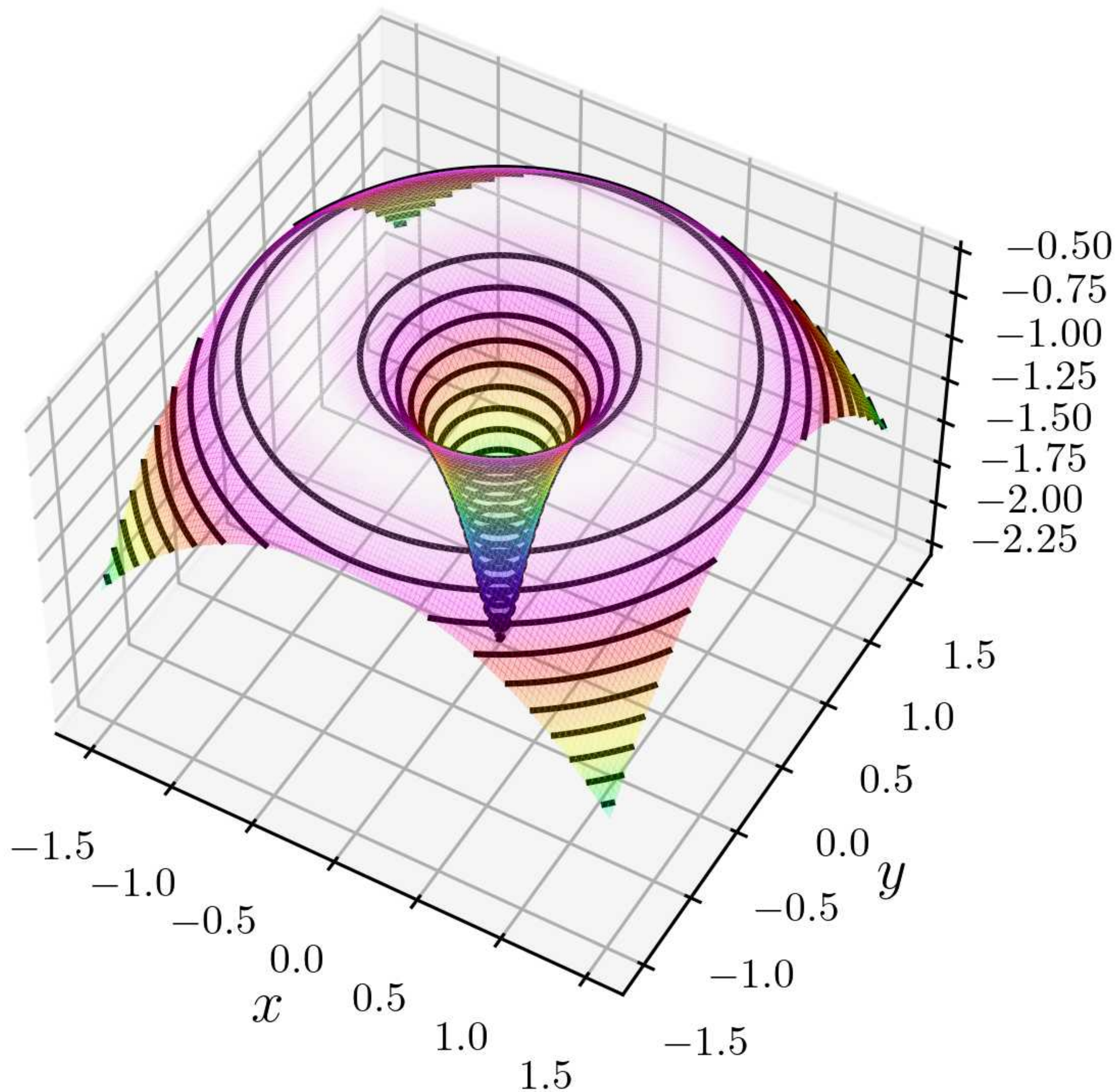
$$\phi_{eff}(\vec{x}) = \phi(\vec{x}) - \frac{1}{2}\Omega^2 R^2$$

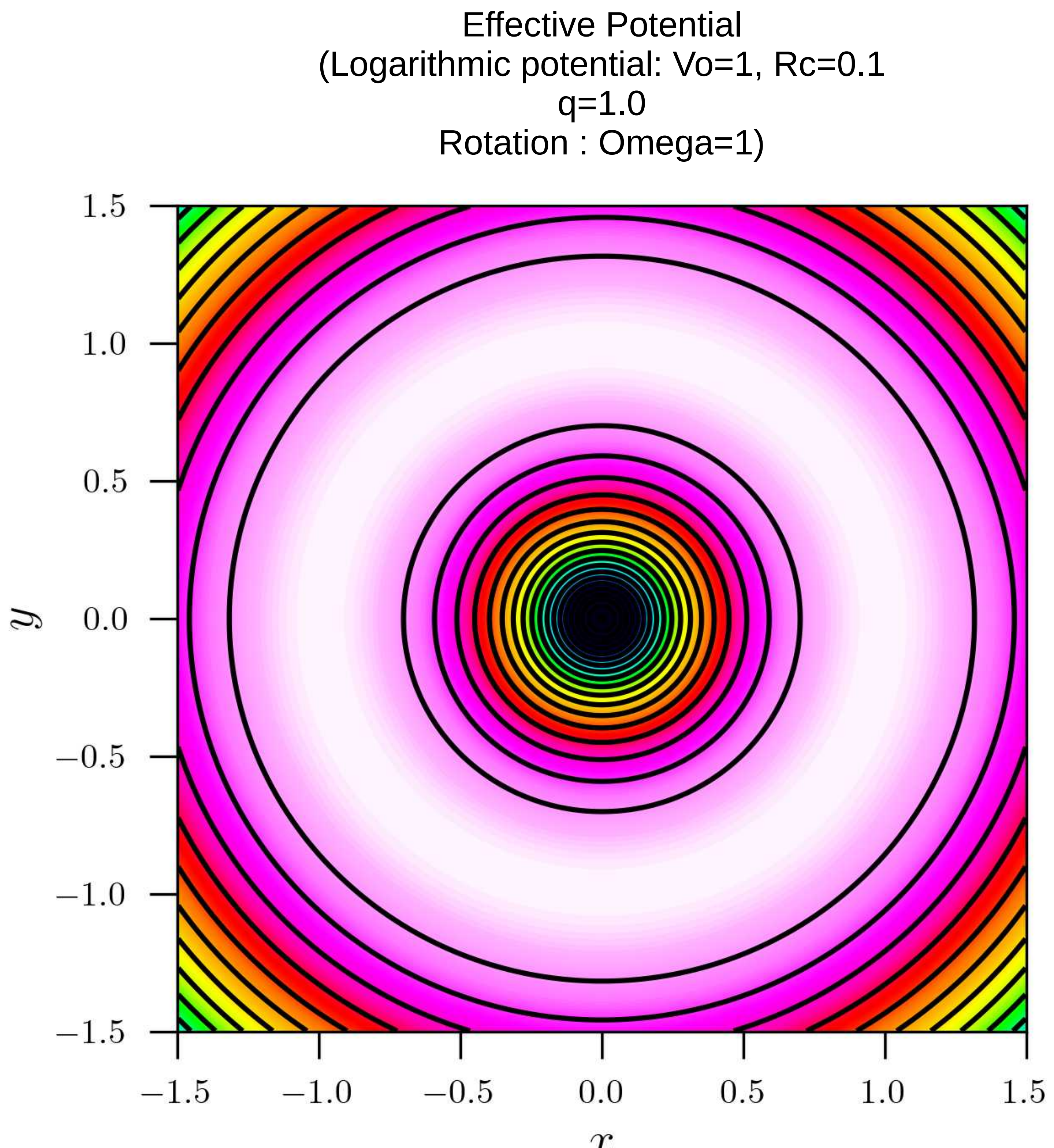


stationary points (corotation radius : centrifugal force = gravity)

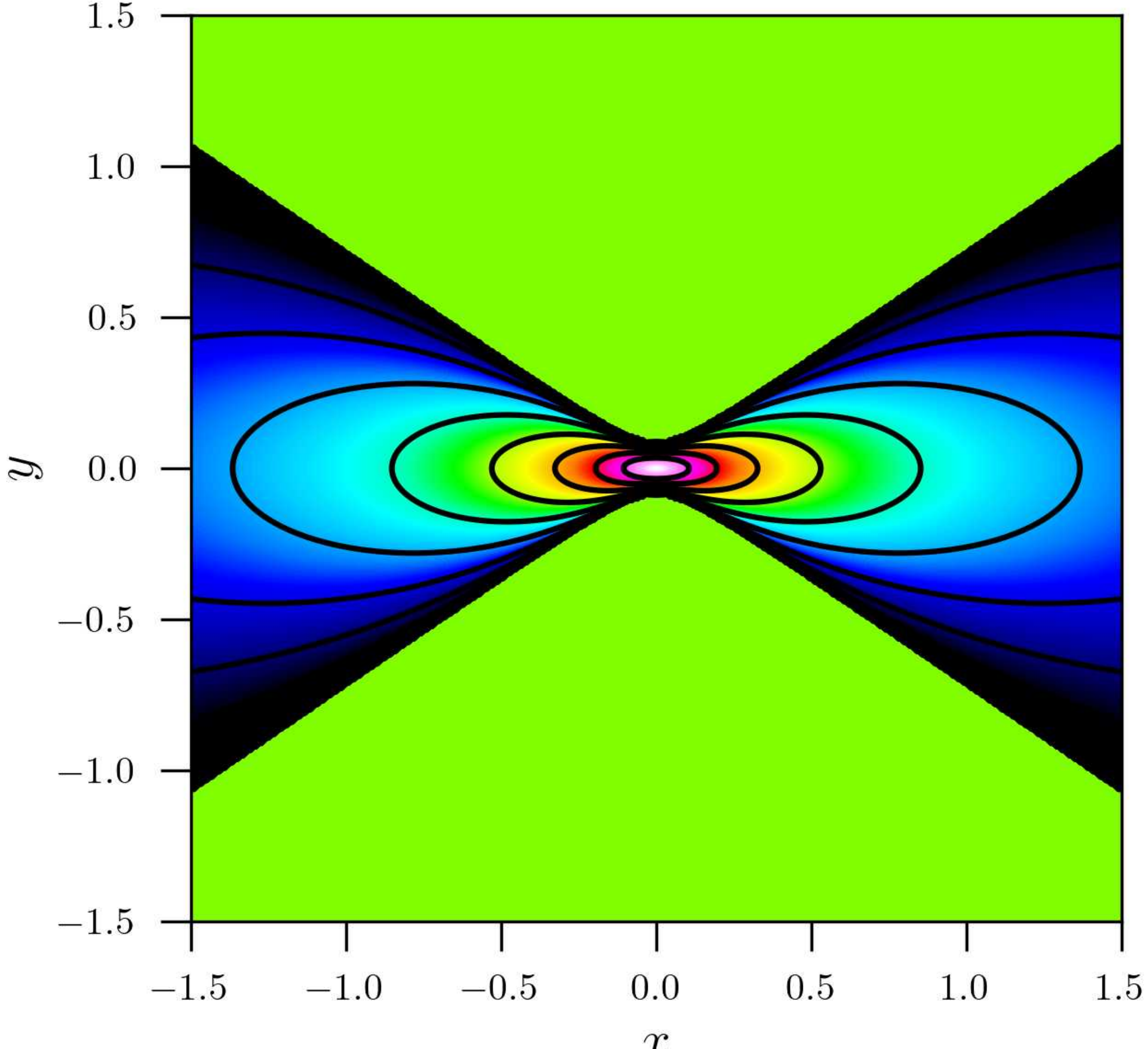

Bar potential (density)
(Logarithmic potential: Vo=1, Rc=0.1
q=1.0)
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
y
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

Effective Potential
(Logarithmic potential: Vo=1, Rc=0.1
q=1.0
Rotation : Omega=1)

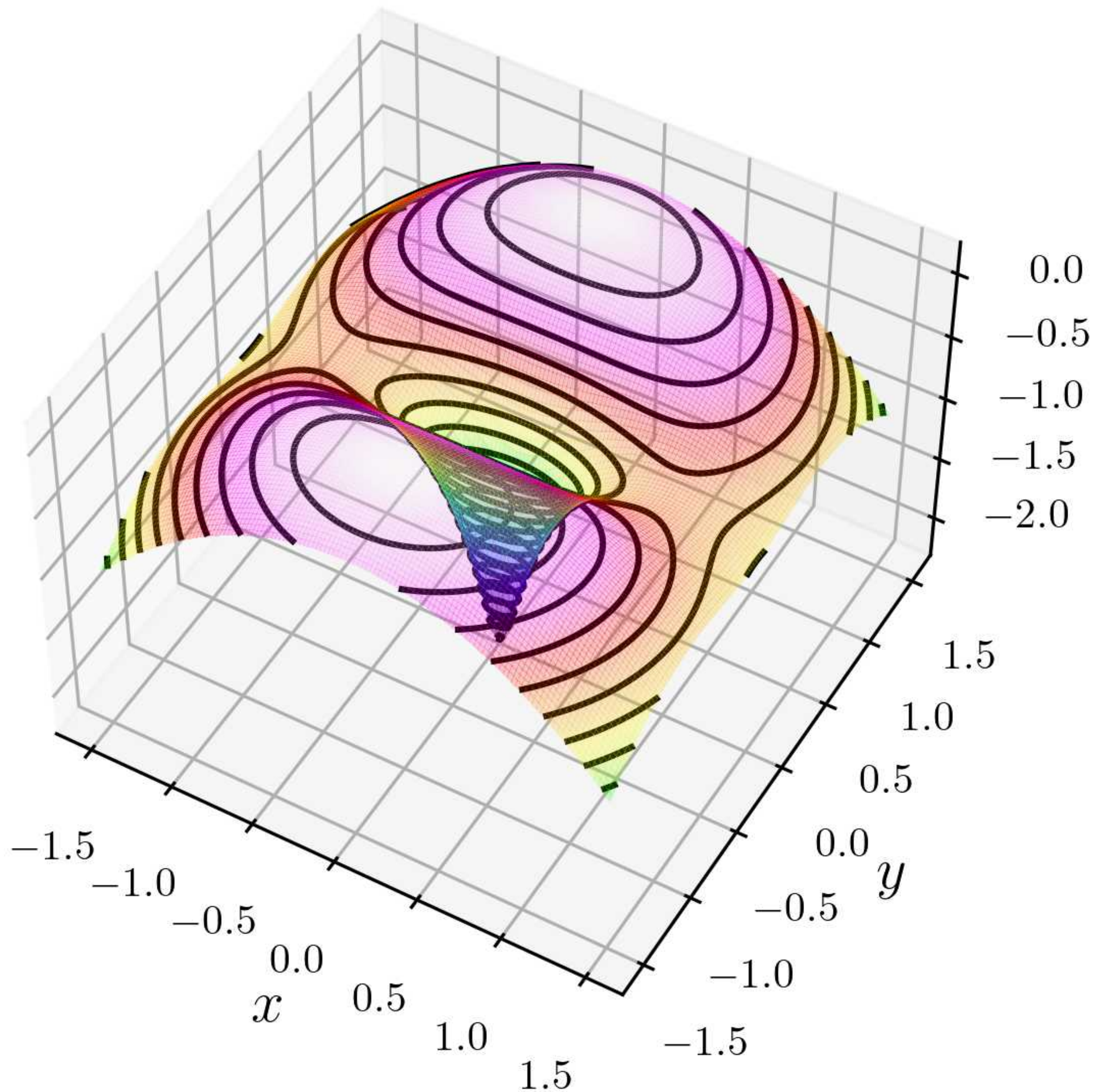



Effective Potential
(Logarithmic potential: Vo=1, Rc=0.1
q=1.0
Rotation : Omega=1)
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
y
x

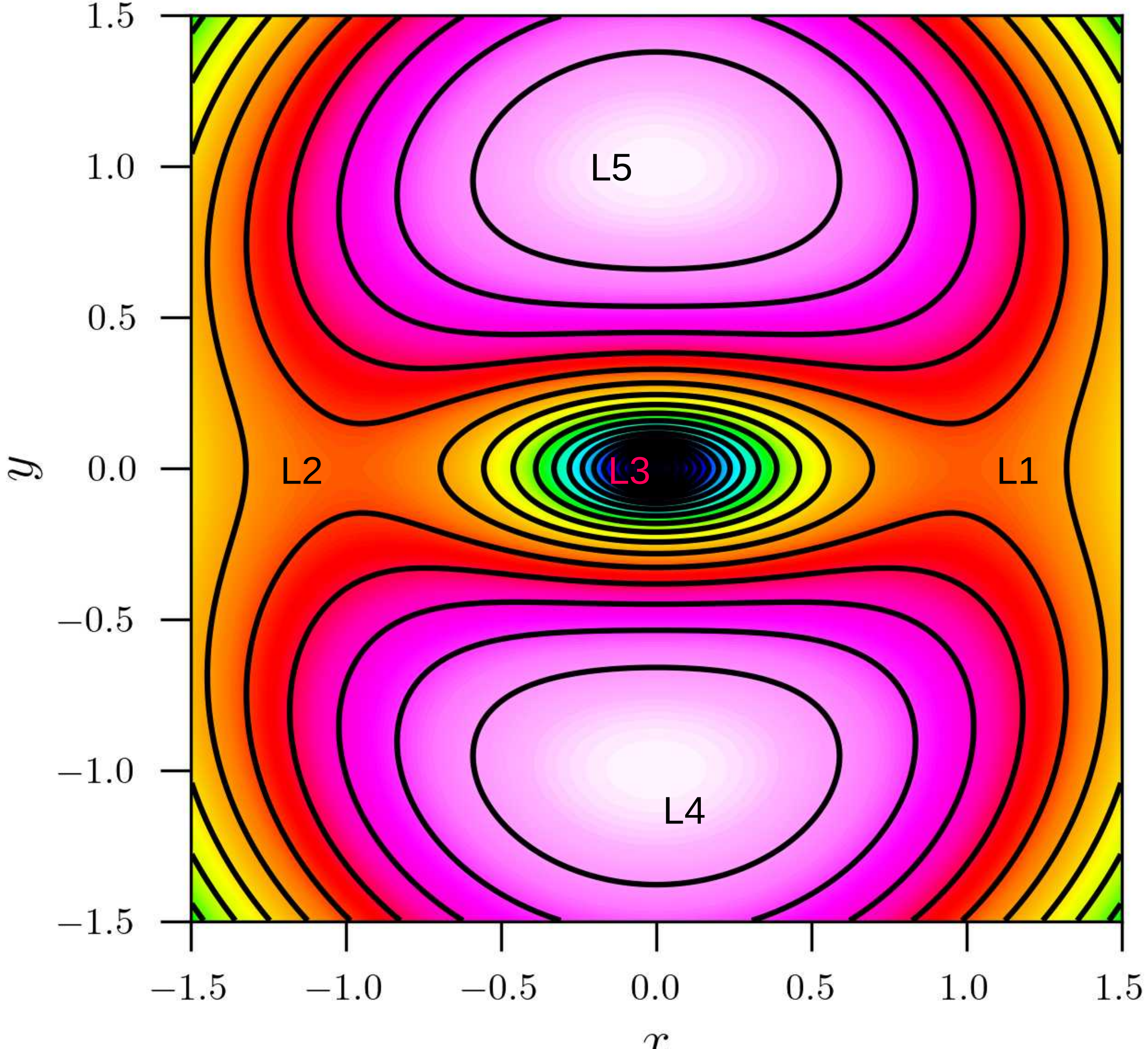
Bar potential
(Logarithmic potential: Vo=1, Rc=0.1
q=0.5)



Effective Potential
(Logarithmic potential: Vo=1, Rc=0.1
q=0.5
Rotation : Omega=1)



Effective Potential
(Logarithmic potential: Vo=1, Rc=0.1
q=0.5
Rotation : Omega=1)



Orbits around Lagrange points

Stability of orbits around Lagrange points

Expand the effective potential in Taylor serie around the Lagrange points (x_L, y_L)

$$\phi_{eff}(x,y) \cong \phi_{eff}(x_L, y_L) + \underbrace{\cancel{\frac{\partial \phi_{eff}}{\partial x}(x - x_L)}}_{=0} + \underbrace{\cancel{\frac{\partial \phi_{eff}}{\partial y}(y - y_L)}}_{=0}$$

$$+ \frac{1}{2}\frac{\partial^2 \phi_{eff}}{\partial x^2}(x - x_L)^2 + \frac{1}{2}\frac{\partial^2 \phi_{eff}}{\partial y^2}(y - y_L)^2 + \underbrace{\cancel{\frac{1}{2}\frac{\partial^2 \phi_{eff}}{\partial x \partial y}}}_{=0}(x - x_L)(y - y_L)$$

by symmetry of the bar, if it is aligned with $\hat{x}$

<u>Now we define</u>

$$\xi := x - x_L \qquad \phi_{xx} := \frac{\partial^2 \phi_{eff}}{\partial x^2}$$

$$\eta := y - y_L \qquad \phi_{yy} := \frac{\partial^2 \phi_{eff}}{\partial y^2}$$

$$\phi_{eff}(\xi, \eta) = \phi_{eff}(0,0) + \frac{1}{2}\phi_{xx}\xi^2 + \frac{1}{2}\phi_{yy}\eta^2$$

Equations of motions

$$\ddot{\vec{x}} = - \vec{\nabla} \phi_{eff} - 2 \left(\vec{\Omega} \times \dot{\vec{x}} \right)$$

in the plane $z = 0$ assuming $\vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$

$$\begin{cases} \ddot{x} = - \dfrac{\partial \phi_{eff}}{\partial x} + 2 \Omega \dot{y} \\ \ddot{y} = - \dfrac{\partial \phi_{eff}}{\partial y} - 2 \Omega \dot{x} \end{cases}$$

$$\boxed{\begin{cases} \ddot{\xi} = + 2 \Omega \dot{\eta} - \phi_{xx} \xi \\ \ddot{\eta} = - 2 \Omega \dot{\xi} - \phi_{yy} \eta \end{cases}}$$

We assume solutions of the form

$$\begin{cases} \xi(t) = X e^{\lambda t} \\ \eta(t) = Y e^{\lambda t} \end{cases} \qquad X, Y, \lambda \in \mathbb{C}$$

The EoM become

$$\begin{cases} (\lambda^2 + \phi_{xx})\, X - (2\lambda\Omega)\, Y = 0 \\ (2\lambda\Omega)\, X + (\lambda^2 + \phi_{yy})\, Y = 0 \end{cases}$$

$$\underbrace{\begin{pmatrix} \lambda^2 + \phi_{xx} & -2\lambda\Omega \\ 2\lambda\Omega & \lambda^2 + \phi_{yy} \end{pmatrix}}_{M} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

simple linear equation

Non trivial solutions (i.e $X \neq 0$, $Y \neq 0$) only if $\mathrm{Det}(M) = 0$

$$\mathrm{Det}\, M = \boxed{\lambda^4 + \lambda^2(\phi_{xx} + \phi_{yy} + 4\Omega^2) + \phi_{xx}\phi_{yy} = 0}$$

"characteristic equation"

Solutions (4 roots, two are coupled)

- if λ is a solution $\Rightarrow$ $-\lambda$ is a solution
- if λ is real

$$\begin{cases} \xi(t) = X e^{\lambda t} & \to \text{exponential growth} \\ \eta(t) = Y e^{\lambda t} & \to \text{exponential growth} \end{cases}$$

$\to$ the star leaves the Lagrange point

$\boxed{\text{UNSTABLE}}$

- if all λ are purely complex $\quad \lambda_1 = \alpha i \quad \lambda_2 = -\alpha i$
  $\lambda_3 = \beta i \quad \lambda_4 = -\beta i \qquad \alpha, \beta \in \mathbb{R}$

$$\begin{aligned}
\xi(t) &= \mathrm{Re}\left(X_1' e^{i\alpha t} + X_2' e^{-i\alpha t} + X_3' e^{i\beta t} + X_4' e^{-i\beta t} \right) \\
&= X_1' \cos(\alpha t) + X_2' \cos(-\alpha t) + X_3' \cos(\beta t) + X_4' \cos(-\beta t) \\
&= X_1 \cos(\alpha t) + X_2 \cos(\beta t)
\end{aligned}$$

idem for $\eta(t)$, so we get

$$\begin{cases} \xi(t) = X_1 \cos(\alpha t) + X_2 \cos(\beta t) \\ \eta(t) = Y_1 \cos(\alpha t) + Y_2 \cos(\beta t) \end{cases}$$

$$\boxed{\text{STABLE}}$$

with

$$\begin{cases} Y_1 = \dfrac{\phi_{xx} - \alpha^2}{2\Omega\alpha} X_1 = \dfrac{2\Omega\alpha}{\phi_{yy} - \alpha^2} X_1 \\[2ex] Y_2 = \dfrac{\phi_{xx} - \beta^2}{2\Omega\beta} X_1 = \dfrac{2\Omega\beta}{\phi_{yy} - \beta^2} X_2 \end{cases}$$

It is possible to demonstrate that :

- At L_3 i.e $\min(\phi_{eff})$ always stable
- At L_2, L_3 i.e the saddles points always unstable
- At L_4, L_5 i.e $\max(\phi_{eff})$ stable or unstable

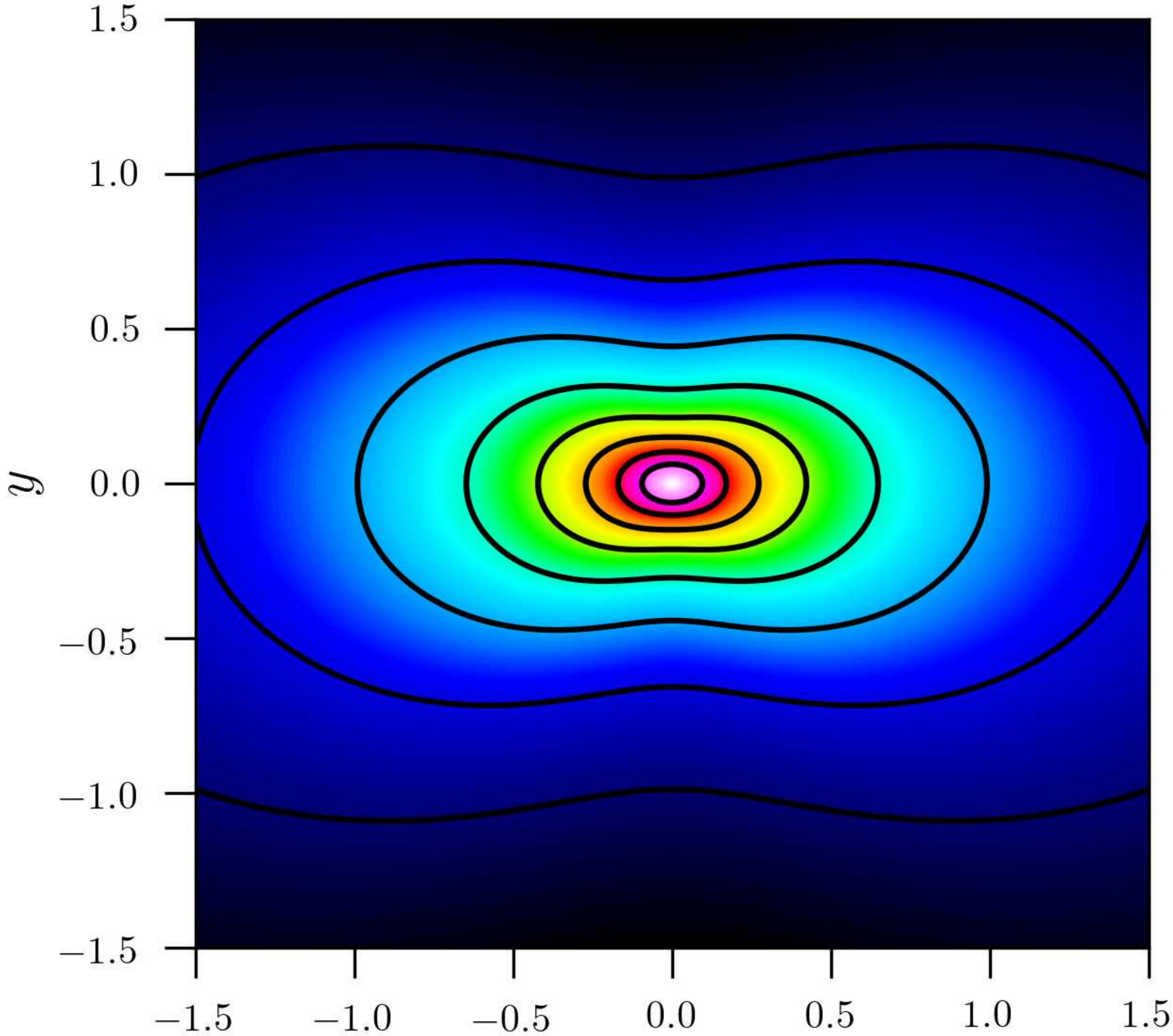
depends on the detail of the potential

Note: The stability comes from the Coriolis force (see Padmanabhan)

Orbits not confined to Lagrange points

Bar model : Logarithmic potential:
Vo=01 Rc=0.1 q=0.8)

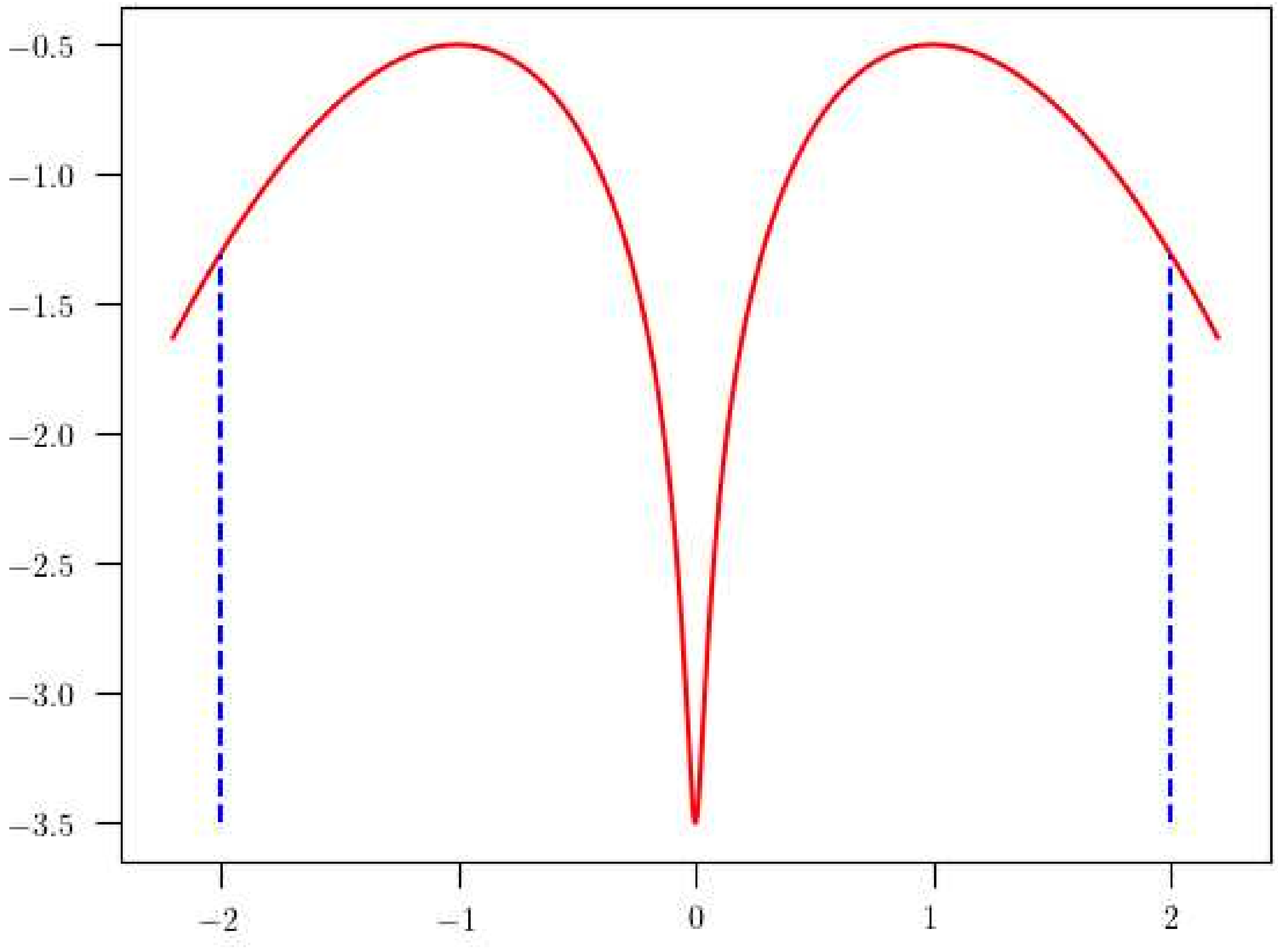
$$\Phi_{\log}(x,y) = \frac{1}{2}V_0^2 \ln\left(R_c^2 + x^2 + \left(\frac{y}{q}\right)^2\right) \qquad \Omega_{\mathrm{p}} \neq 0$$



Low energy orbits

$$R \ll R_{\mathrm{corot}}$$

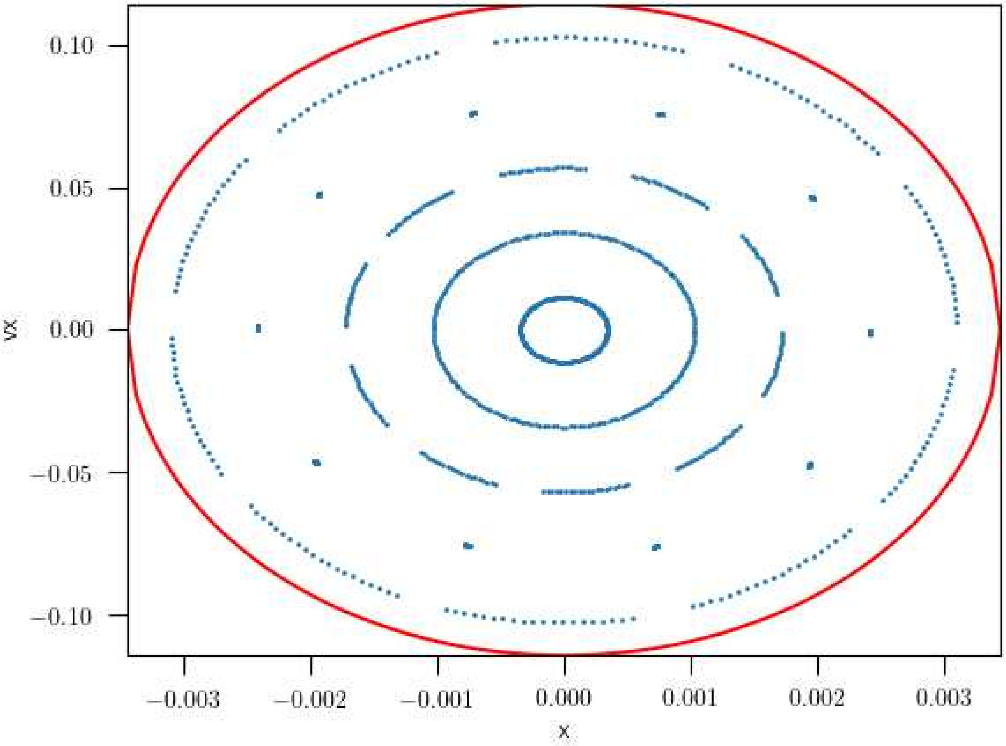
Potential and energy




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 --plotpotential
```

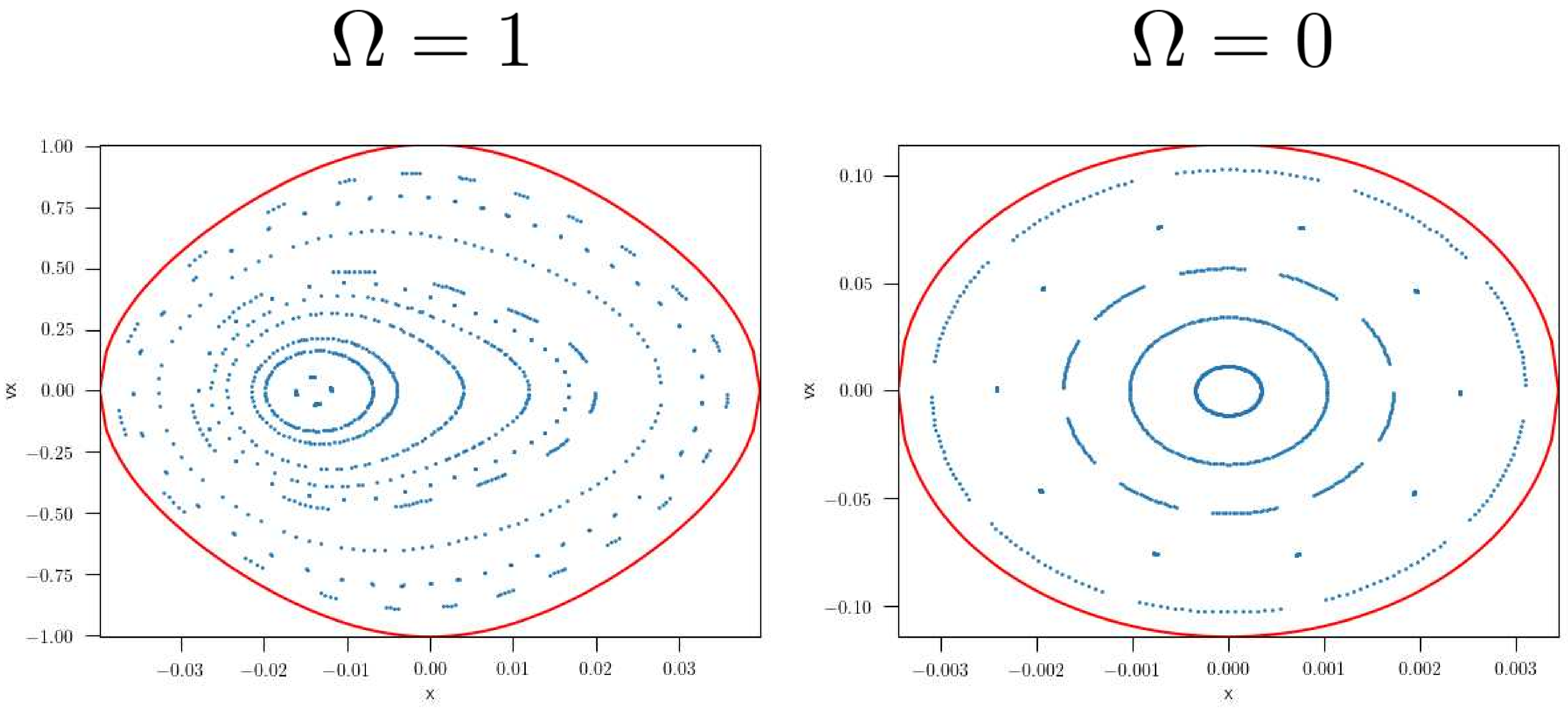
Orbits around L_3

$$\Omega = 0$$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

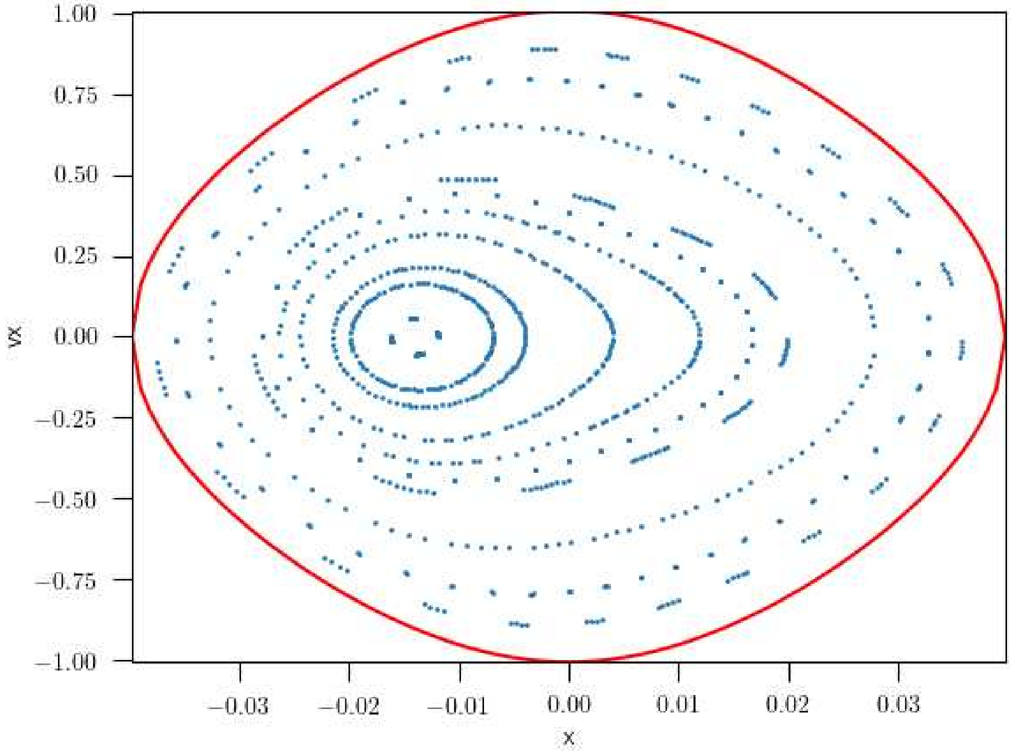
Orbits around L_3

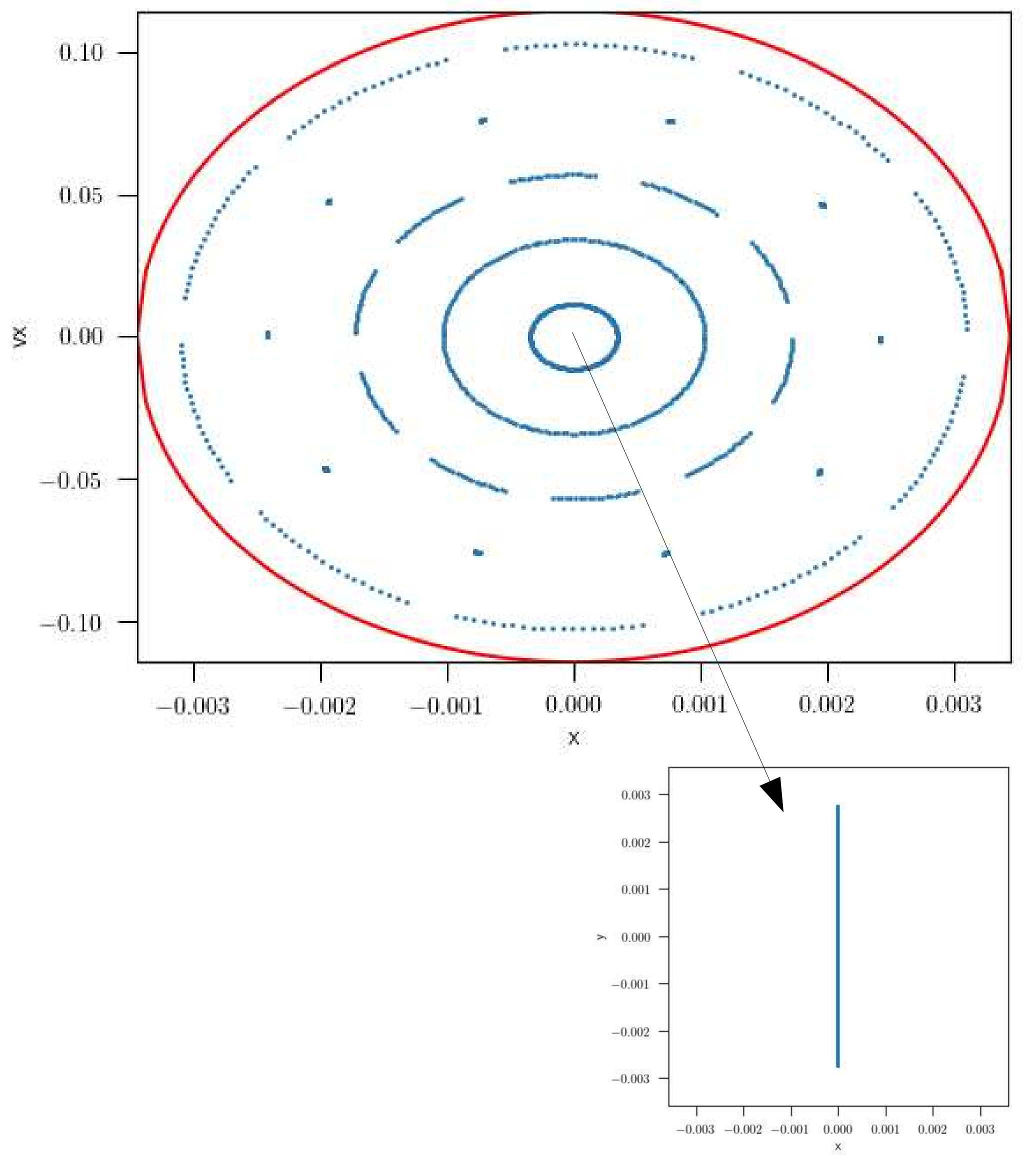



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Orbits around L_3

$\Omega = 1$ $\qquad\qquad$ $\Omega = 0$



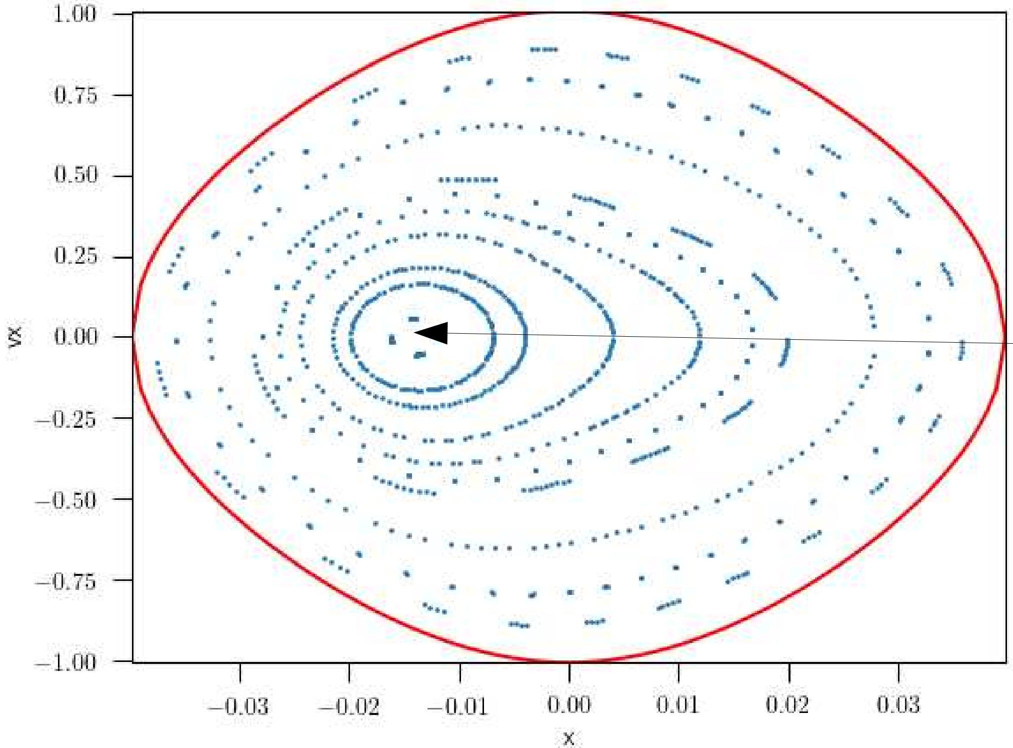


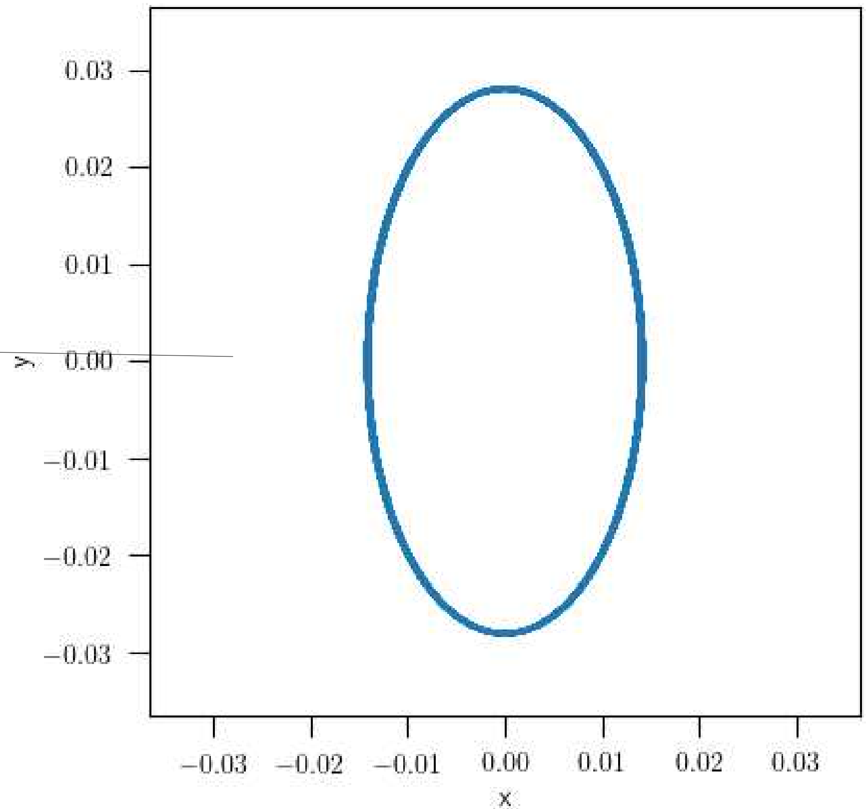

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Short axis (Y) orbits (periodic)

$\Omega = 1$

$\Omega = 0$





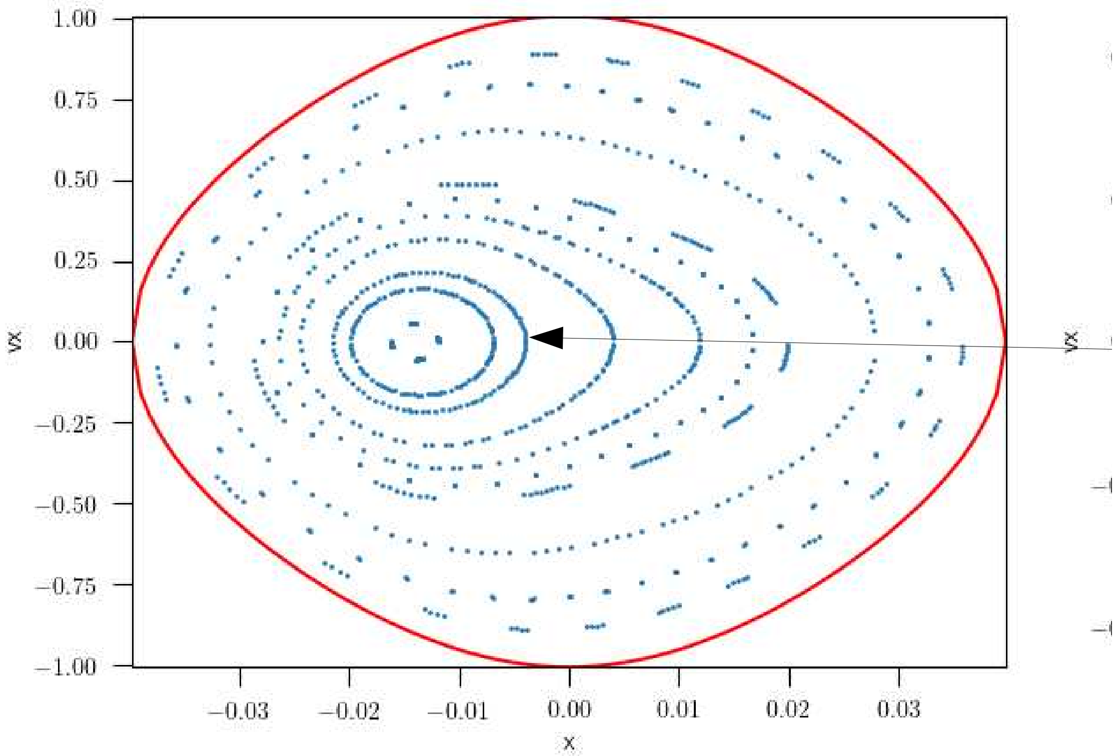
X4

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x -0.014
```

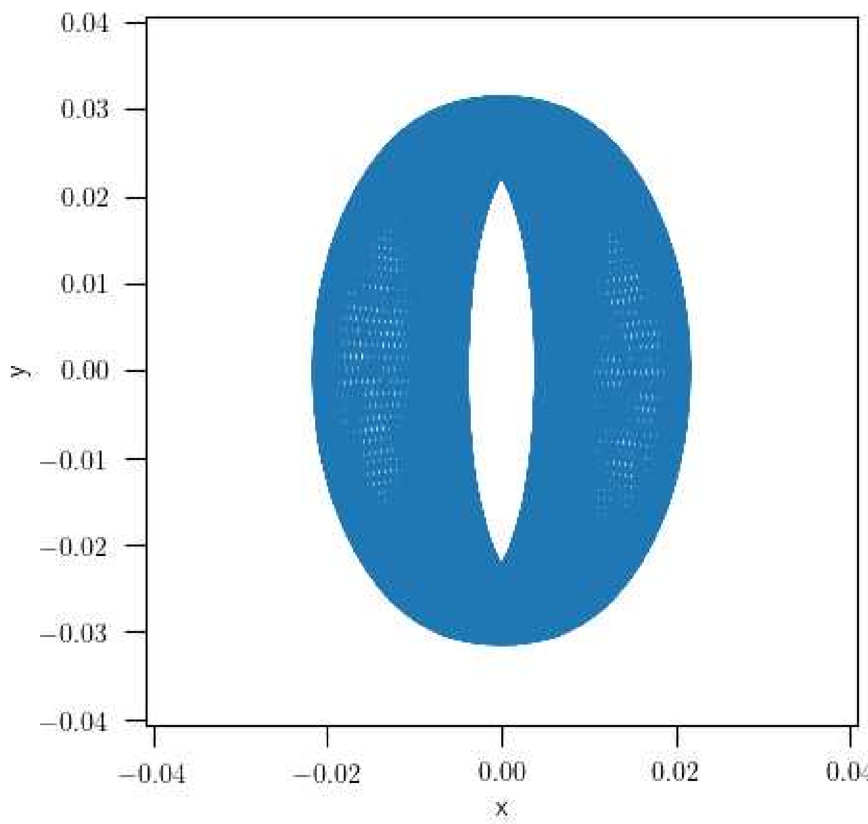
Apparition of a periodic loop orbit (replace the radial orbit, perpendicular to the bar)

Short axis (Y) orbits (periodic)

$\Omega = 1$



$\Omega = 0$

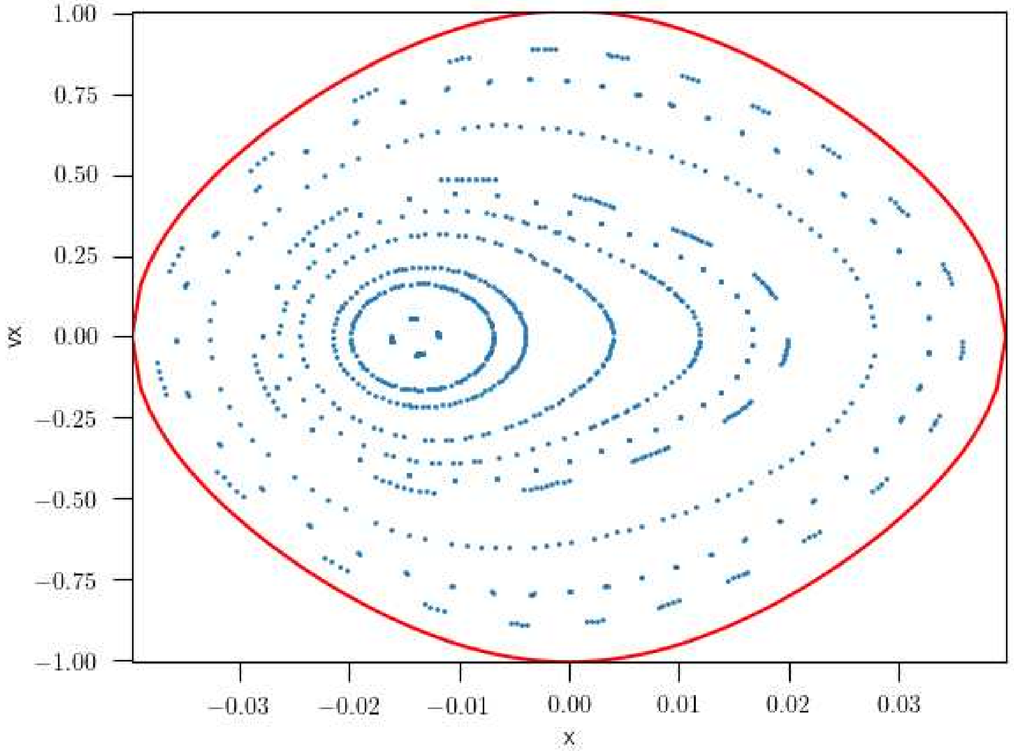


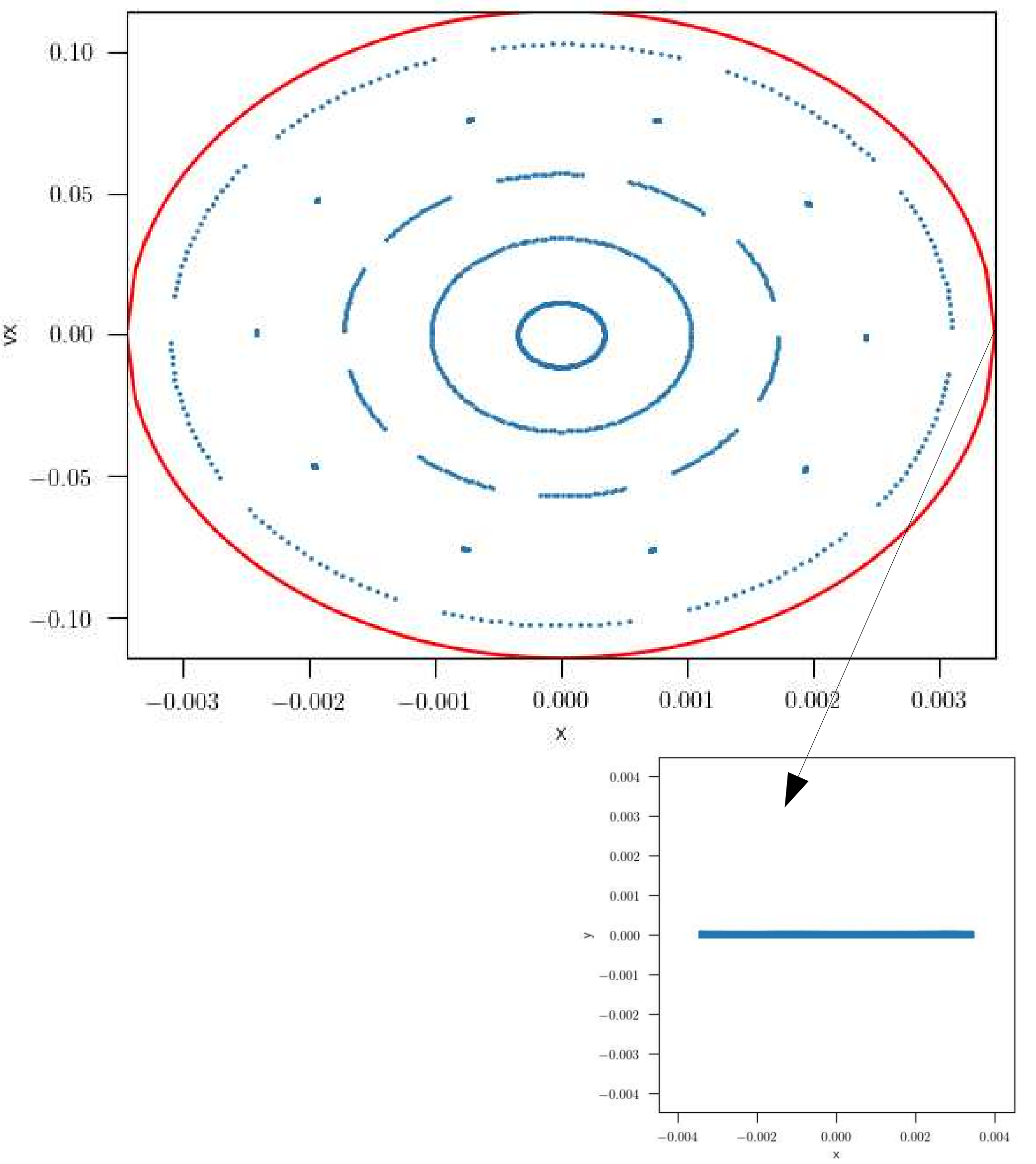
X4

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x -0.004
```

Orbits around L_3

$\Omega = 1$ $\Omega = 0$

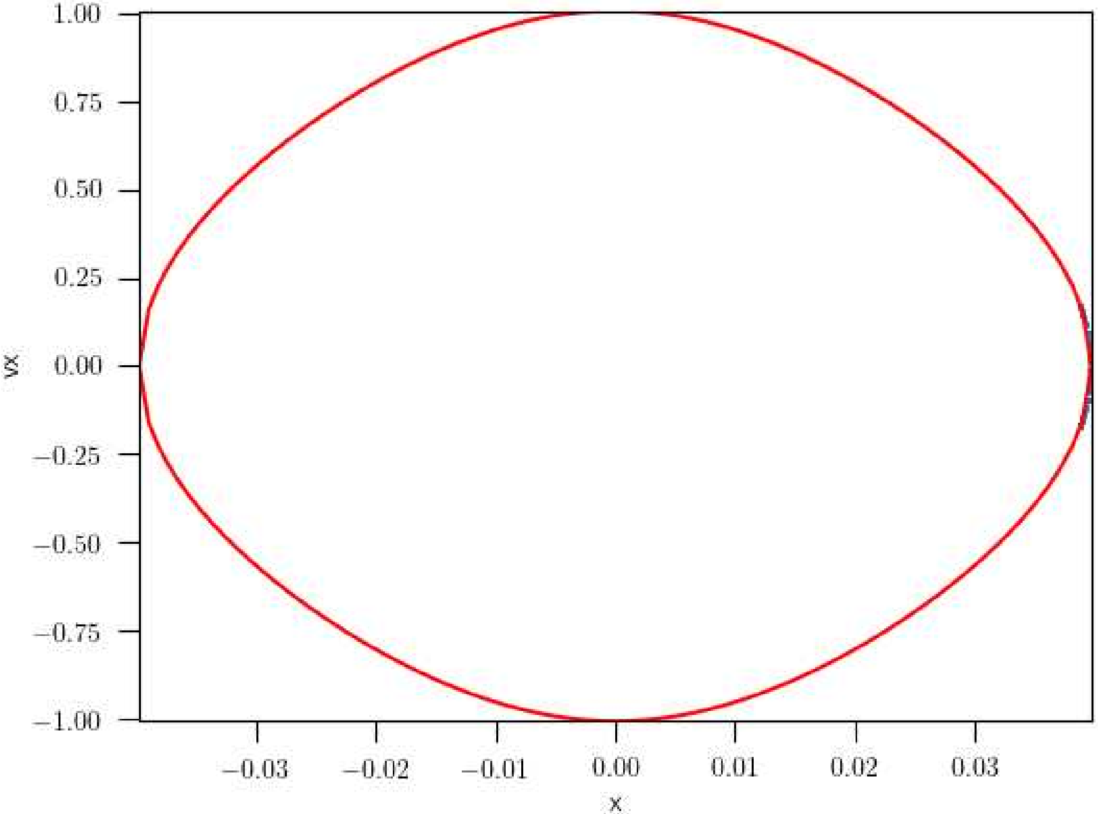


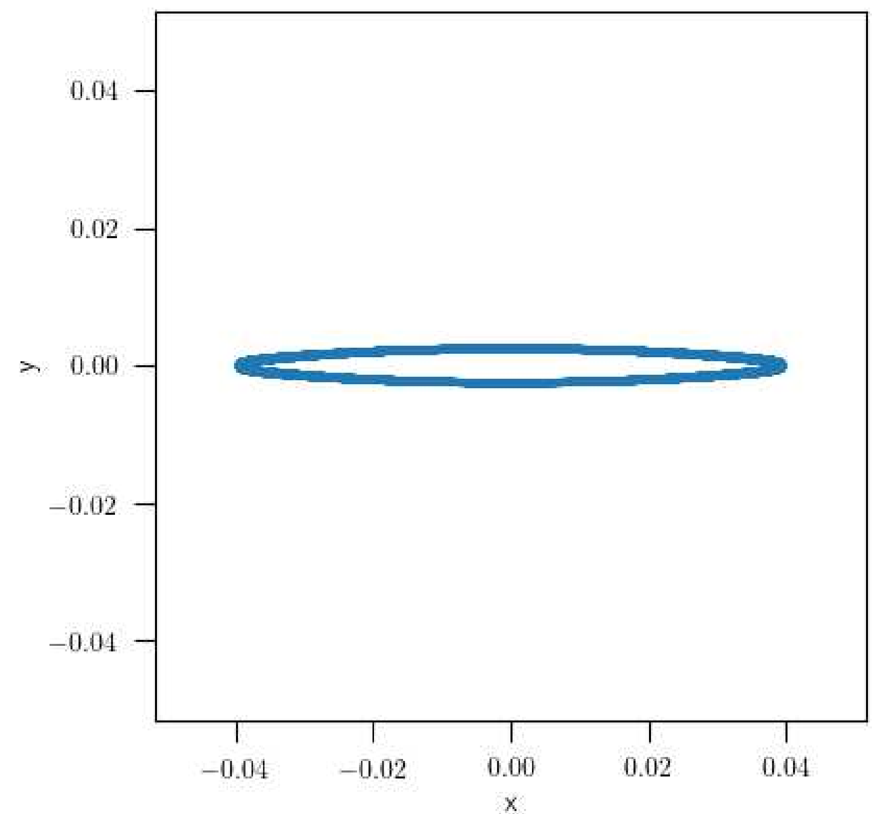



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Long axis (X) orbits (periodic)

$$\Omega = 1$$





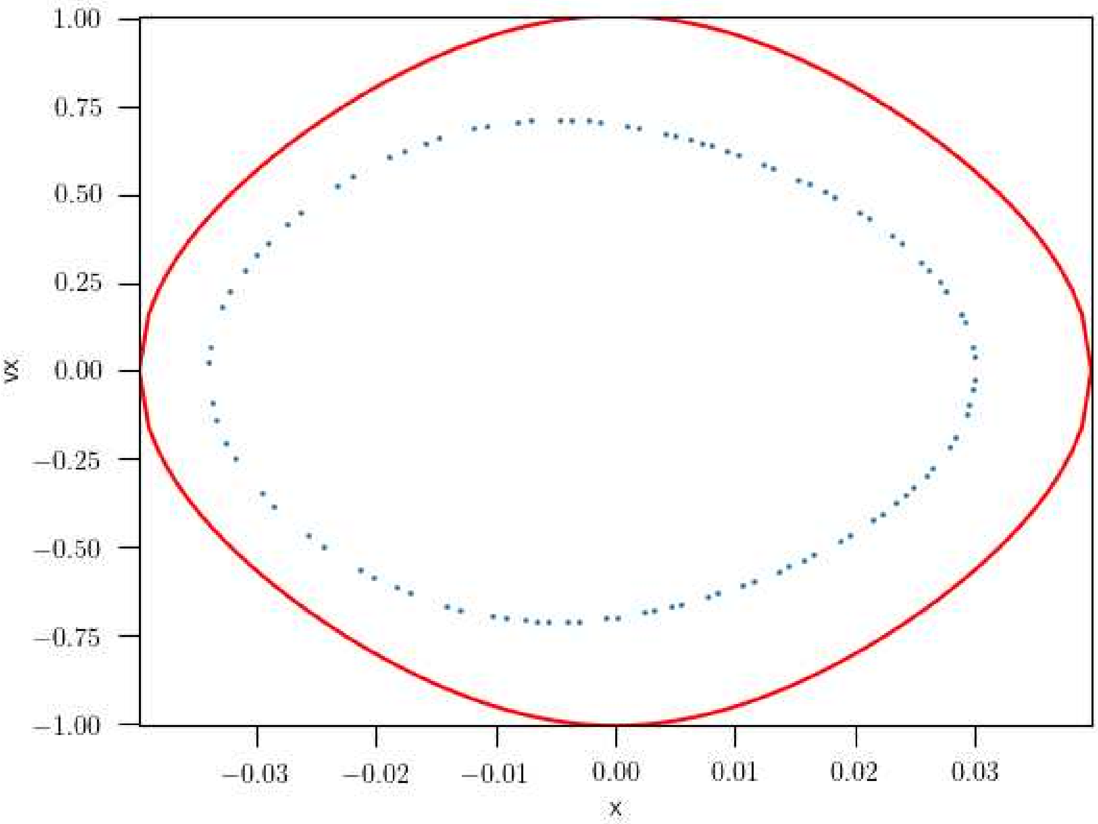
x1

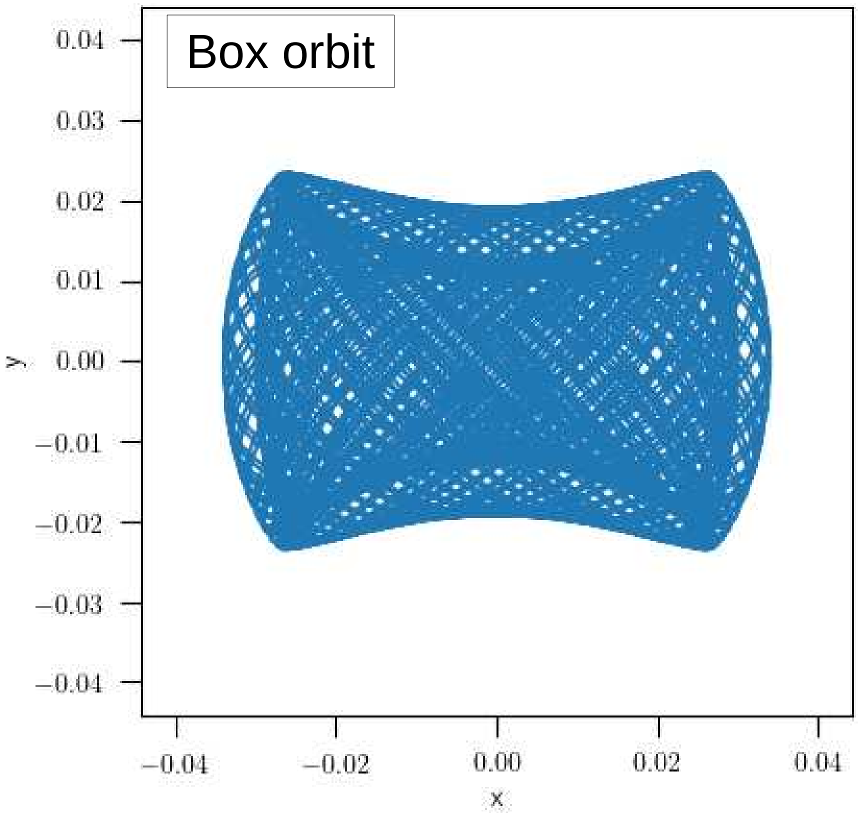
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x 0.03975
```

Apparition of a periodic loop orbit (replace the radial orbit, parallel to the bar)

Long axis (X) orbits (non periodic)

$$\Omega = 1$$





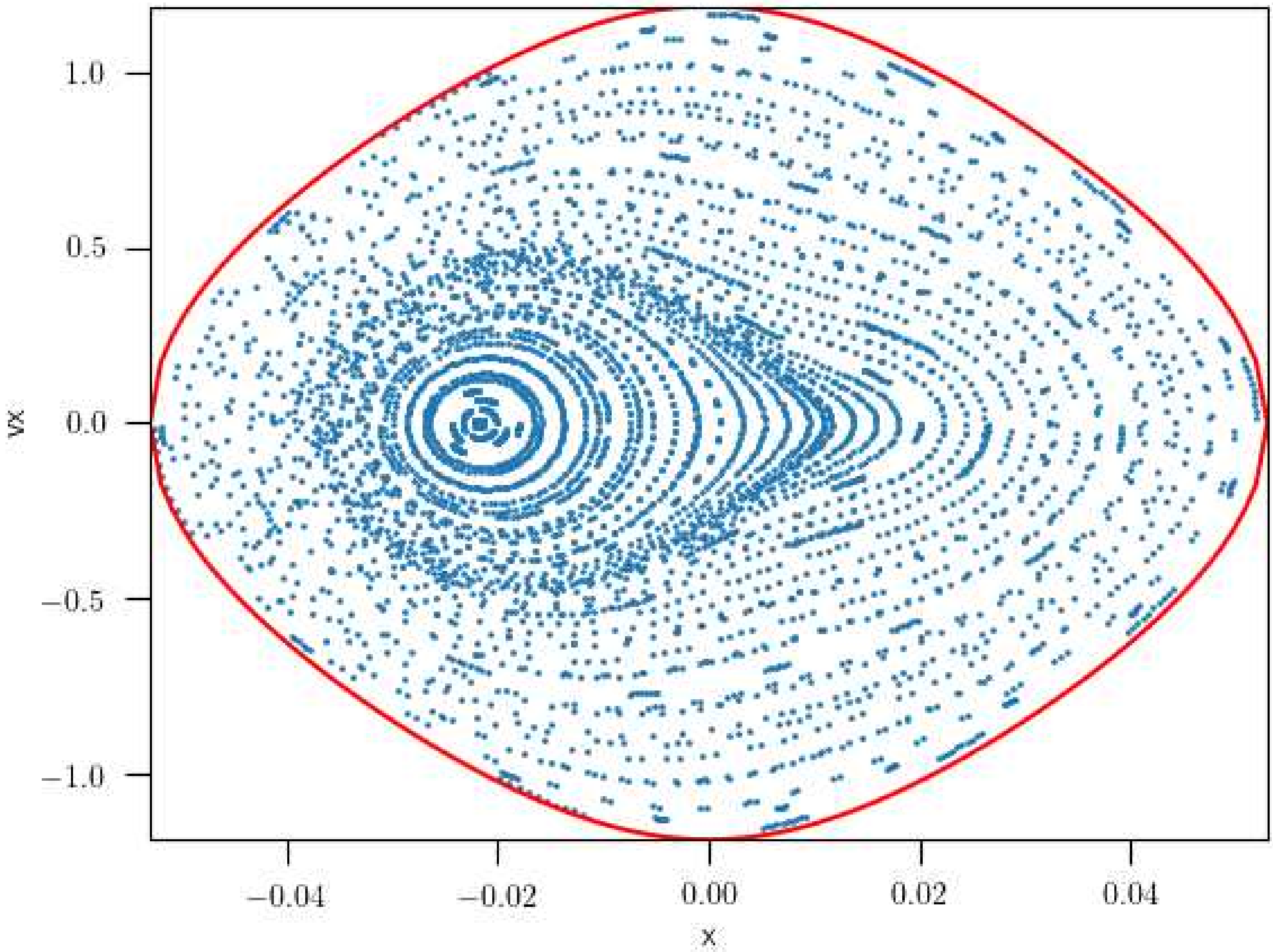
x1

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x 0.03
```

Increasing the energy

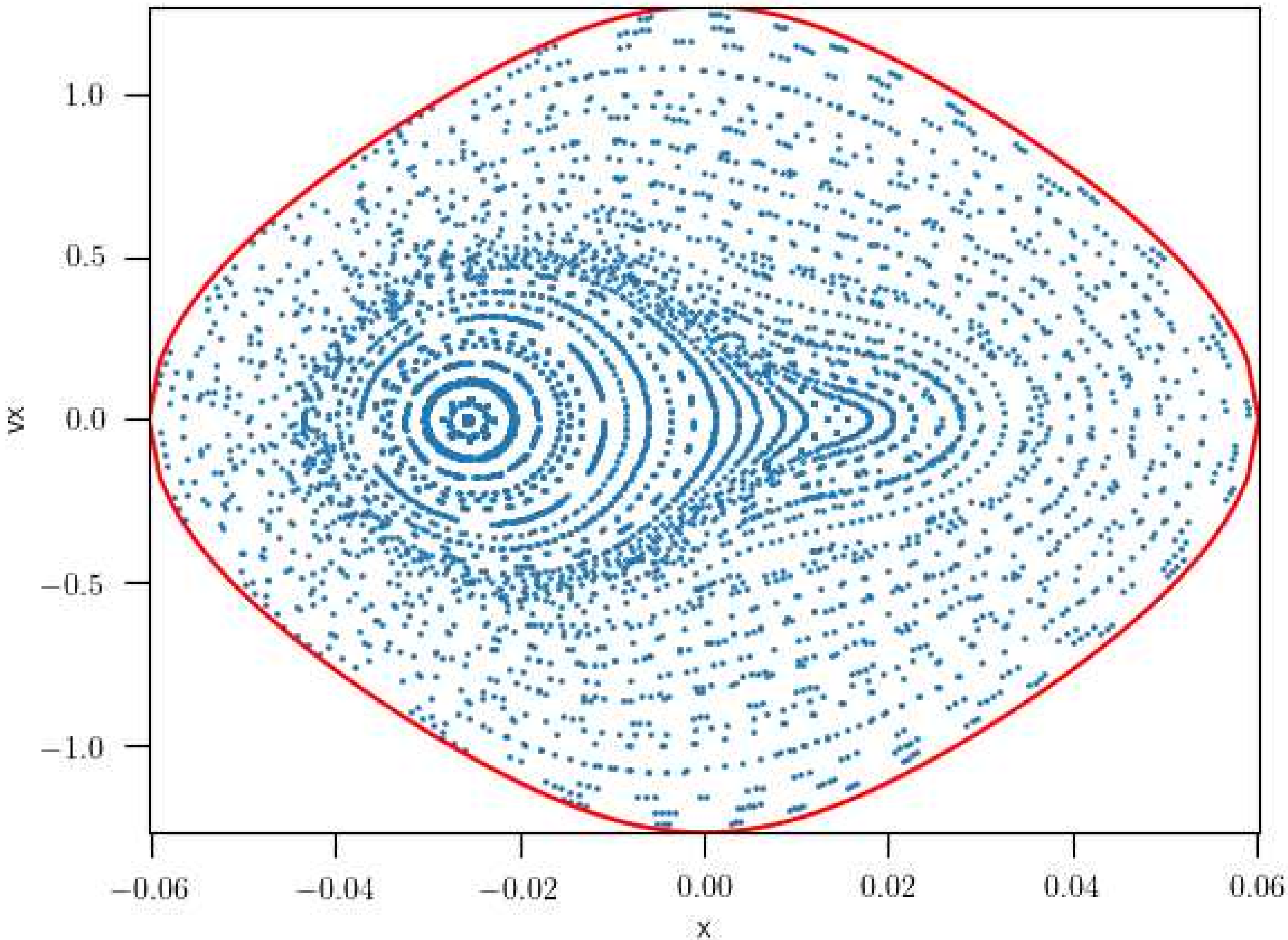
$$E = -2.8$$

$$E = -2.8$$



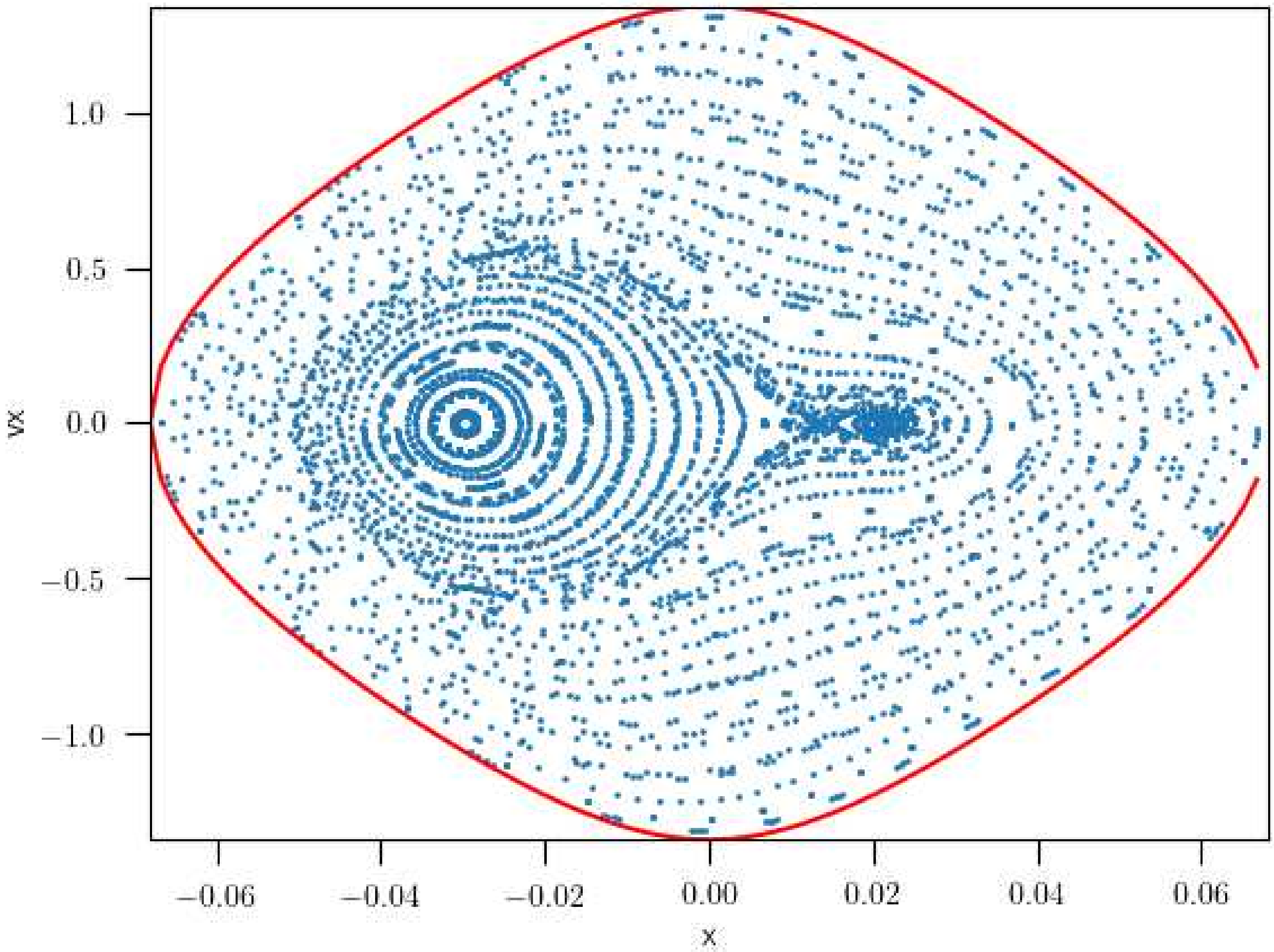

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.8 --norbits 50
```

$$E = -2.7$$



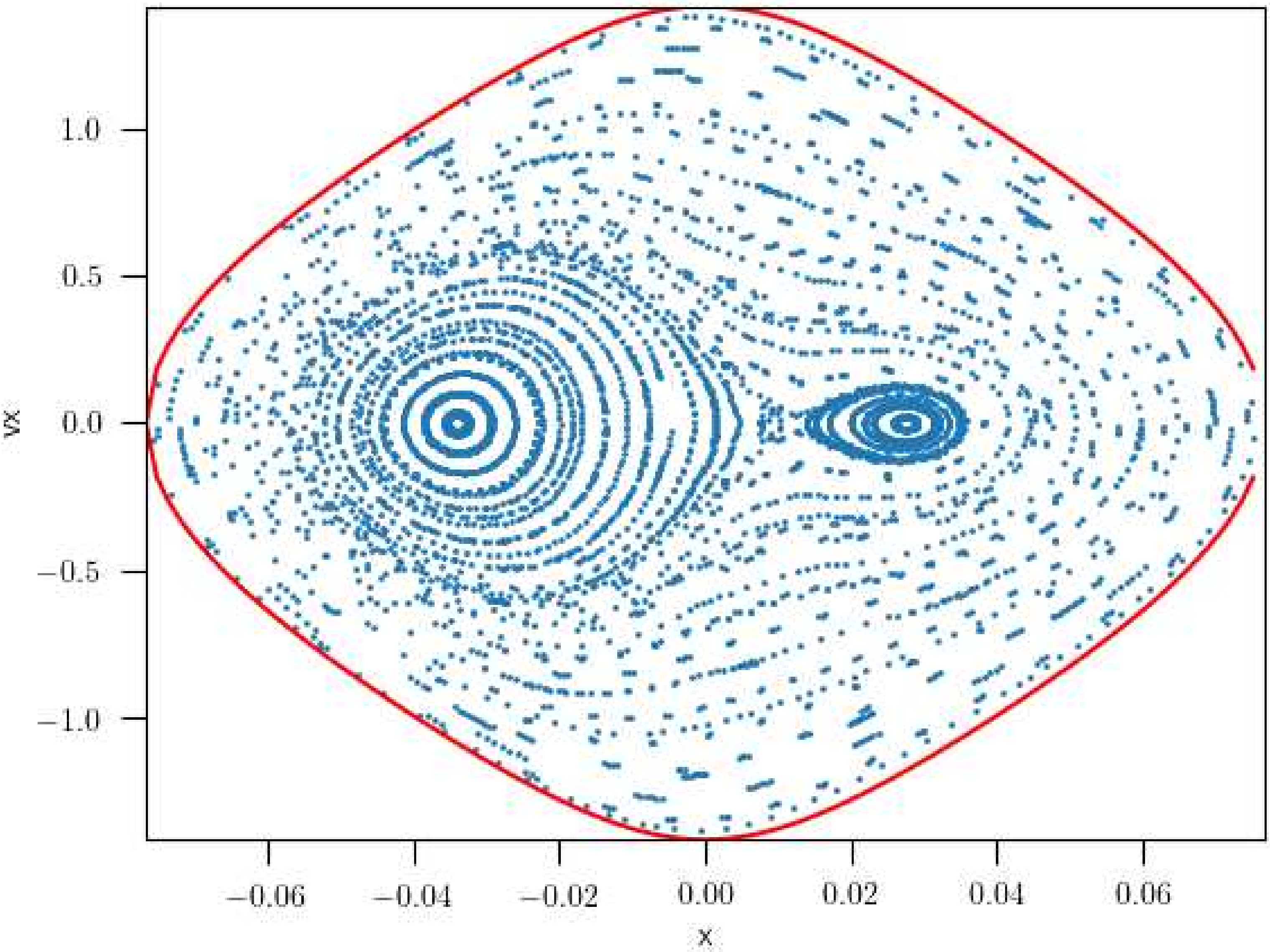

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.7 --norbits 50
```

$$E = -2.6$$



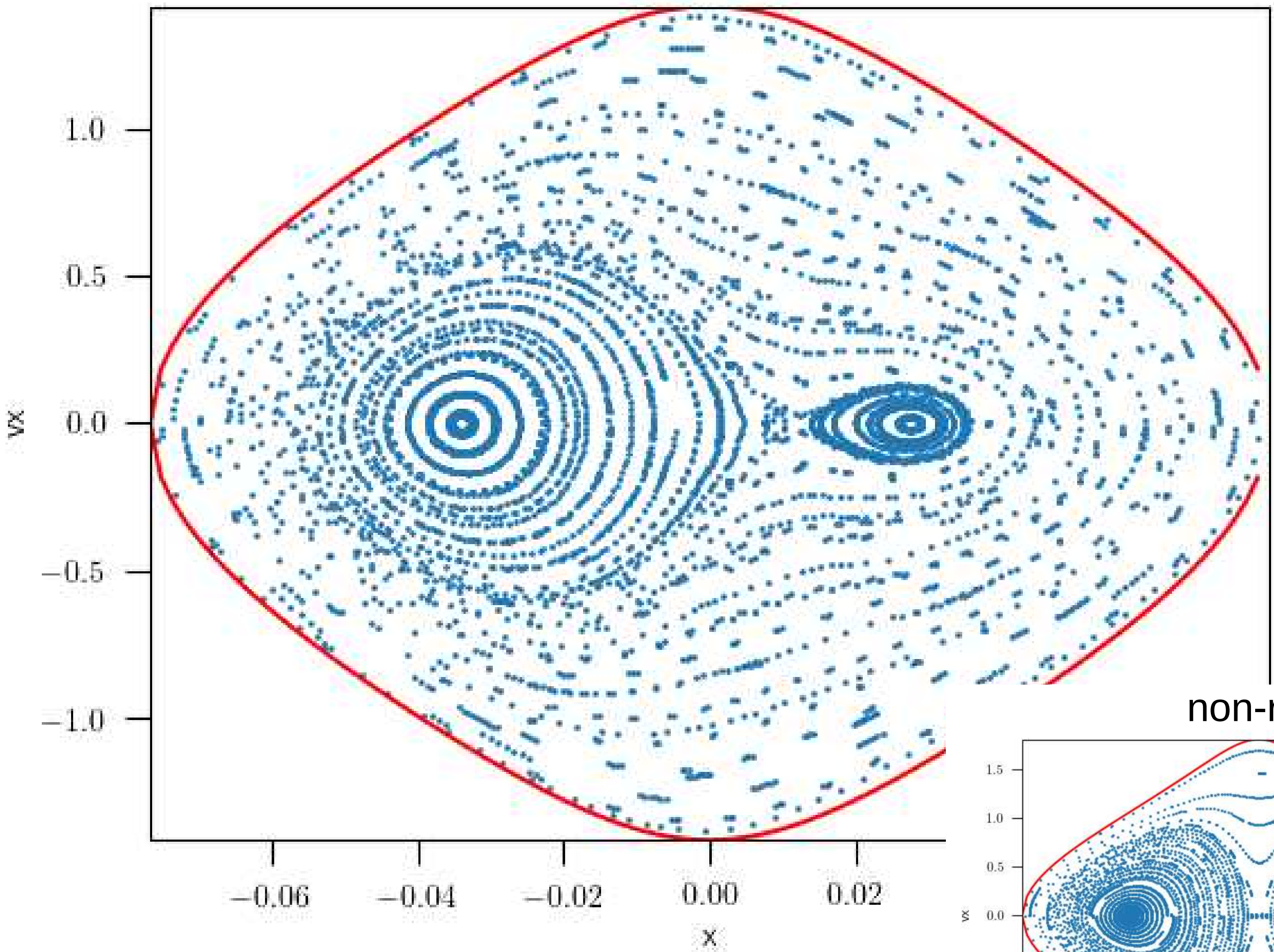

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.6 --norbits 50
```

$$E = -2.5$$

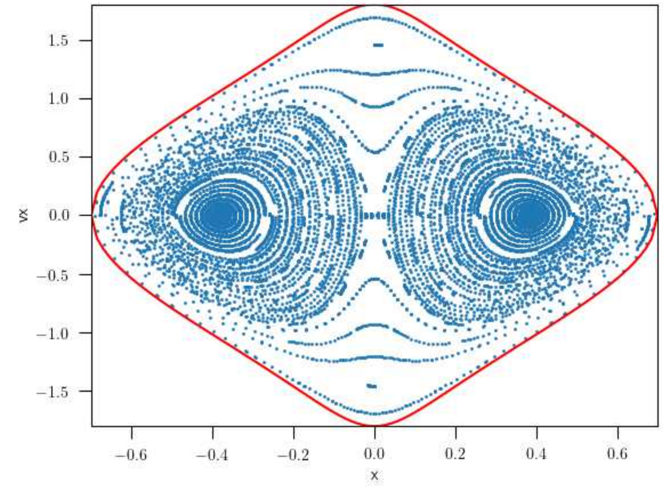



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --norbits 50
```

$E = -2.5$

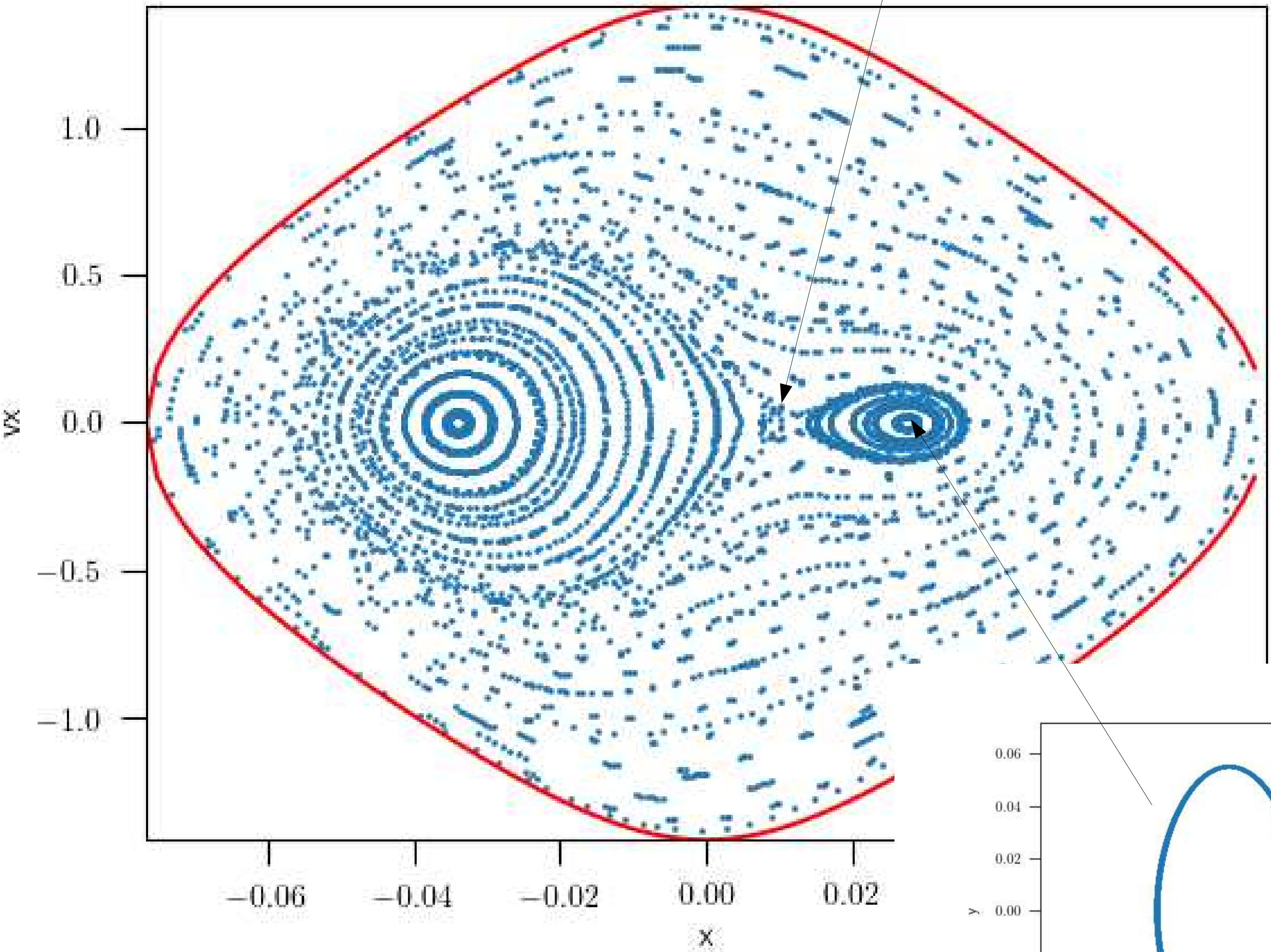


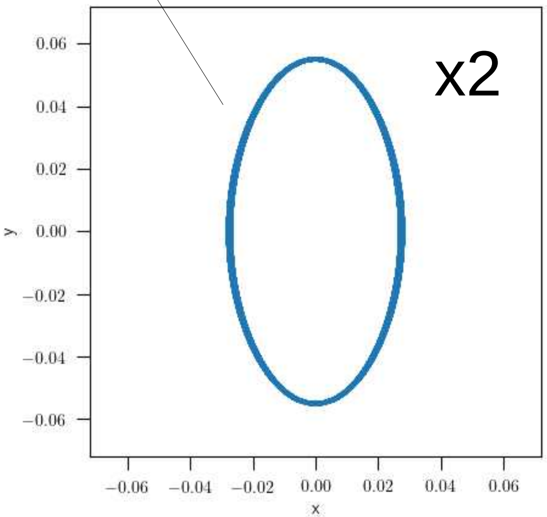
non-rotating case




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --norbits 50
```

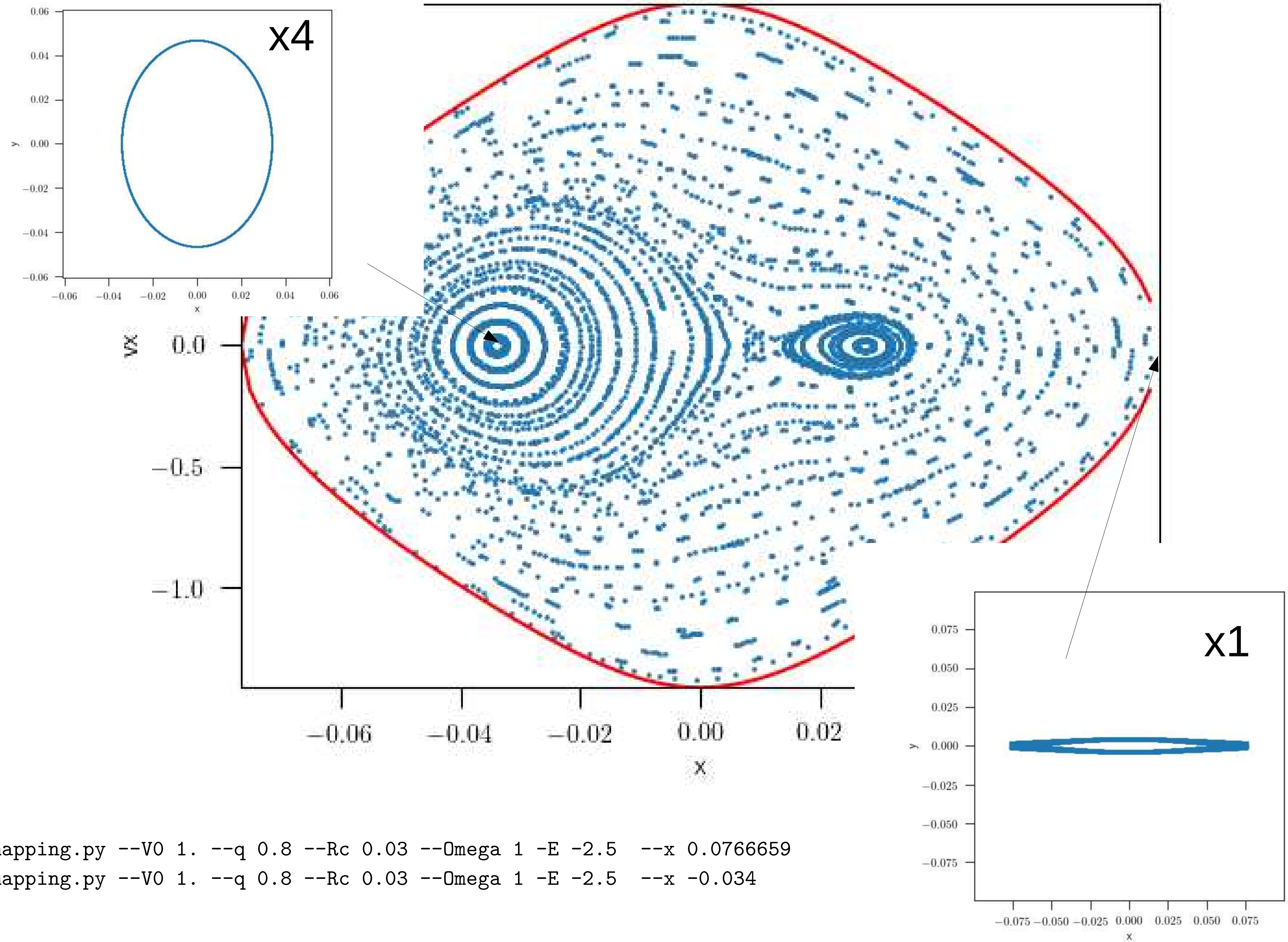
Bifurcation : apparition of x2 (stabe)/x3 (unstable) orbits






```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0268
```

x1 : prograde x4 : retrograde

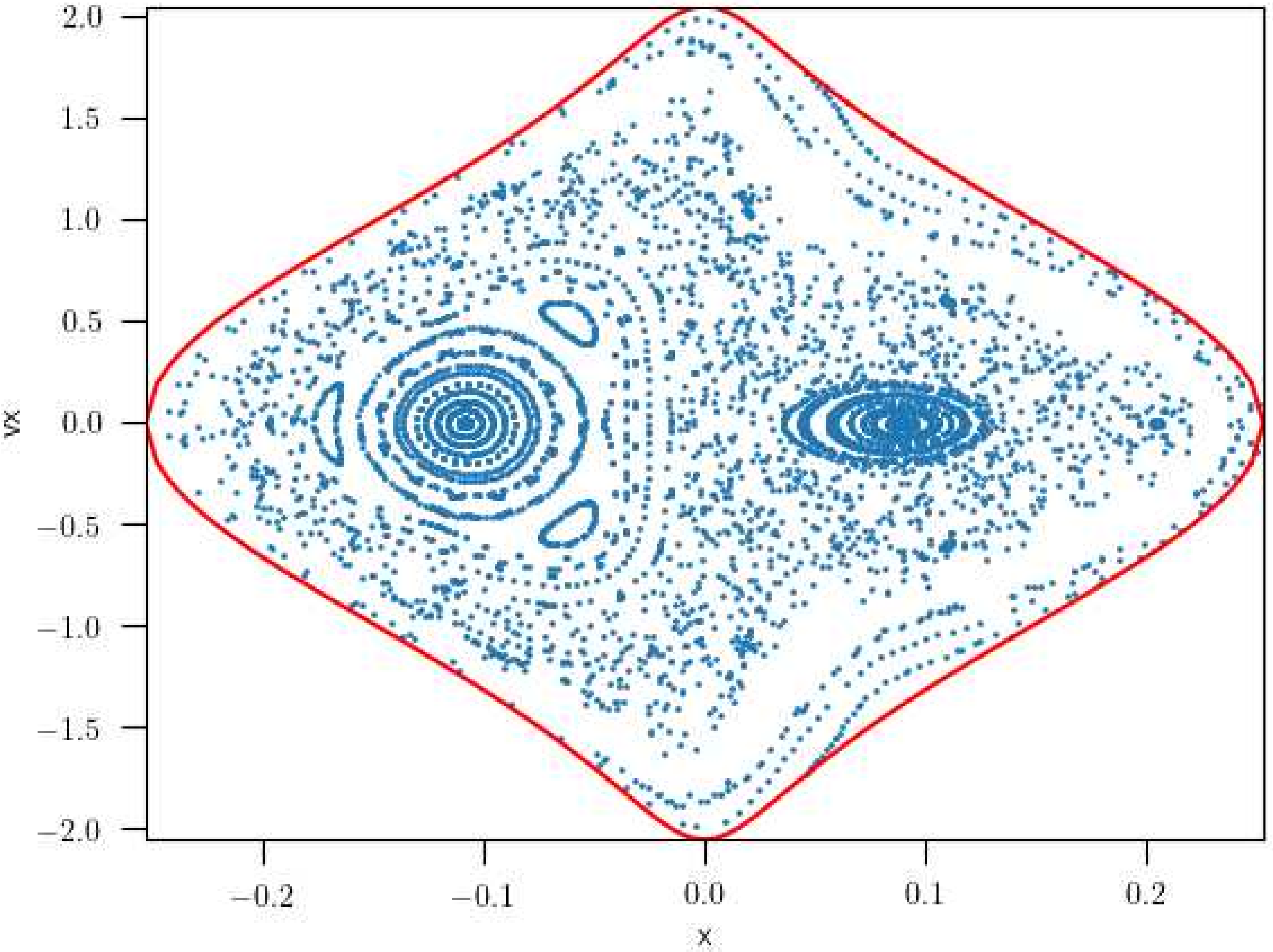



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x -0.034
```

Increasing the energy further

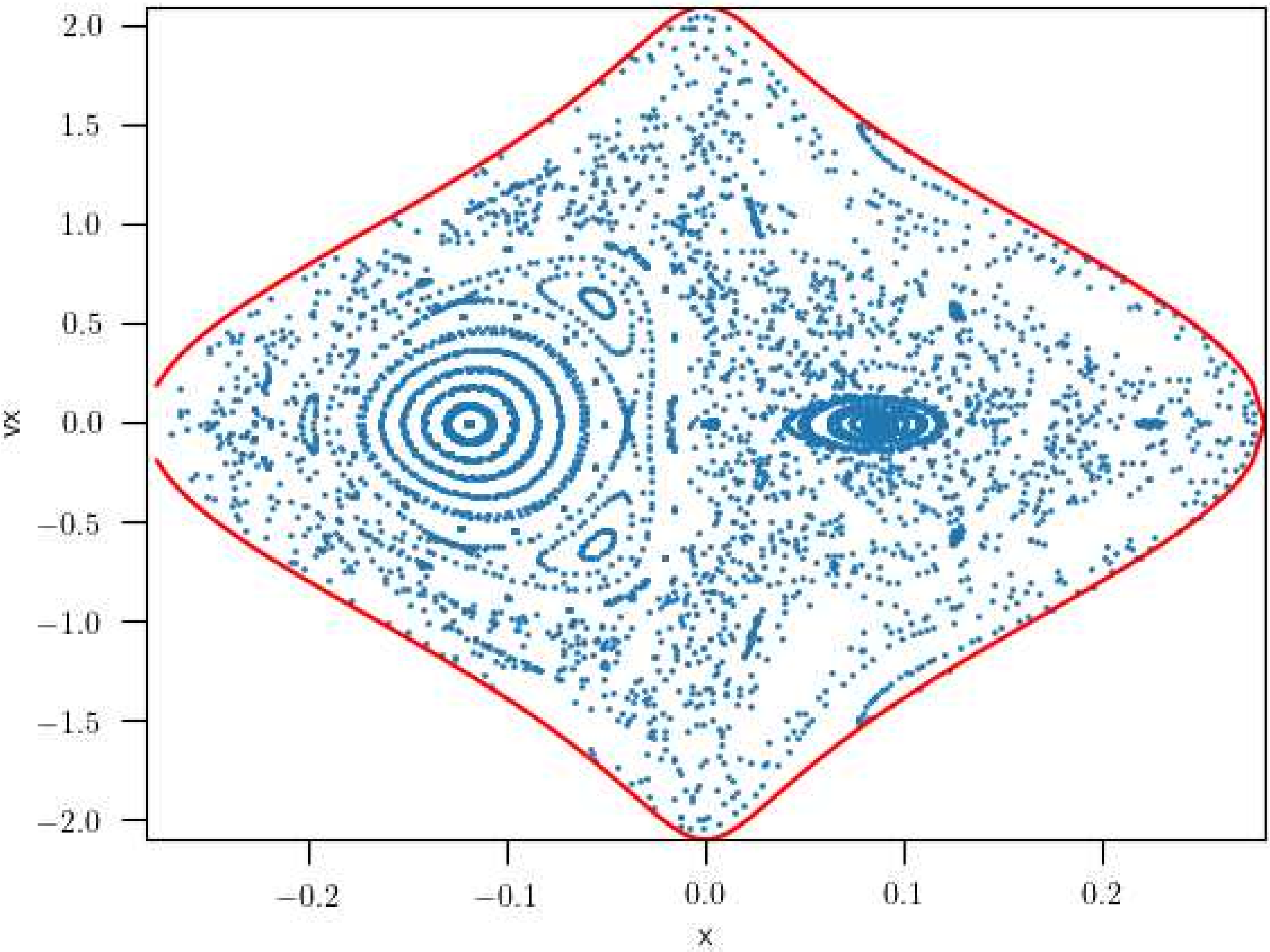
$$E = -1.4$$

$E = -1.4$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.4 --norbits 50
```

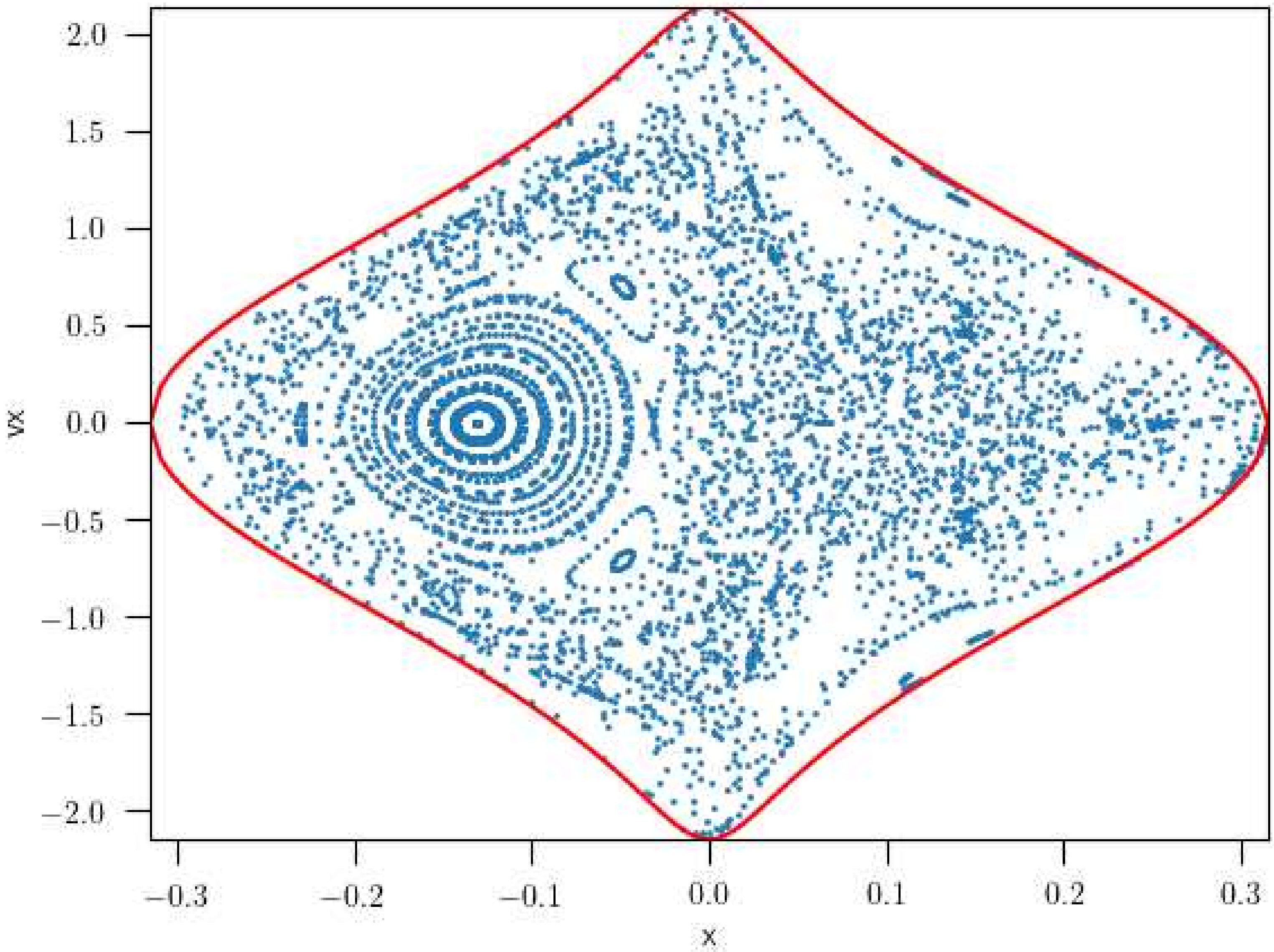
$E = -1.3$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.3 --norbits 50
```

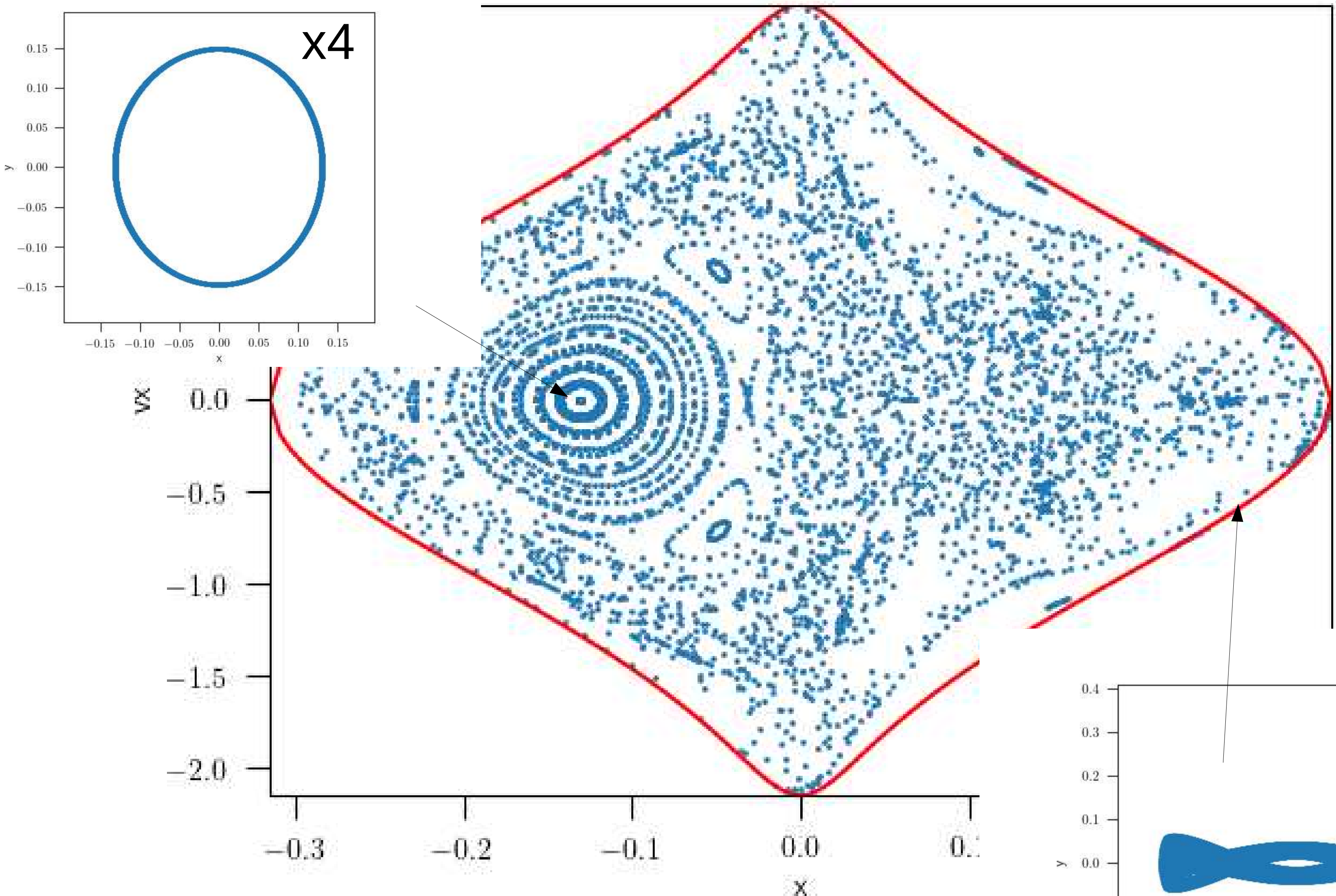
$$E = -1.2$$

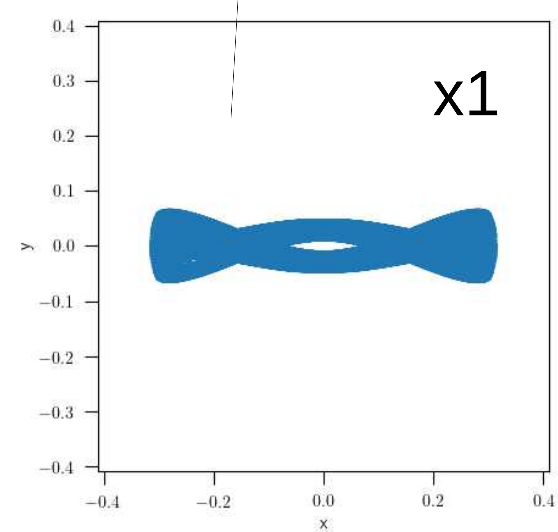
Bifurcation : x2/x3 disappeared




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --norbits 50
```

$$E = -1.2$$

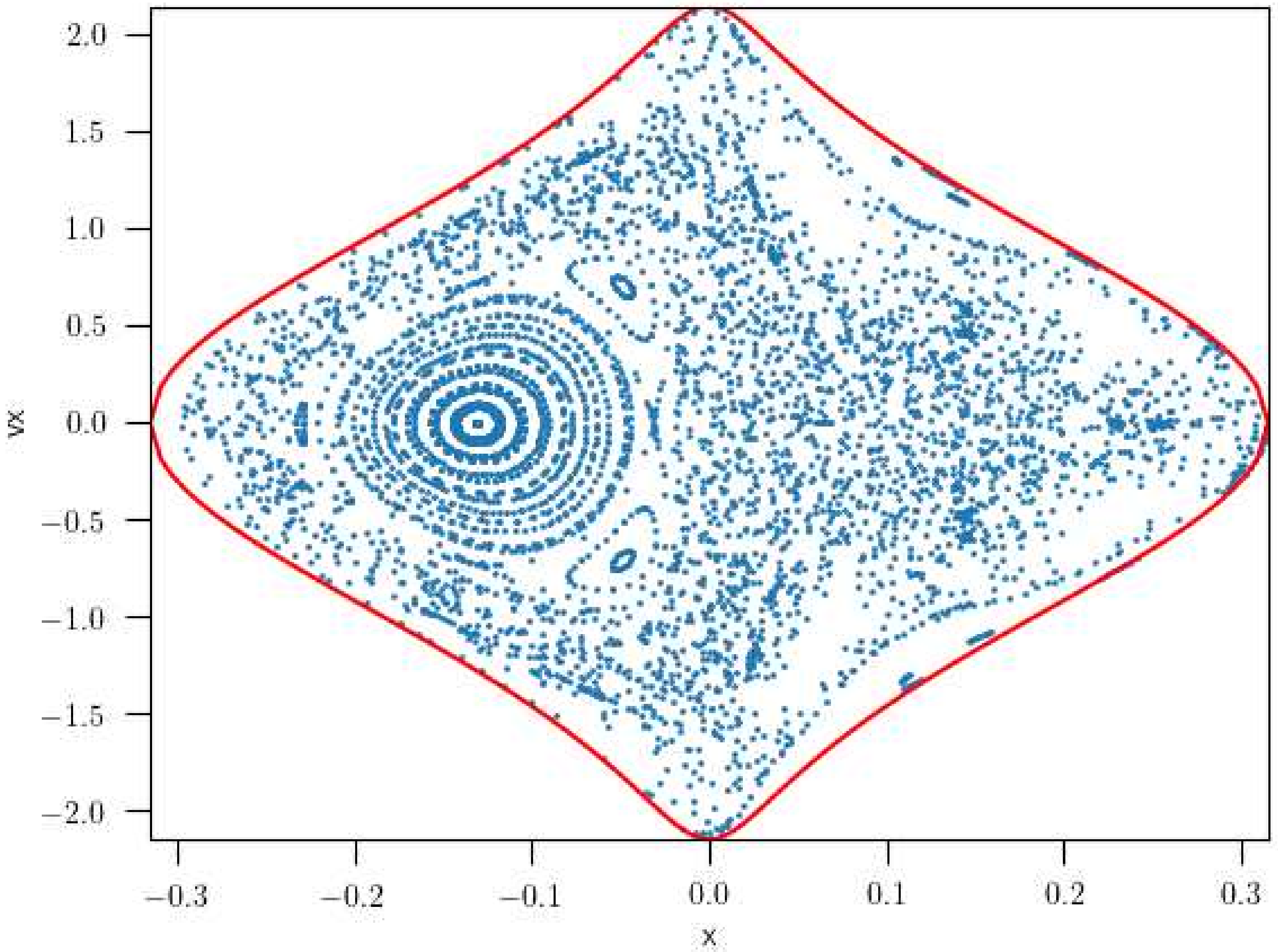




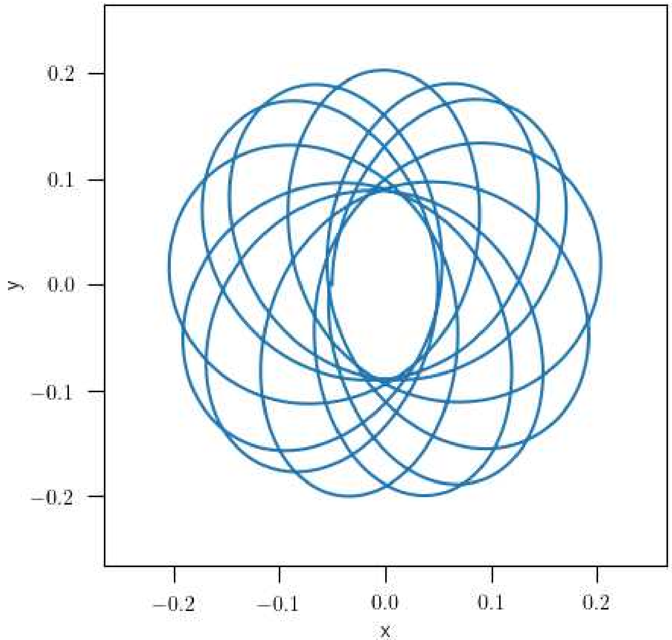

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x 0.315099
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x -0.1283
```

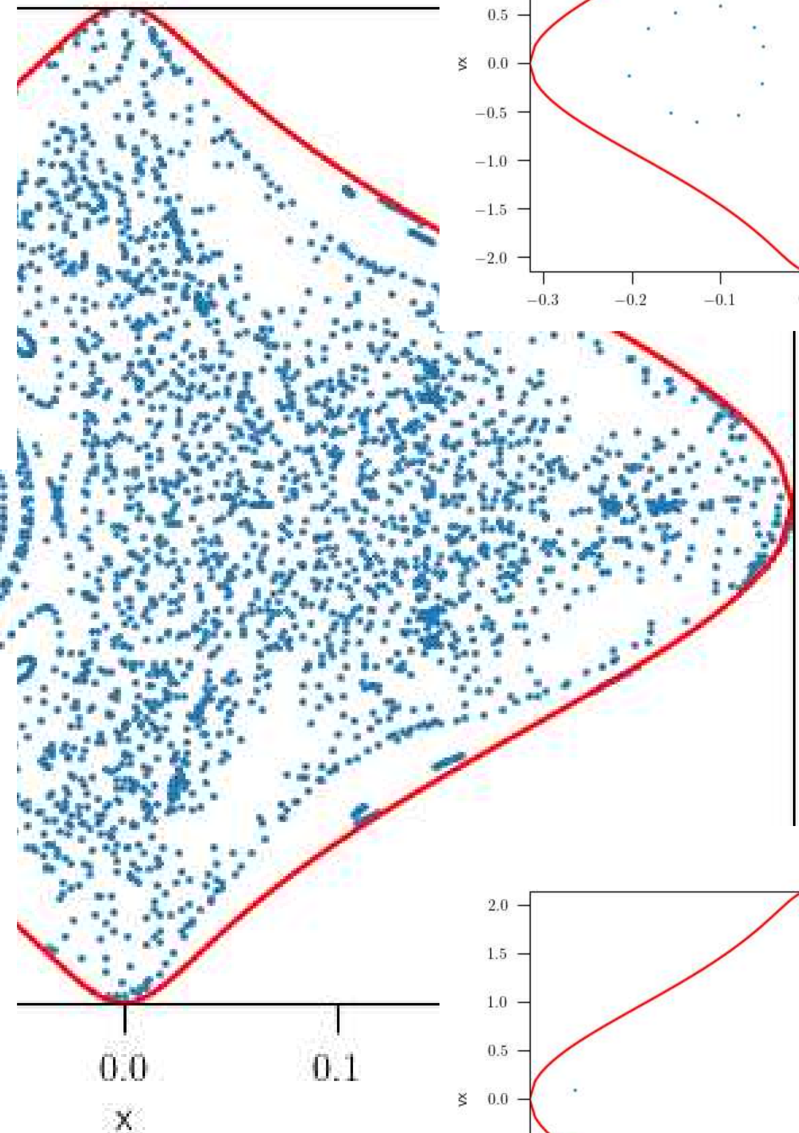
Chaotic orbits

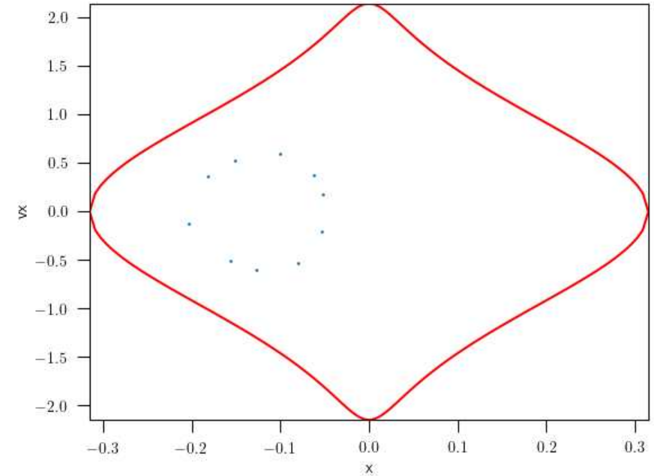
Chaotic orbits

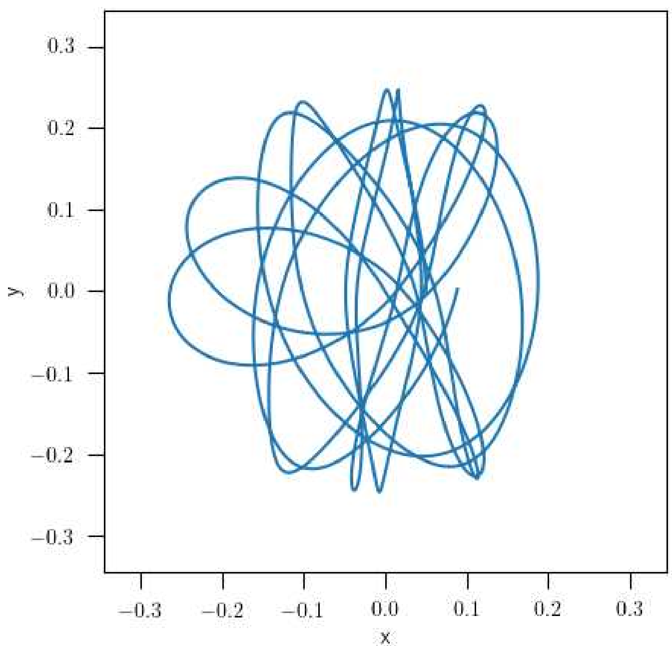


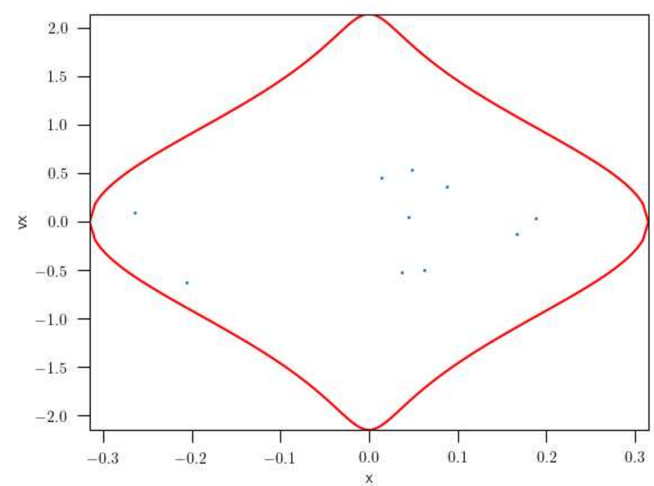
aotic orbits



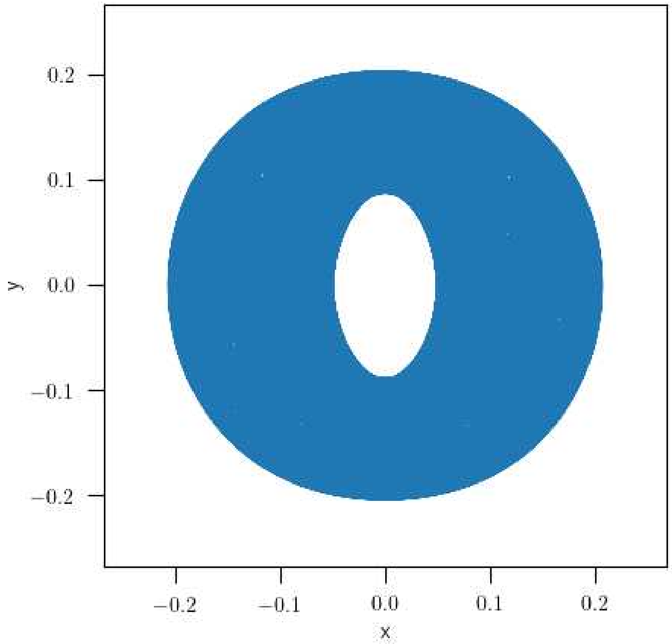


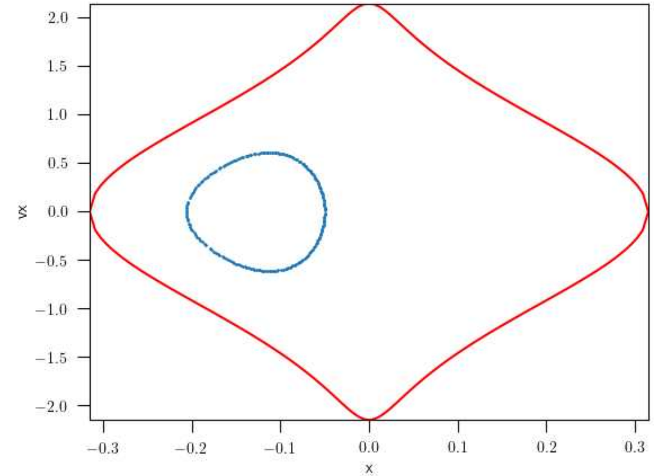


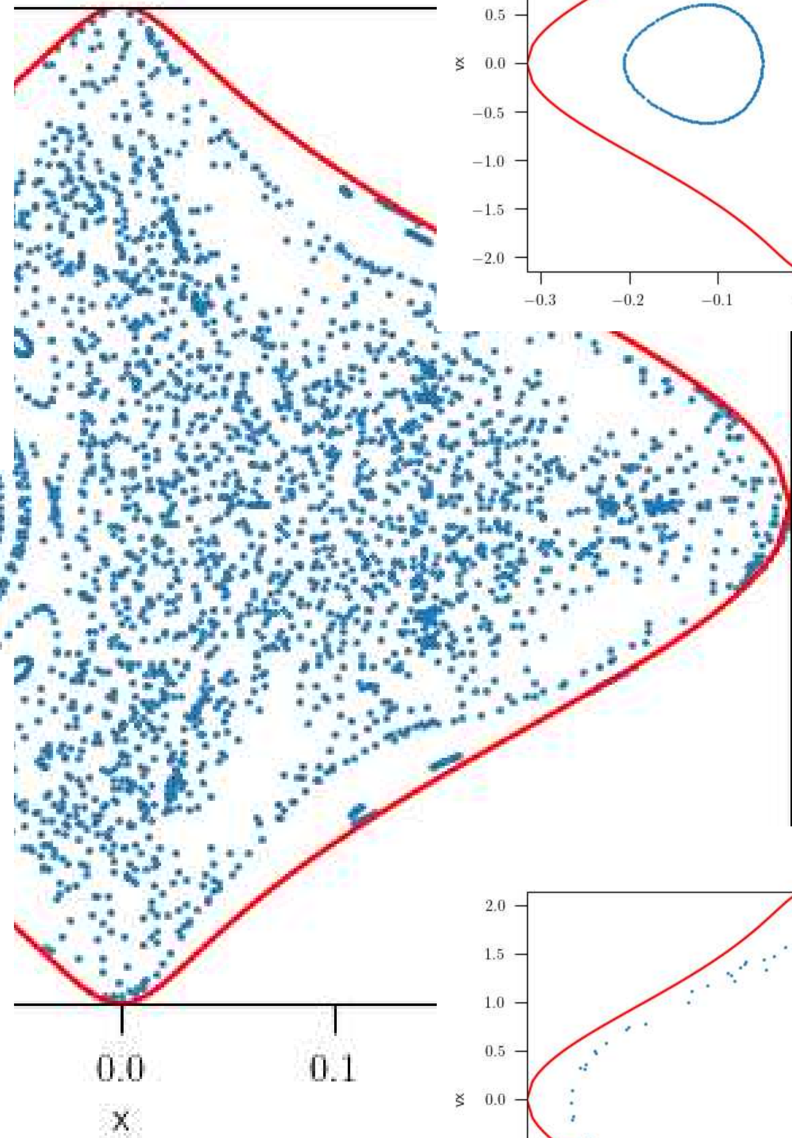


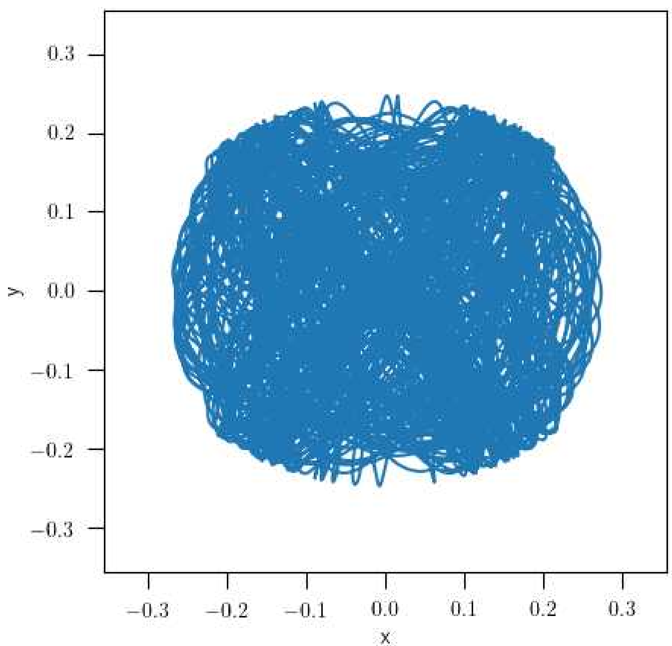


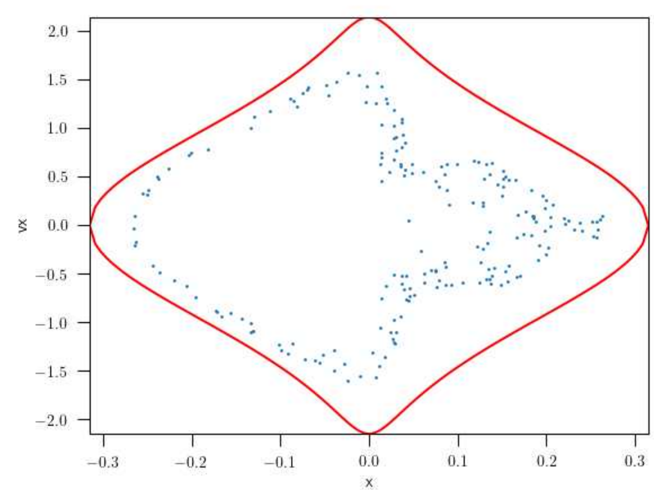
aotic orbits



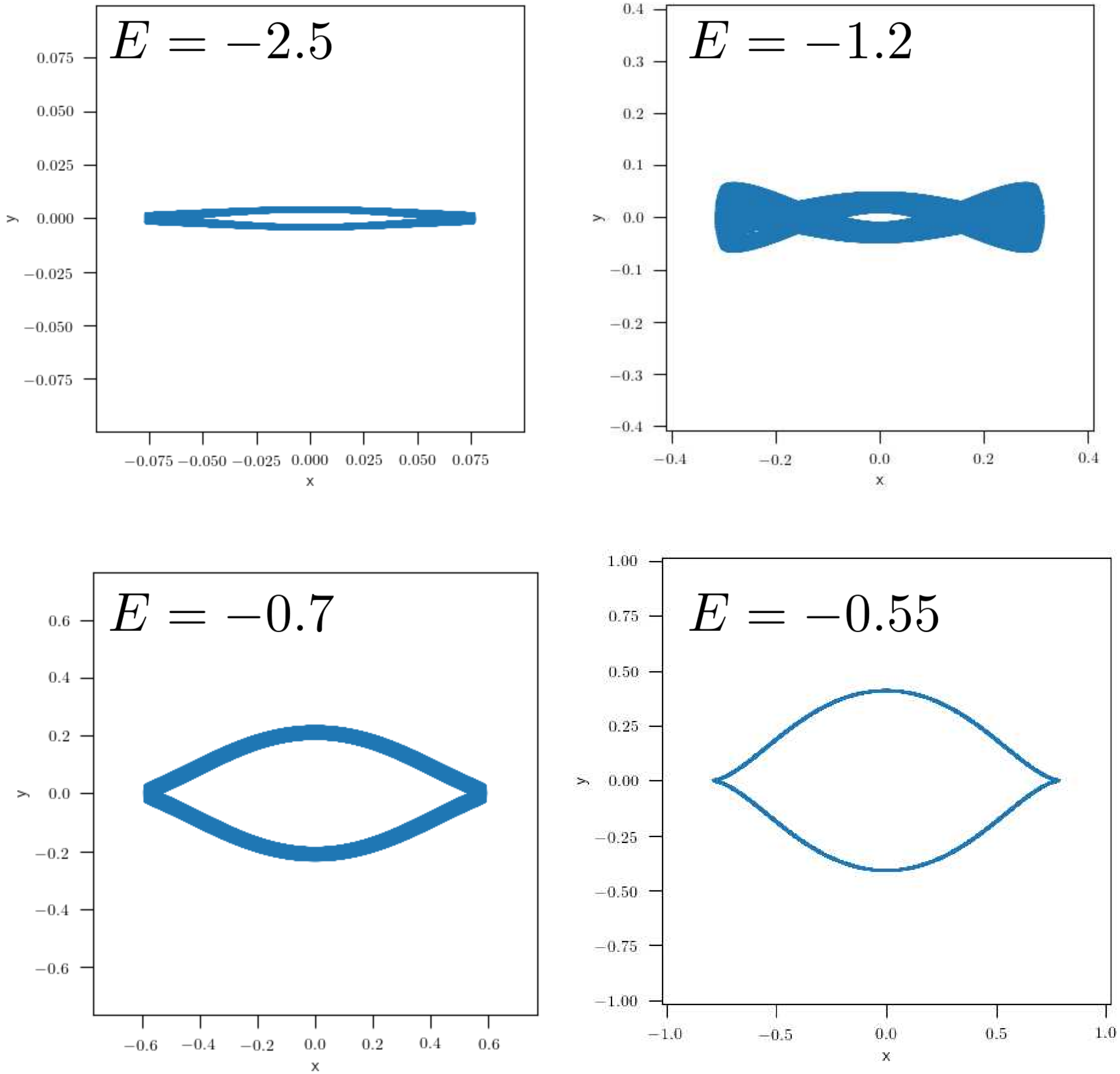








Evolution of the x1 orbit with increasing energy




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x 0.315099
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -0.7 --x 0.590356
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -0.55 --x 0.783882
```

The X-orbit families
(characteristics curves)

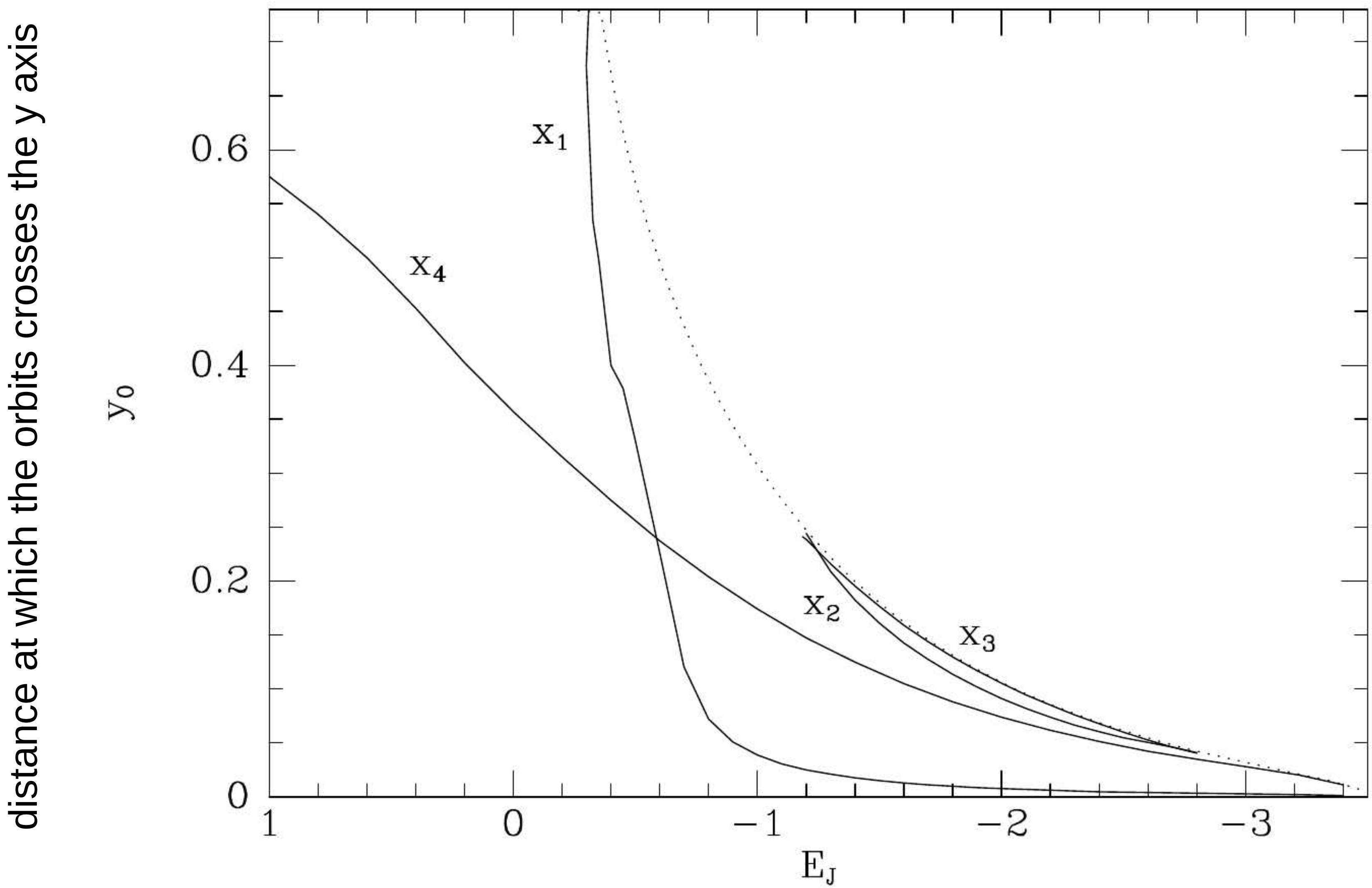


Figure 3.18 A plot of the Jacobi constant E_J of closed orbits in $\Phi_L(q = 0.8, R_c = 0.03, \Omega_b = 1)$ against the value of y at which the orbit cuts the potential's short axis. The dotted curve shows the relation $\Phi_{eff}(0, y) = E_J$. The families of orbits x_1–x_4 are marked.

Orbits
in weak rotating bars

Objective

- Split a loop orbit in two parts:

  - a circular motion of a guiding center

  - oscillations around the guiding center

Orbits in weak rotating bars (planar potentials)

- the bared potential rotates with a pattern speed Ω_b

Lagrangian :

$$\mathcal{L}(\vec{x},\dot{\vec{x}}) = \frac{1}{2}(\dot{\vec{x}} + \vec{\Omega}_b \times \vec{x})^2 - \phi(\vec{x})$$

In 2-D, with $\vec{\Omega}_b = \Omega_b \vec{e}_z$

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x} - y\Omega_b)^2 + \frac{1}{2}(\dot{y} + x\Omega_b)^2 - \phi(x,y)$$

In cylindrical coordinates

$$\boxed{\mathcal{L}(R, \varphi, \dot{R}, \dot{\varphi}) = \frac{1}{2}\dot{R}^2 + \frac{1}{2}(R(\dot{\varphi} + \Omega_b))^2 - \phi(R,\varphi)}$$

Equations of motion in cylindrical coordinates (Euler-Lagrange)

$$\boxed{\begin{cases} \ddot{R} = R\left(\dot{\varphi} + \Omega_b\right)^2 - \dfrac{\partial \phi}{\partial R} \\ \dfrac{d}{dt}\left(R^2\left(\dot{\varphi} + \Omega_b\right)\right) = - \dfrac{\partial \phi}{\partial \varphi} \end{cases}}$$

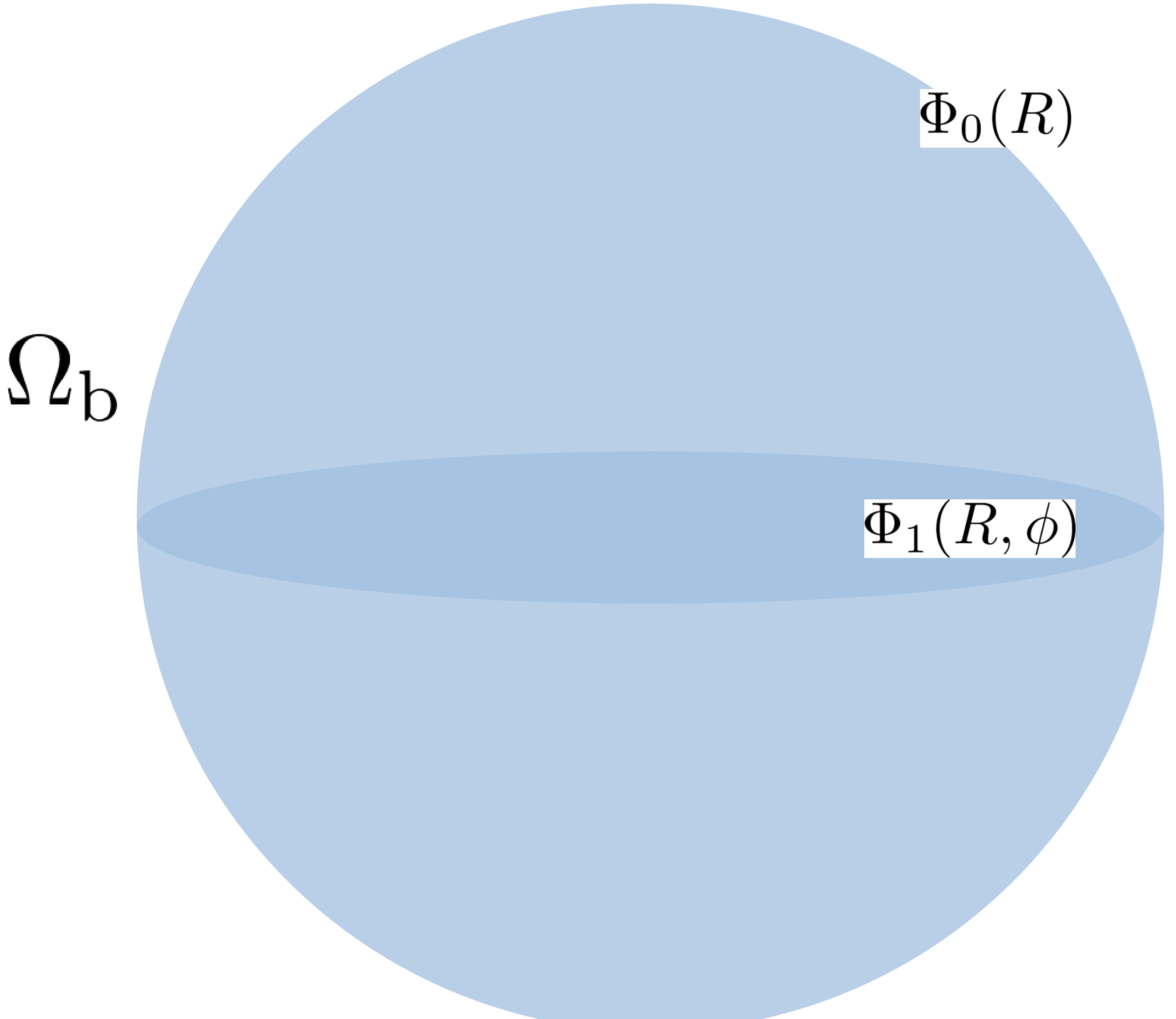
Assumptions

① A weak perturbation : $\phi(R,\varphi) = \underbrace{\phi_0(R)}_{\text{cyl. symmetry}} + \underbrace{\phi_1(R,\varphi)}_{\text{perturbation}}$ $\quad \dfrac{|\phi_1|}{|\phi_0|} \ll 1$

$$\boxed{\phi_1(R,\varphi) = \underbrace{\phi_b(R)}_{\text{radial dependency}} \underbrace{\cos(m\varphi)}_{\text{azimuthal dependency}}}$$

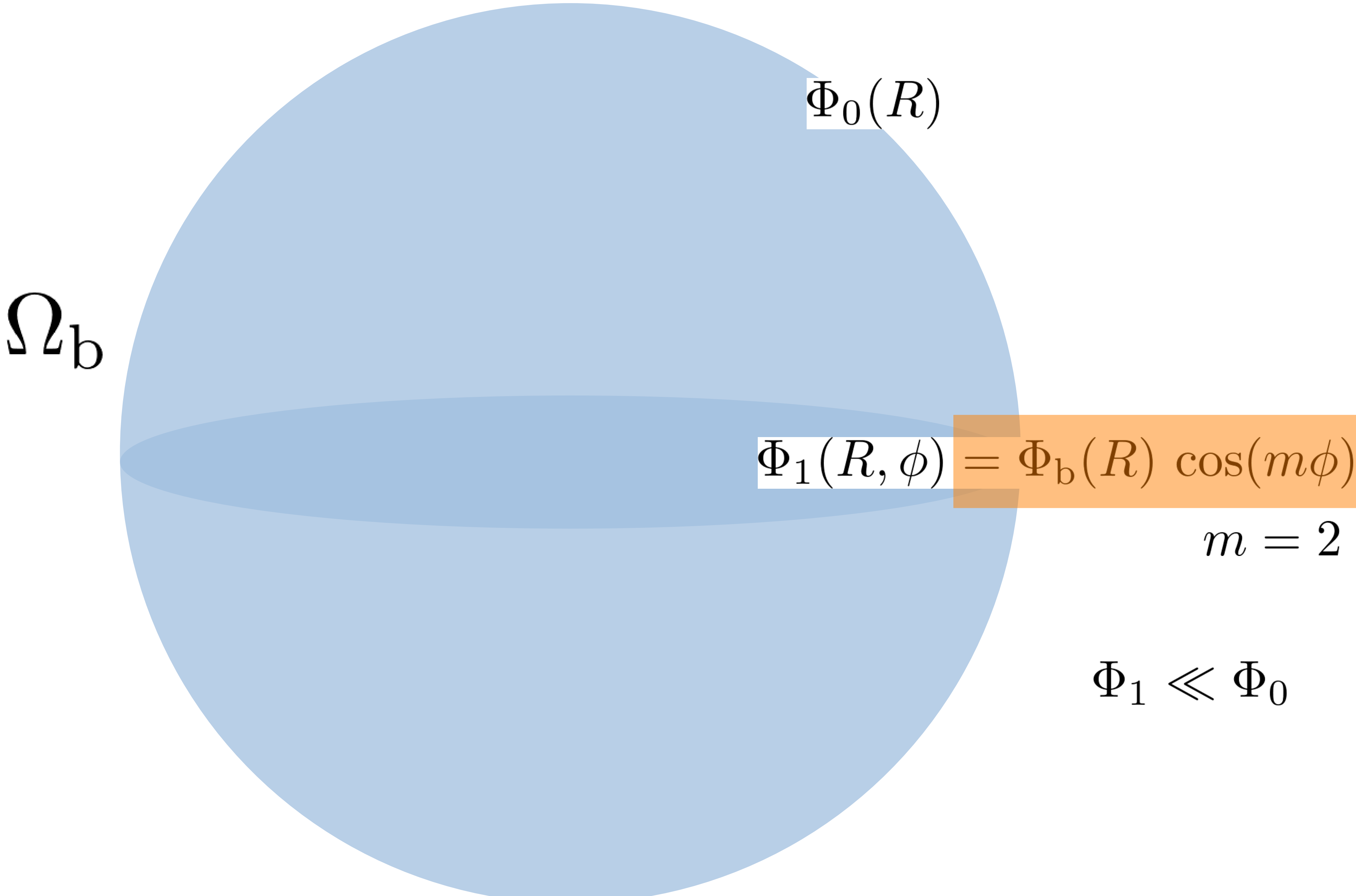
m : perturbation mode

The weakly-bared galaxy model



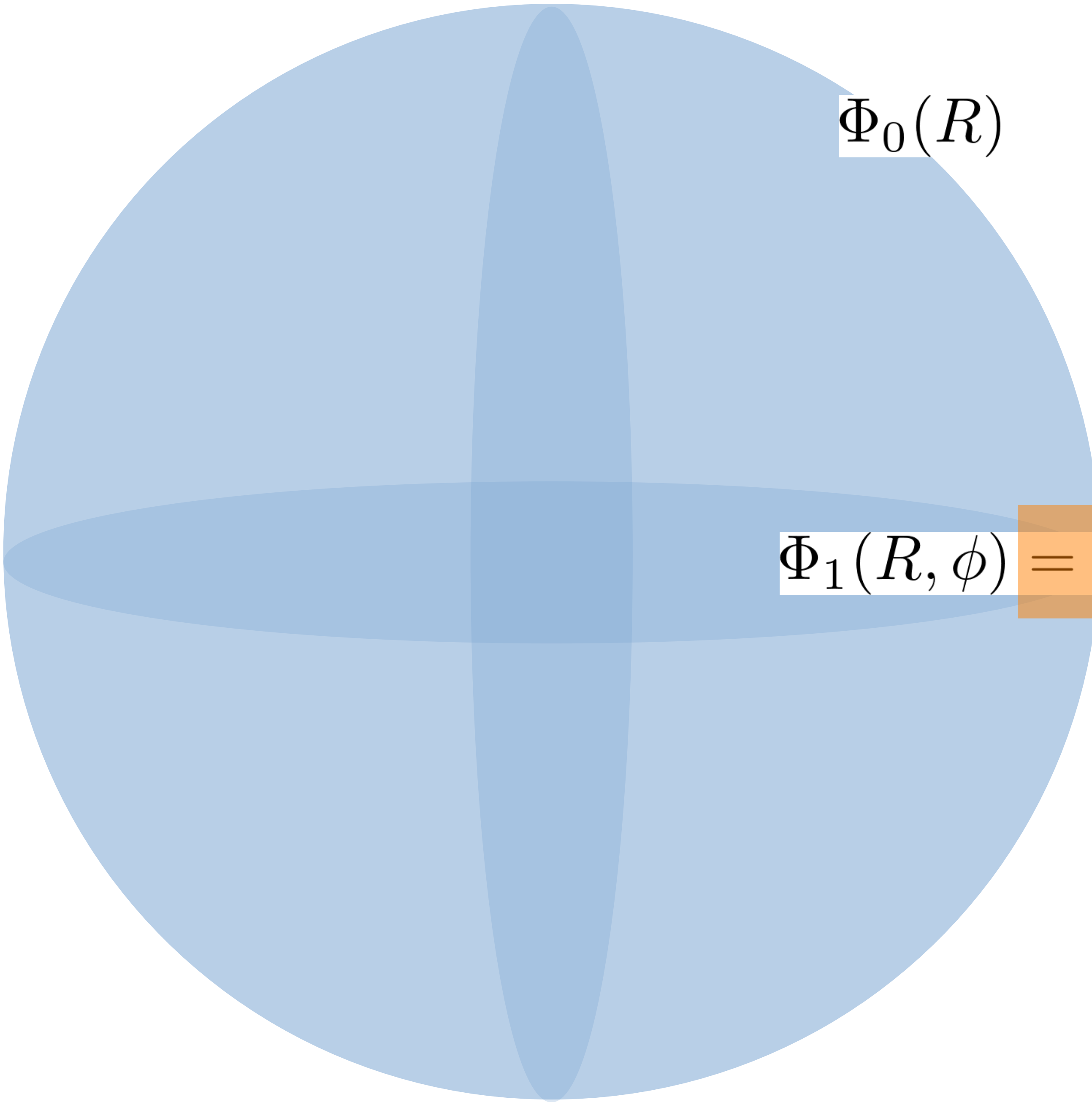
$\Phi_1 \ll \Phi_0$

The weakly-bared galaxy model



The weakly-bared galaxy model

Ω_{b}



$\Phi_0(R)$

$$\Phi_1(R,\phi) = \Phi_{\mathrm{b}}(R)\,\cos(m\phi)$$

$$m = 4$$

$$\Phi_1 \ll \Phi_0$$

Assumptions

② The motion may be decomposed into two parts

1) circular motion
2) perturbation

$$\boxed{\begin{cases} R(t) = R_0(t) + R_1(t) \\ \varphi(t) = \varphi_0(t) + \varphi_1(t) \end{cases}}$$

$R_1 \ll R_0$

$\varphi_1 \ll \varphi_0$

Note

$$\begin{cases} R_0(t) = R_0 & (R_0 = \text{radius of the guiding center}) \\ \varphi_0(t) = (\Omega_0 - \Omega_b)\, t & (\Omega_0 = \text{circular frequency}) \end{cases}$$

Solution of the EoM (2nd order terms)

Radial motion

$$\boxed{R_1(\varphi_0) = C_1 \cos\left(\frac{\varkappa_0 \varphi_0}{\Omega_0 - \Omega_b} + \alpha\right) - \left[\frac{d\Phi_b}{dR} + \frac{2\Omega\Phi_b}{R(\Omega - \Omega_b)}\right]_{R_0} \frac{\cos(m\varphi_0)}{\varkappa_0^2 - m^2(\Omega_0 - \Omega_b)^2}}$$

C_1, α : arbitrary constants

$\varkappa_0$: radial epicycle frequency

Azimuthal motion

$$\boxed{\varphi_1(t) = -2\Omega_0 \frac{R_1}{R_0} - \frac{\Phi_b(R_0)}{R_0^2(\Omega_0 - \Omega_b)} \cos\left(m(\Omega_0 - \Omega_b)t\right) + \text{cte}}$$

Orbits in weak rotating bars (planar potentials)

- the barred potential rotates with a pattern speed Ω_s

Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\dot{\vec{x}} + \vec{\Omega}_s \times \vec{x}\right)^2 - \phi(\vec{x})$$

2D, with $\vec{\Omega} = \Omega_s \vec{e}_z$

$$\mathcal{L}(x, \dot{x}, y, \dot{y}) = \frac{1}{2}\left(\dot{x} - y\,\Omega_s\right)^2 + \frac{1}{2}\left(\dot{y} + x\,\Omega_s\right)^2 - \phi(x)$$

In cylindrical coordinates

$$\boxed{L(R, \dot{R}, \varphi, \dot{\varphi}) = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\left[R(\dot{\varphi} + \Omega_s)\right]^2 - \phi(R, \varphi)}$$

Complement

Equations of motion

Euler - Lagrange $\quad \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{R}} = \dfrac{\partial \mathcal{L}}{\partial R}$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{R}} = \boxed{\ddot{R} = R(\dot{\varphi} + \Omega_s)^2 - \frac{\partial \phi}{\partial R}} = \frac{\partial \mathcal{L}}{\partial R}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \boxed{\frac{d}{dt}\Big(R^2(\underbrace{\dot{\varphi} + \Omega_s}_{})\Big) = -\frac{\partial \phi}{\partial \varphi}} = \frac{\partial \mathcal{L}}{\partial \varphi}$$

$\dot{\varphi}^I = \Omega$ angular speed in the inertial frame

• if ϕ is axisymmetric, L_z conservation

We assume a weak bar

$$\phi(R,\varphi) = \underbrace{\phi_0(R)}_{\text{cylindrical symmetry}} + \underbrace{\phi_1(R,\varphi)}_{\text{perturbation}} \quad \text{with } \left|\frac{\phi_1}{\phi_0}\right| \ll 1$$

Split the motion into two parts

$$\begin{cases} R(t) = R_0 + R_1(t) \\ \varphi(t) = \varphi_0(t) + \varphi_1(t) \end{cases}$$

R_0 : radius of the guiding center

Equations of motion at first order

$$\phi(R,\varphi) \simeq \phi_0(R_0) + \phi_1(R_0,\varphi) + \left.\frac{\partial \phi_0}{\partial R}\right)_{R_0}(R-R_0) + \left.\frac{\partial \phi_1}{\partial R}\right)_{R_0}(R-R_0) + \frac{1}{2}\frac{\partial^2\phi_0}{\partial R^2}(R-R_0)^2 + \frac{1}{2}\frac{\partial^2\phi_1}{\partial R^2}(R-R_0)^2$$

then
$$\begin{cases} \dfrac{\partial \phi}{\partial R} = \left.\dfrac{\partial \phi_0}{\partial R}\right)_{R_0} + \left.\dfrac{\partial^2 \phi_0}{\partial R^2}\right)_{R_0} R_1 + \left.\dfrac{\partial \phi_1}{\partial R}\right)_{R_0} + O(2) \\ \dfrac{\partial \phi}{\partial \varphi} = \left.\dfrac{\partial \phi_1}{\partial \varphi}\right)_{R_0} + O(2) \qquad \left(\text{as } \dfrac{\partial \phi_0}{\partial \varphi} = 0\right) \end{cases}$$

+ keep only the first order term

Zero order terms

① Radial equation

$$\boxed{\ddot{r} = r\,(\dot{\varphi} + \Omega_s)^2 - \frac{\partial \phi}{\partial r}} \longrightarrow \boxed{r_0\,(\dot{\varphi}_0 + \Omega_s)^2 = \left.\frac{\partial \phi_0}{\partial r}\right)_{R_0}}$$

$$\dot{\varphi}_0^{I\,2} = \frac{1}{R_0}\left.\frac{\partial \phi_0}{\partial r}\right)_{R_0} = \Omega_0^2$$

② Azimuthal equation

$$\boxed{\frac{d}{dt}\left(r^2(\dot{\varphi} + \Omega_s)\right) = -\frac{\partial \phi}{\partial \varphi}} \longrightarrow \frac{d}{dt}\left((\dot{\varphi}_0 + \Omega_s)(r_0^2 + 2r_0)\right) = 0$$

$$(r_0^2 + 2r_0)\,\frac{d}{dt}(\dot{\varphi}_0 + \Omega_s) = 0$$

$$\Rightarrow \Omega_0 = \dot{\varphi}_0^{I} = cte \qquad \boxed{\dot{\varphi}_0 = cte}$$

Interpretation

$$\boxed{R_0 (\dot{\varphi}_0 + \Omega_b)^2 = \left.\frac{\partial \Phi_0}{\partial R}\right)_e}$$ $$\boxed{\dot{\varphi}_0 = \text{cte}}$$

$\Omega(R)$: $\Omega^2(R) = \frac{1}{R}\frac{\partial \Phi_0}{\partial R}$ circular frequency in absense of perturbation

Thus : $\dot{\varphi}_0 + \Omega_b = \Omega(R_0) = \Omega_0$

$$\boxed{\dot{\varphi}_0 = \Omega_0 - \Omega_b} \quad (= \text{cte})$$

$$\boxed{\varphi_0(t) = (\Omega_0 - \Omega_b)\, t}$$ angular frequency in the rotating rest frame

Note

- Lagrange points are stationary points
  $\dot{\varphi}_0 = 0$ $\Omega_0 = \Omega_b$ (corotation)
- Elsewhere

  if $\Omega_0 > \Omega_b$ $\rightarrow$ $\cdot$ prograde orbits

  if $\Omega_0 < \Omega_b$ $\rightarrow$ $\cdot$ retrograde orbits

First order terms

① Radial equation

$$\ddot{r} = r(\dot{\varphi} + \Omega_s)^2 - \frac{\partial \phi}{\partial r} \longrightarrow$$

$$\ddot{r}_1 + r_1\left(\frac{\partial^2 \phi_0}{\partial r^2} - \Omega^2\right)_{R_0} - 2\, R_0\, \dot{\varphi}_1\, \Omega_0 = -\left.\frac{\partial \phi_1}{\partial R}\right)_{R_0}$$

② Azimuthal equation

$$\frac{d}{dt}\left(r^2(\dot{\varphi} + \Omega_s)\right) = -\frac{\partial \phi}{\partial \varphi} \longrightarrow$$

$$\ddot{\varphi}_1 + 2\,\Omega_0\,\frac{\dot{R}_1}{R_0} = -\frac{1}{R_0^2}\left(\frac{\partial \phi_1}{\partial \varphi}\right)_{R_0}$$

First order terms

① Radial equation

$$\ddot{r} = R(\dot{\varphi} + \Omega_s)^2 - \frac{\partial \phi}{\partial R}$$

$$\ddot{r}_1 + r_1\left(\frac{\partial^2 \phi_0}{\partial R^2} - \Omega^2\right)_{R_0} - 2\,R_0\,\dot{\varphi}_1\,\Omega_0 = -\left(\frac{\partial \phi_1}{\partial R}\right)_{R_0}$$

② Azimuthal equation

$$\frac{d}{dt}\left(R^2(\dot{\varphi} + \Omega_s)\right) = -\frac{\partial \phi}{\partial \varphi}$$

$$\ddot{\varphi}_1 + 2\,\Omega_0\,\frac{\dot{R}_1}{R_0} = -\frac{1}{R_0^2}\left(\frac{\partial \phi_1}{\partial \varphi}\right)_{R_0}$$

We restrict to simple perturbations of the type

$$\boxed{\phi_1(R,\varphi) = \underbrace{\phi_b(R)}_{\substack{\text{radial}\\ \text{dependency}}} \underbrace{\cos(m\varphi)}_{\substack{\text{azimuthal}\\ \text{dependency}}}}$$

- $m = 2 \Rightarrow$ a bar
- note: any perturbation can be obtained by summing over m

<u>We get</u> assuming $\boxed{\varphi_1 \ll \varphi_0}$

$$\frac{\partial \phi_1}{\partial R} = \frac{\partial \phi_b}{\partial R} \cos(m\varphi) \simeq \frac{\partial \phi_b}{\partial R} \cos(m\varphi_0) = \frac{\partial \phi_b}{\partial R} \cos(m(\Omega_0 - \Omega_b)t)$$

$\varphi_1 \ll \varphi_0$

$$\frac{\partial \phi_1}{\partial \varphi} = -\phi_b(R) \sin(m\varphi)\, m \simeq -\phi_b(R)\, m \sin(m(\Omega_0 - \Omega_b)t)$$

$$\begin{cases} \ddot{R}_1 + R_1\left(\dfrac{\partial^2\phi_0}{\partial R^2} - \Omega^2\right)_{R_0} - 2R_0\dot{\varphi}_1\Omega_0 = \left.\dfrac{\partial\phi_b}{\partial R}\right)_{R_0}\cos(m(\Omega_0-\Omega_b)t) \\ \ddot{\varphi}_1 + 2\Omega_0\dfrac{\dot{R}_1}{R_0} = \dfrac{m\,\phi_b(R_0)}{R_0^2}\sin(m(\Omega_0-\Omega_b)t)) \end{cases}$$

We can integrate $\ddot{\varphi}_1$

$$\boxed{\dot{\varphi}_1 = -2\Omega_0\frac{R_1}{R_0} - \frac{\phi_b(R_0)}{R_0^2(\Omega_0-\Omega_b)}\cos(m(\Omega_0-\Omega_b)t)) + cte}$$

Introducing $\dot{\varphi}_1$ in the "$\ddot{R}_1$" equation

$$\boxed{\ddot{R}_1 + \varkappa_0^2 R_1 = -\frac{d\phi_b}{dR} + \left[\frac{2\Omega\phi_b}{R(\Omega-\Omega_b)}\right]_{R_0}\cos(m(\Omega_0-\Omega_b)t)) + cte}$$

with:

$$\varkappa_0^2 = \left(\frac{d^2\phi_0}{\partial R^2} + 3\Omega^2\right)_{R_0} = \left(R\frac{d\Omega^2}{dR} + 4\Omega^2\right)_{R_0}$$

the radial epicycle frequency

Complement

General solution (harmonic oscillator of frequ. $\varkappa_0$ driven at frequ. $m(\Omega_0-\Omega_s)$)

$$\boxed{R_n(t) = C_n \cos(\varkappa_0 t + \alpha) - \left[\frac{d\Phi_s}{dR} + \frac{2\Omega\Phi_b}{R(\Omega-\Omega_b)}\right]_{R_0} \frac{\cos(m(\Omega_0-\Omega_s)t)}{\varkappa_0^2 - m^2(\Omega_0-\Omega_s)^2}}$$

using $\varphi_0(t) = (\Omega_0 - \Omega_b)t$

$$\boxed{R_n(\varphi_0) = C_n \cos\left(\frac{\varkappa_0\varphi_0}{\Omega_0-\Omega_s} + \alpha\right) - \left[\frac{d\Phi_s}{dR} + \frac{2\Omega\Phi_b}{R(\Omega-\Omega_b)}\right]_{R_0} \frac{\cos(m\varphi_0)}{\varkappa_0^2 - m^2(\Omega_0-\Omega_s)^2}}$$

C_n , α arbitrary constants

Discussion

$$\boxed{R_1(\varphi_0) = C_1 \cos\left(\frac{\kappa_0 \varphi_0}{\Omega_0 - \Omega_b} + \alpha\right) - \left[\frac{d\Phi_b}{dR} + \frac{2\Omega\Phi_b}{R(\Omega-\Omega_b)}\right]_{R_0} \frac{\cos(m\varphi_0)}{\kappa_0^2 - m^2(\Omega_0-\Omega_b)^2}}$$

① if $\Phi_b(R) = 0$ (no perturbation) $\quad\underbrace{\text{Epicycles motions}}$

$$\boxed{R_1(t) = C_1 \cos(\kappa_0 t + \alpha)} \equiv x(t) \quad \text{radial oscillations}$$

$$\boxed{\dot{\varphi}_1(t) = -2\Omega_0 \frac{R_1(t)}{R_0}} \Rightarrow y(t) \quad \text{oscillations along the orbit}$$

② if $C_1 = 0 \qquad \Phi_b \neq 0$

$$\boxed{R_1(\varphi_0) = -\underbrace{\left[\frac{d\Phi_b}{dR} + \frac{2\Omega\Phi_b}{R(\Omega-\Omega_b)}\right]_{R_0}}_{\text{cte}} \underbrace{\frac{\cos(m\varphi_0)}{\kappa_0^2 - m^2(\Omega_0-\Omega_b)^2}}_{\text{periodic in } \varphi_0 \ \left(\frac{2\pi}{m}\right)}}$$

$\boxed{\Rightarrow \text{closed orbit}}$

③ if $C_1 \neq 0$ oscillations around the closed orbit (same family)

$\boxed{\text{The orbit is not necessary closed}}$

Resonances

! two problematic terms $\frac{1}{\Omega_0 - \Omega_b}$ and $\frac{1}{\varkappa_0^2 - m^2(\Omega_0 - \Omega_b)^2}$

$\Rightarrow$ R_1 may diverge !

1) $\boxed{\Omega_0 = \Omega_b}$ Corotation

we are at a radius where the circular frequency is similar to the pattern speed of the bar

as $\dot{\varphi}_0 = \Omega_0 - \Omega_b \Rightarrow \underline{\dot{\varphi}_0 = 0}$ → static in the rotating frame

2) $\boxed{\underline{m(\Omega_0 - \Omega_b)} = \pm \varkappa_0}$ Lindblad resonances

frequ. at which the star encounter the potential minimum

→ the frequency at wich a star encounter a potential minimum is similar to its radial frequency

$\Rightarrow$ exitation

$\equiv \Omega_b = \Omega \pm \frac{\varkappa}{2}$

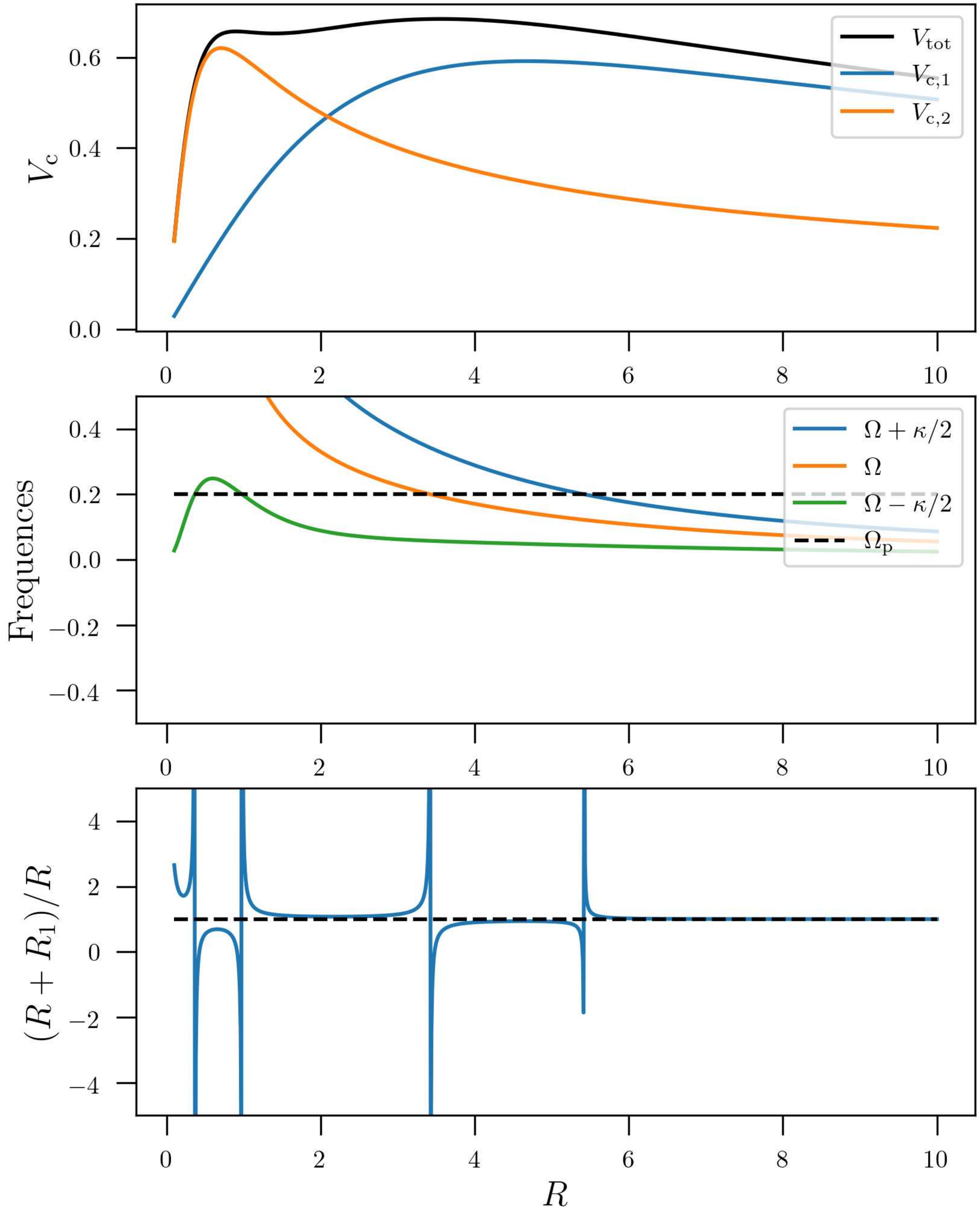
A circular orbit has two natural frequencies

① $\varkappa$: radial frequ.

② 0 : azimuthal frequ.

(no change $\Rightarrow$ frequ $= 0$)

Resonances occur when the forcing frequency $m(\Omega_0 - \Omega_b)$ is equal to one of these frequencies.



Disk : Miyamoto-Nagai
Bulge : Plummer

Inner Lindblad resonnances
(ILR1, ILR2)

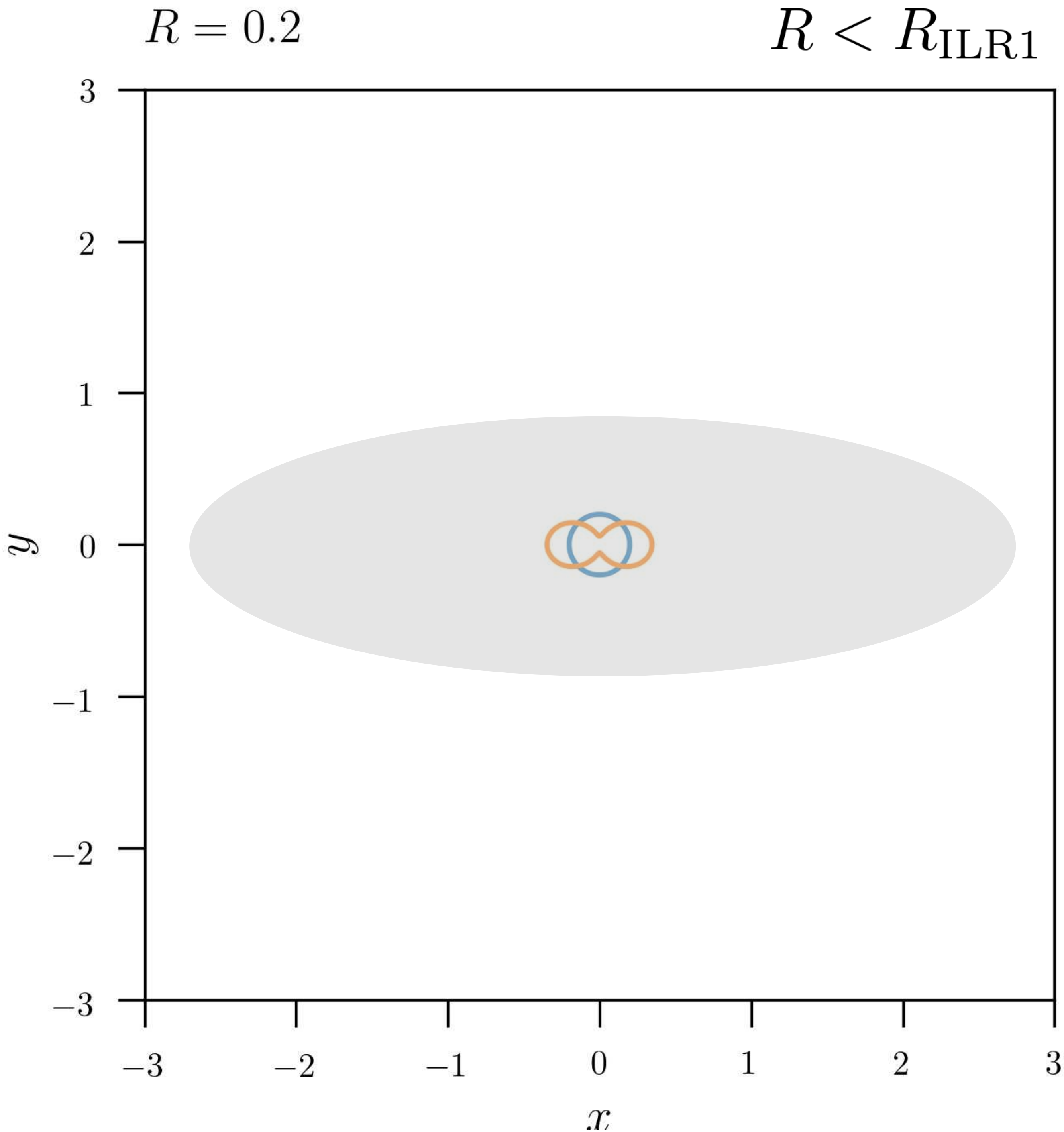
$$\Omega_{\mathrm{b}} = \Omega - \kappa/2$$

Outer Lindblad resonnance
(OLR)

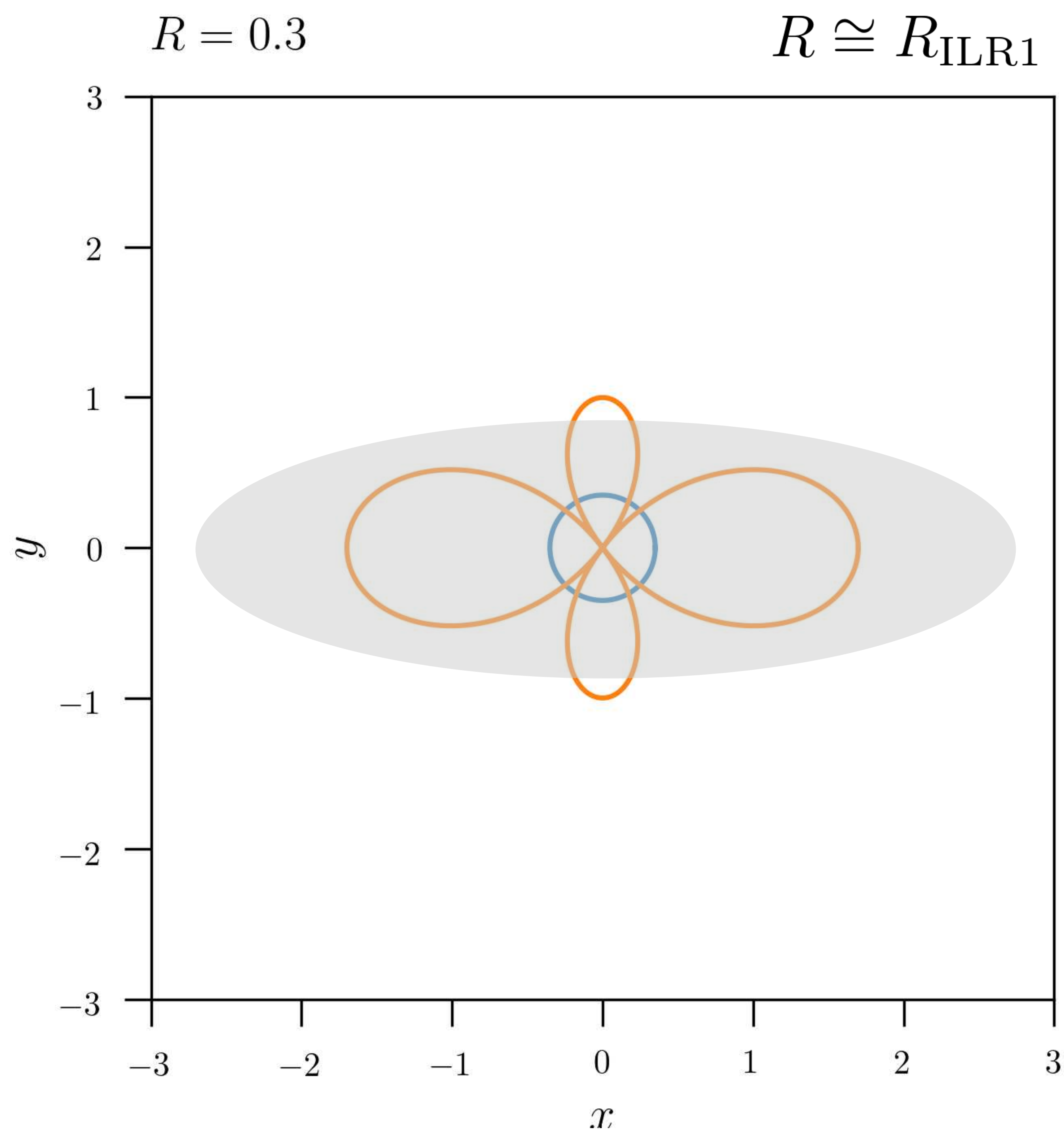
$$\Omega_{\mathrm{b}} = \Omega + \kappa/2$$

Corotation (CR)

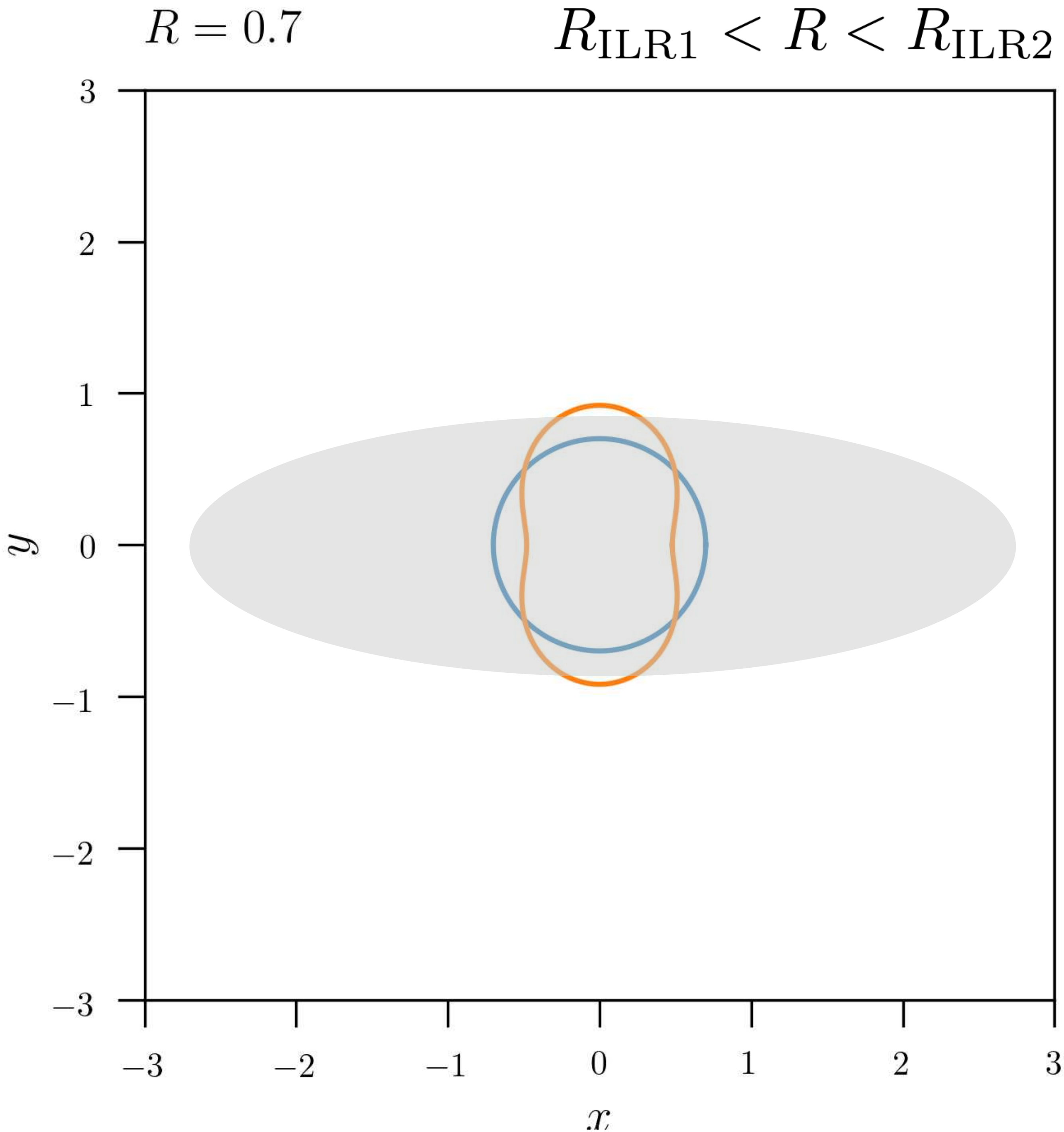
$$\Omega_{\mathrm{b}} = \Omega$$



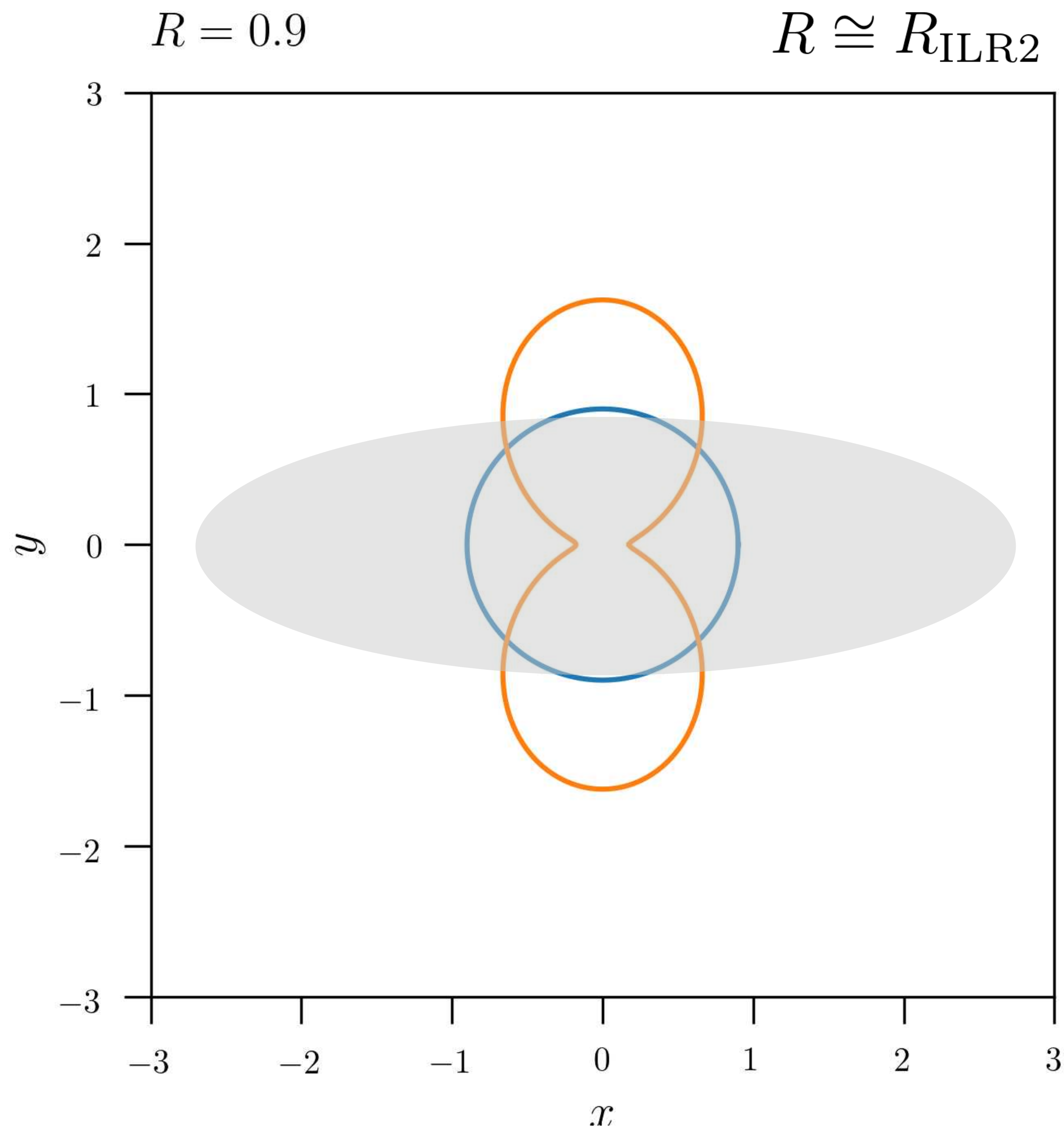
$R = 0.2$
$R < R_{\rm ILR1}$
3
2
1
0
−1
−2
−3
y
−3
−2
−1
0
1
2
3
x

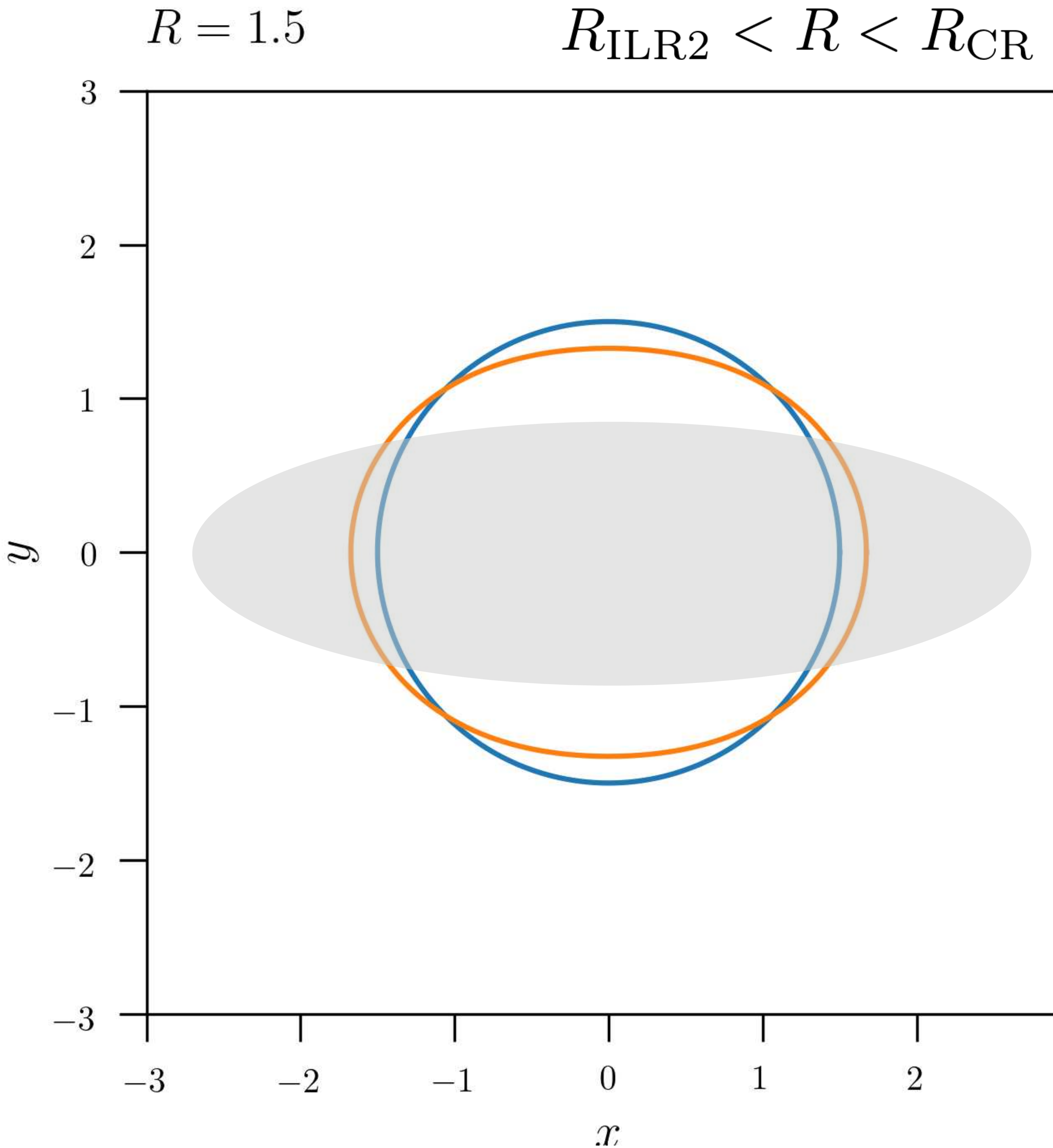


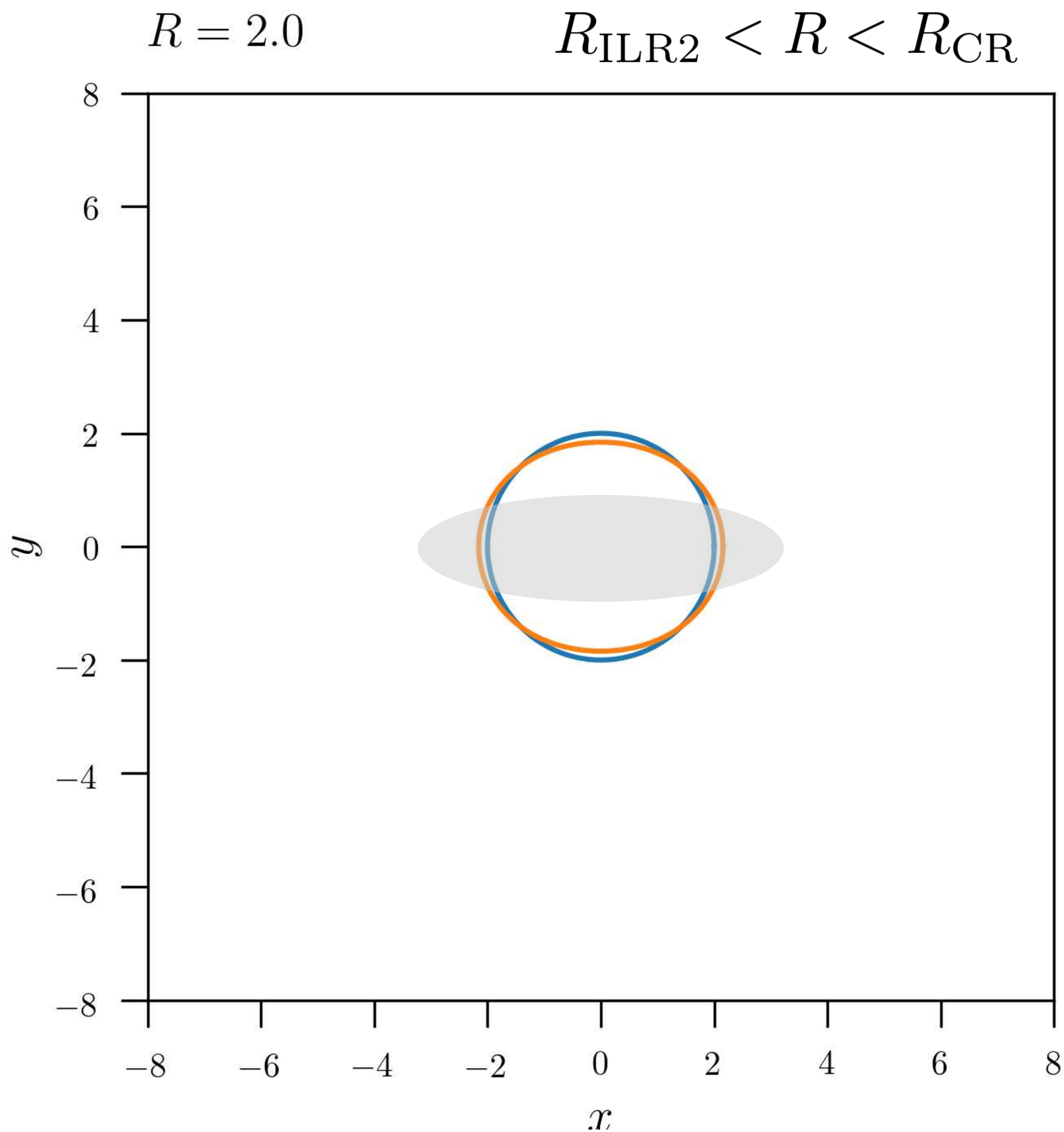
$R = 0.3$
$R \cong R_{\mathrm{ILR1}}$
3
2
1
0
−1
−2
−3
y
−3
−2
−1
0
1
2
3
x

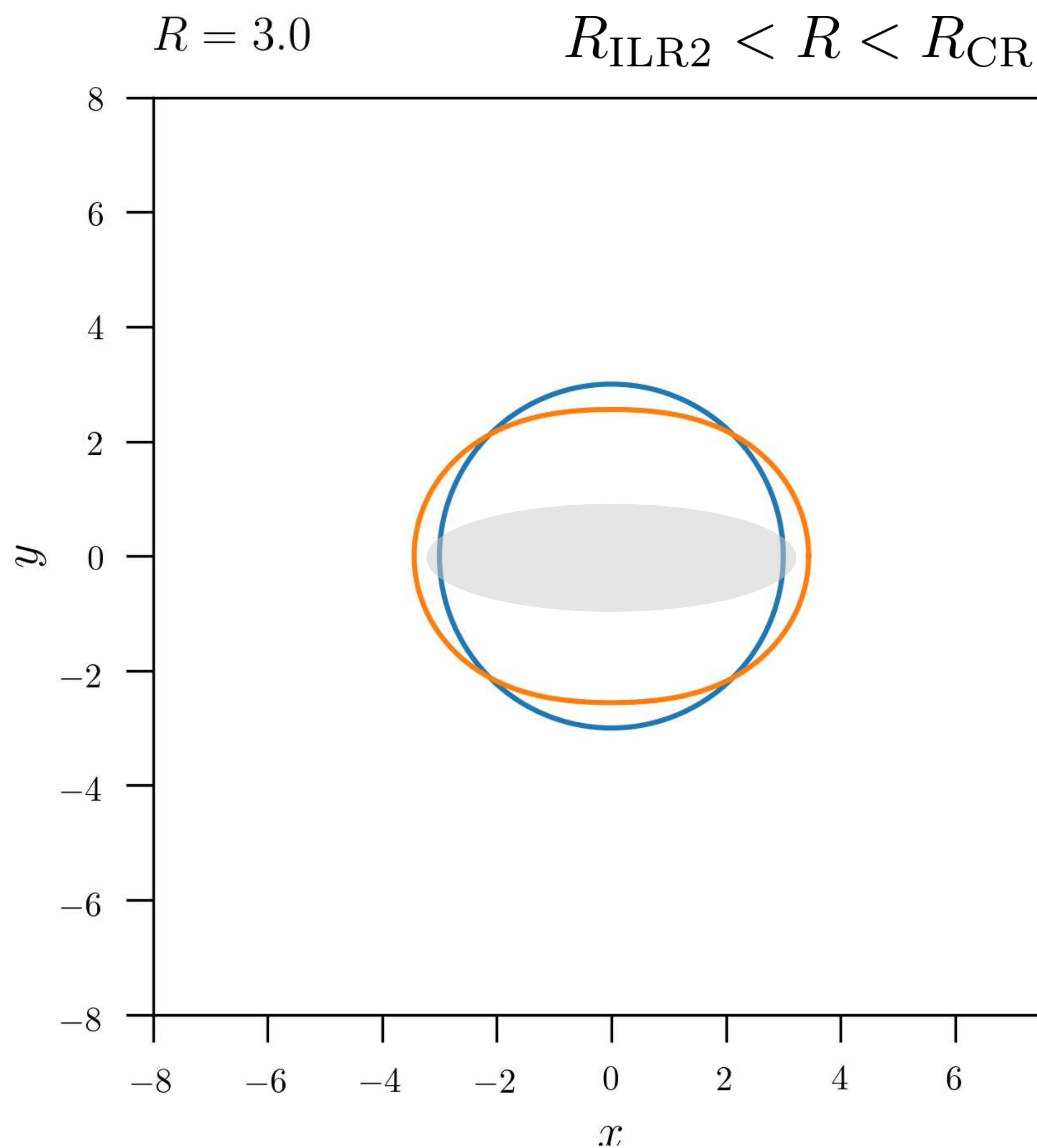


$R = 0.7$
$R_{\mathrm{ILR1}} < R < R_{\mathrm{ILR2}}$
y
x
3
2
1
0
−1
−2
−3


$R = 0.9$
$R \cong R_{\mathrm{ILR2}}$
3
2
1
0
−1
−2
−3
y
−3
−2
−1
0
1
2
3
x


$R = 1.5$
$R_{\mathrm{ILR2}} < R < R_{\mathrm{CR}}$
3
2
1
0
−1
−2
−3
y
−3
−2
−1
0
1
2
3
x

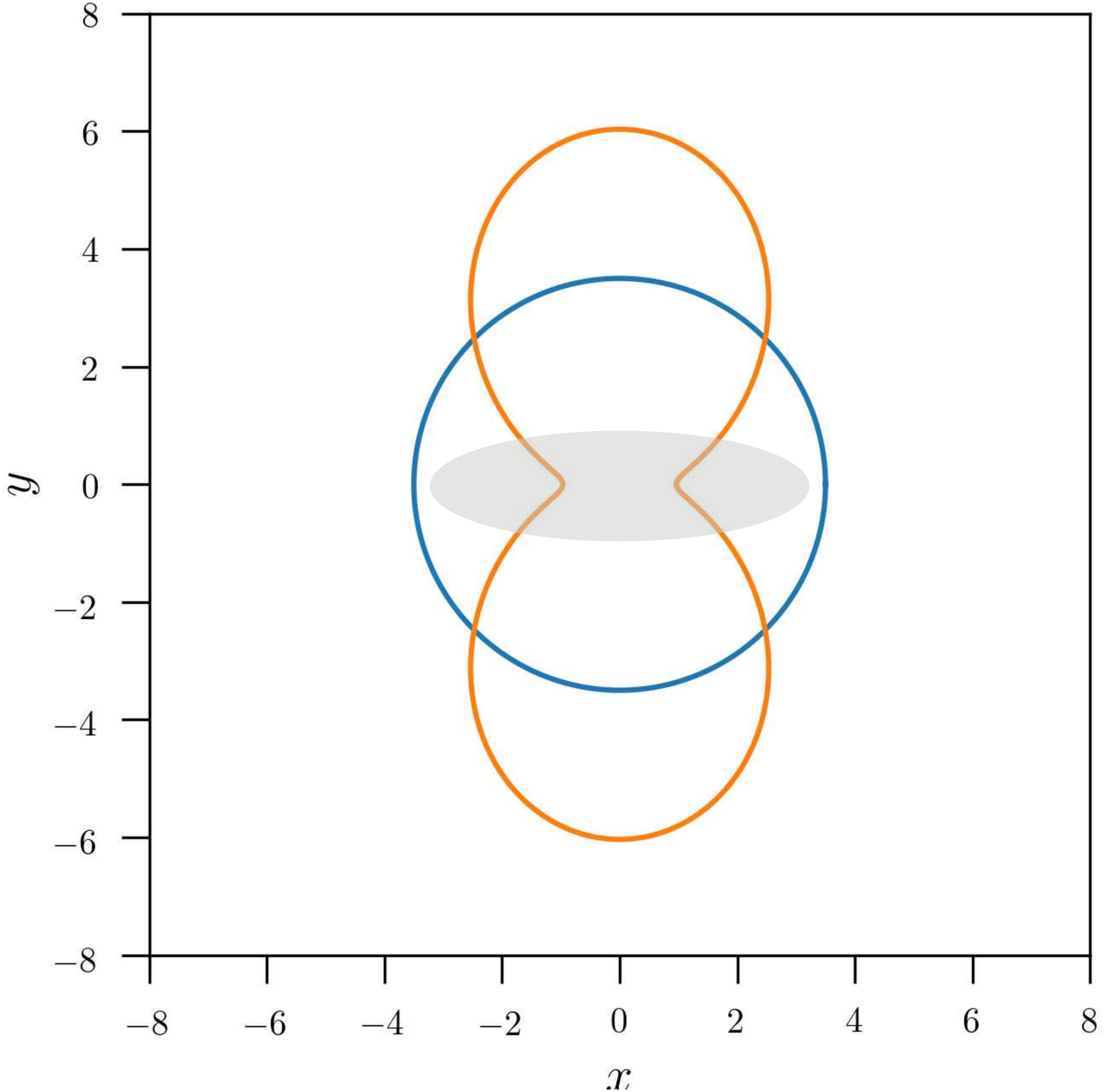

$R = 2.0$
$R_{\mathrm{ILR2}} < R < R_{\mathrm{CR}}$
y
x



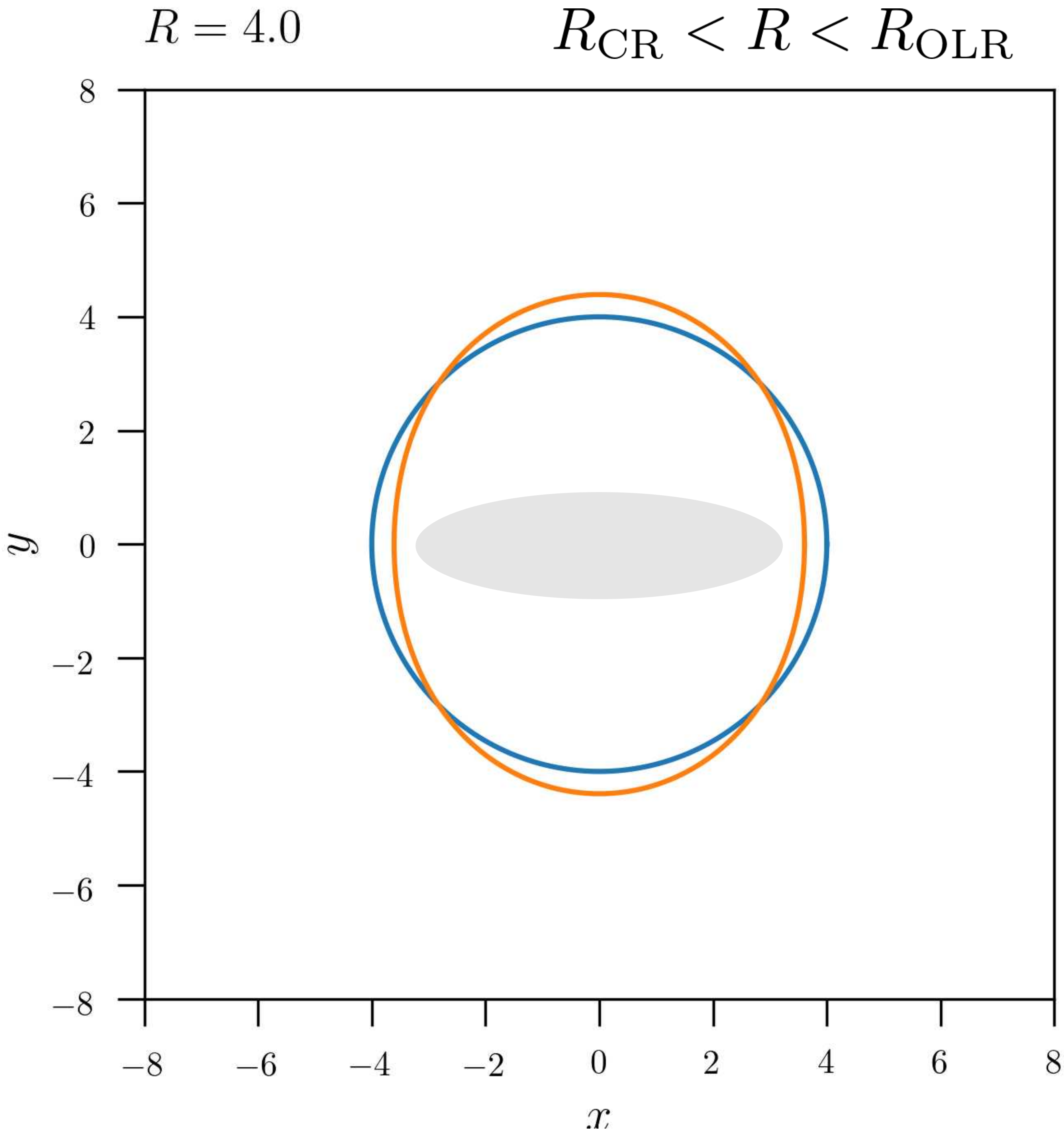
$R = 3.0$
$R_{\mathrm{ILR2}} < R < R_{\mathrm{CR}}$
8
6
4
2
0
−2
−4
−6
−8
y
−8
−6
−4
−2
0
2
4
6
8
x

$R = 3.5$

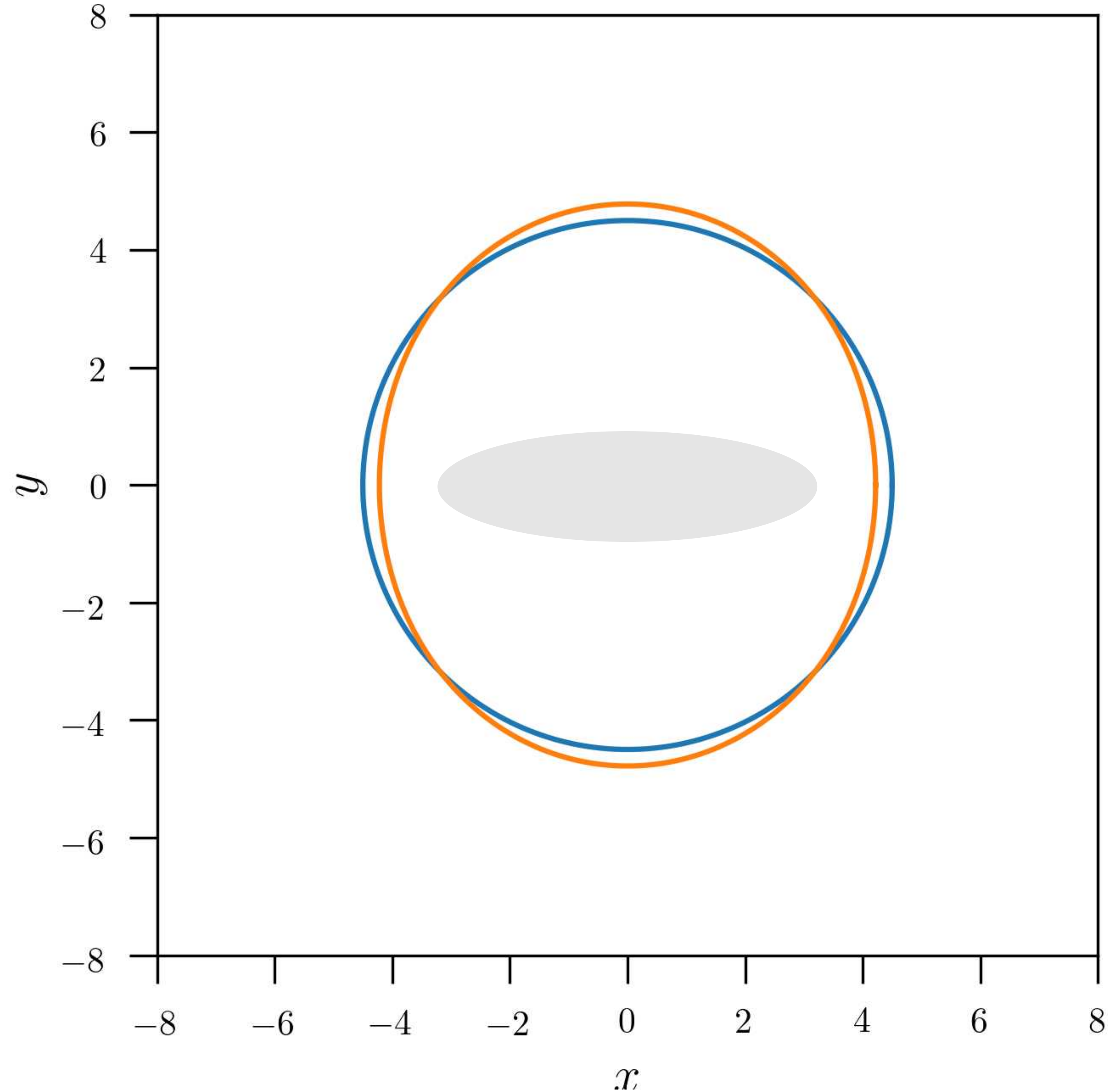
$R \cong R_{\rm CR}$

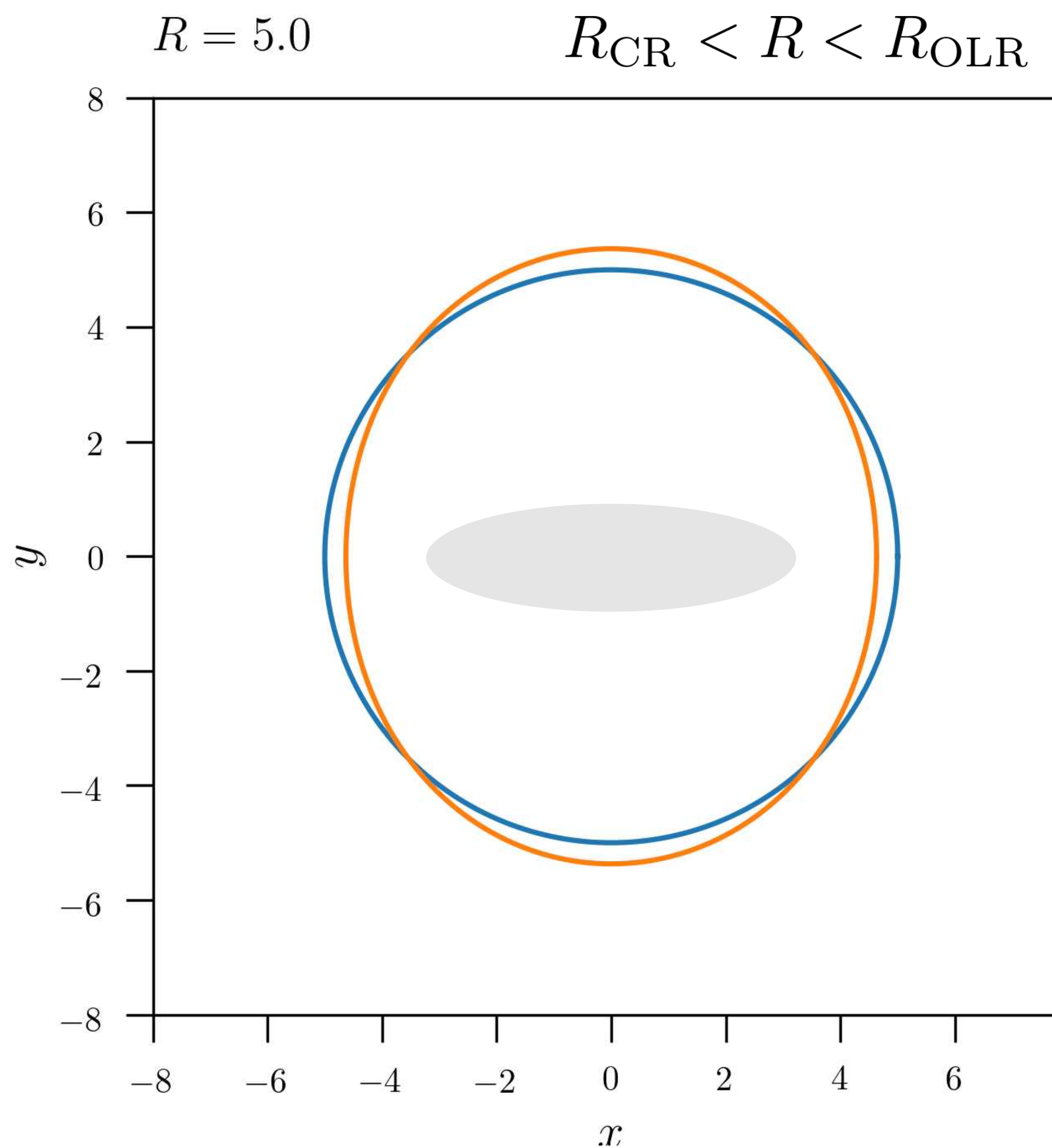


+ closed orbits
around L4, L5

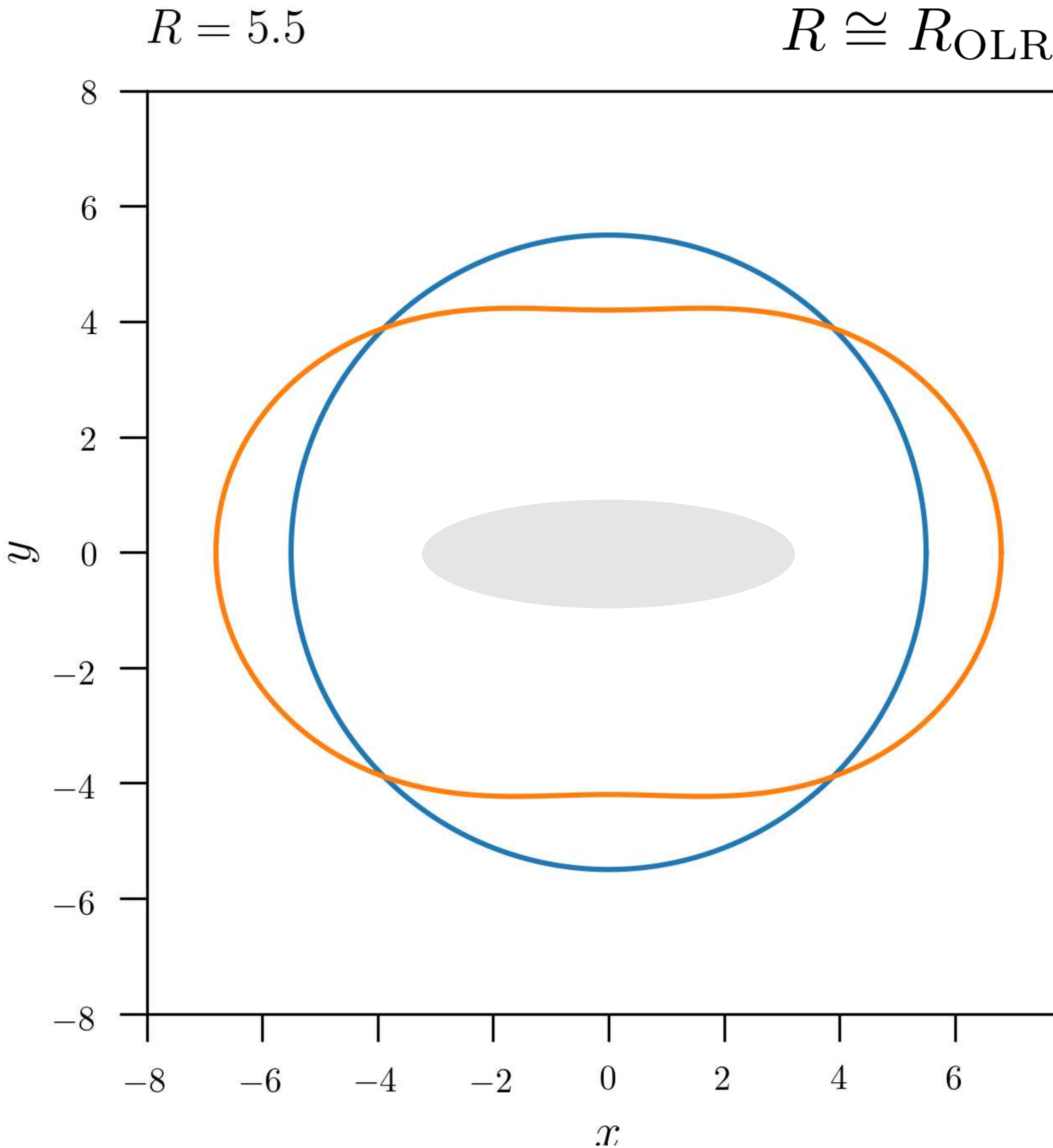

$R = 4.0$
$R_{\mathrm{CR}} < R < R_{\mathrm{OLR}}$
y
x
8
6
4
2
0
−2
−4
−6
−8

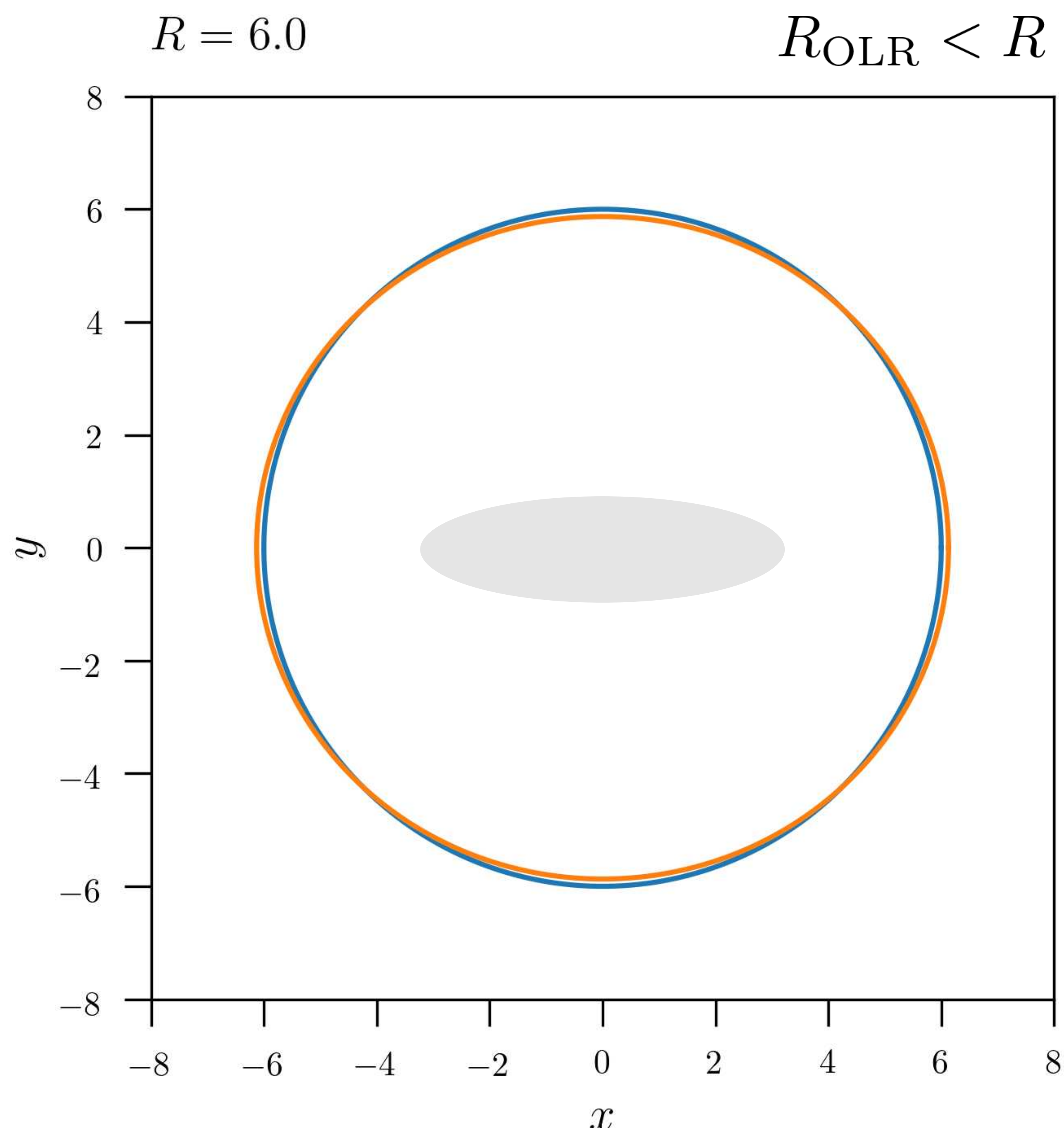
$R = 4.5$ $R_{\mathrm{CR}} < R < R_{\mathrm{OLR}}$

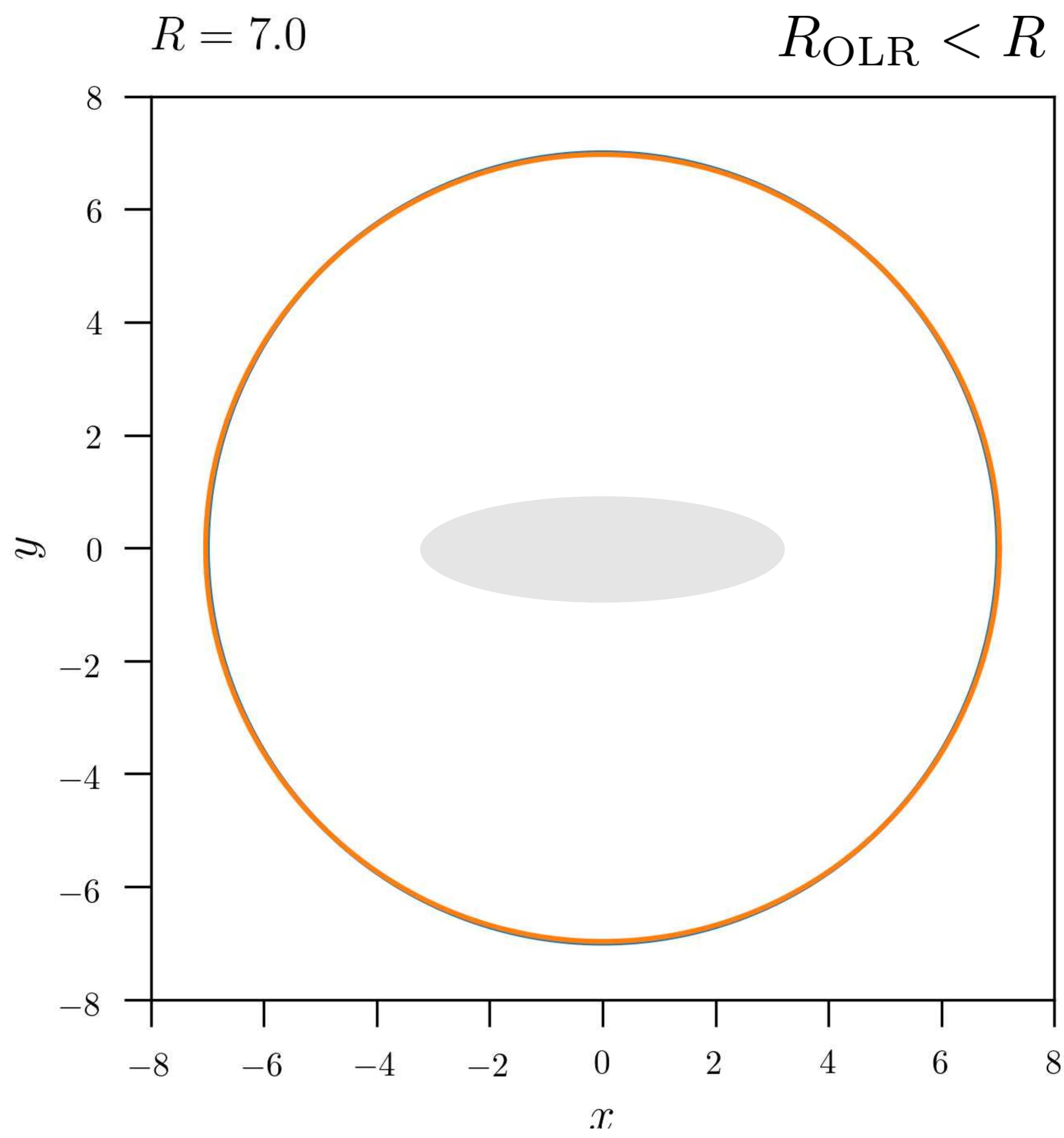




$R = 5.0$
$R_{\rm CR} < R < R_{\rm OLR}$
y
x
8
6
4
2
0
−2
−4
−6
−8


$R = 5.5$
$R \cong R_{\mathrm{OLR}}$
y
x
8
6
4
2
0
−2
−4
−6
−8

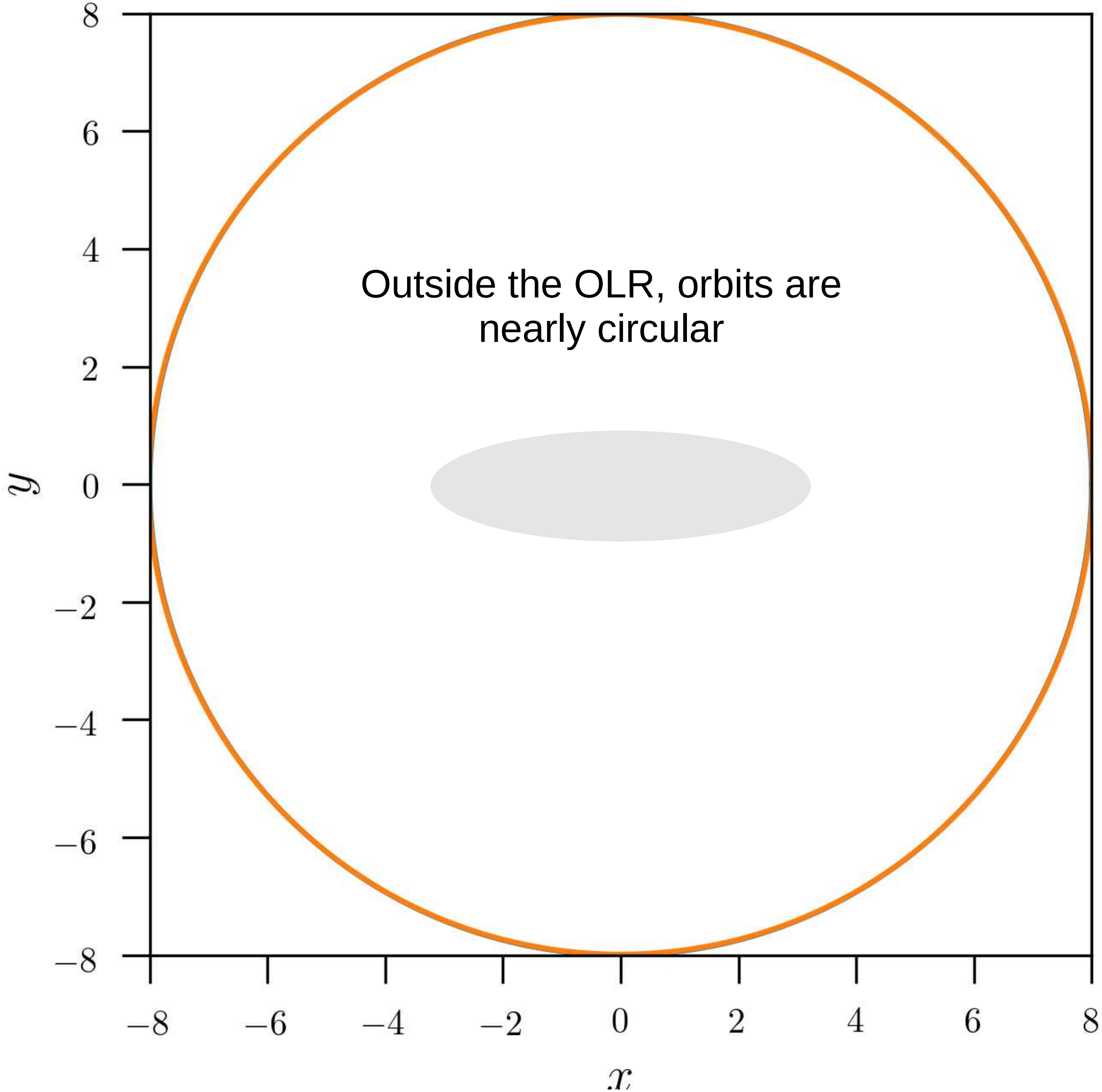

$R = 6.0$
$R_{\mathrm{OLR}} < R$
8
6
4
2
0
−2
−4
−6
−8
y
−8
−6
−4
−2
0
2
4
6
8
x

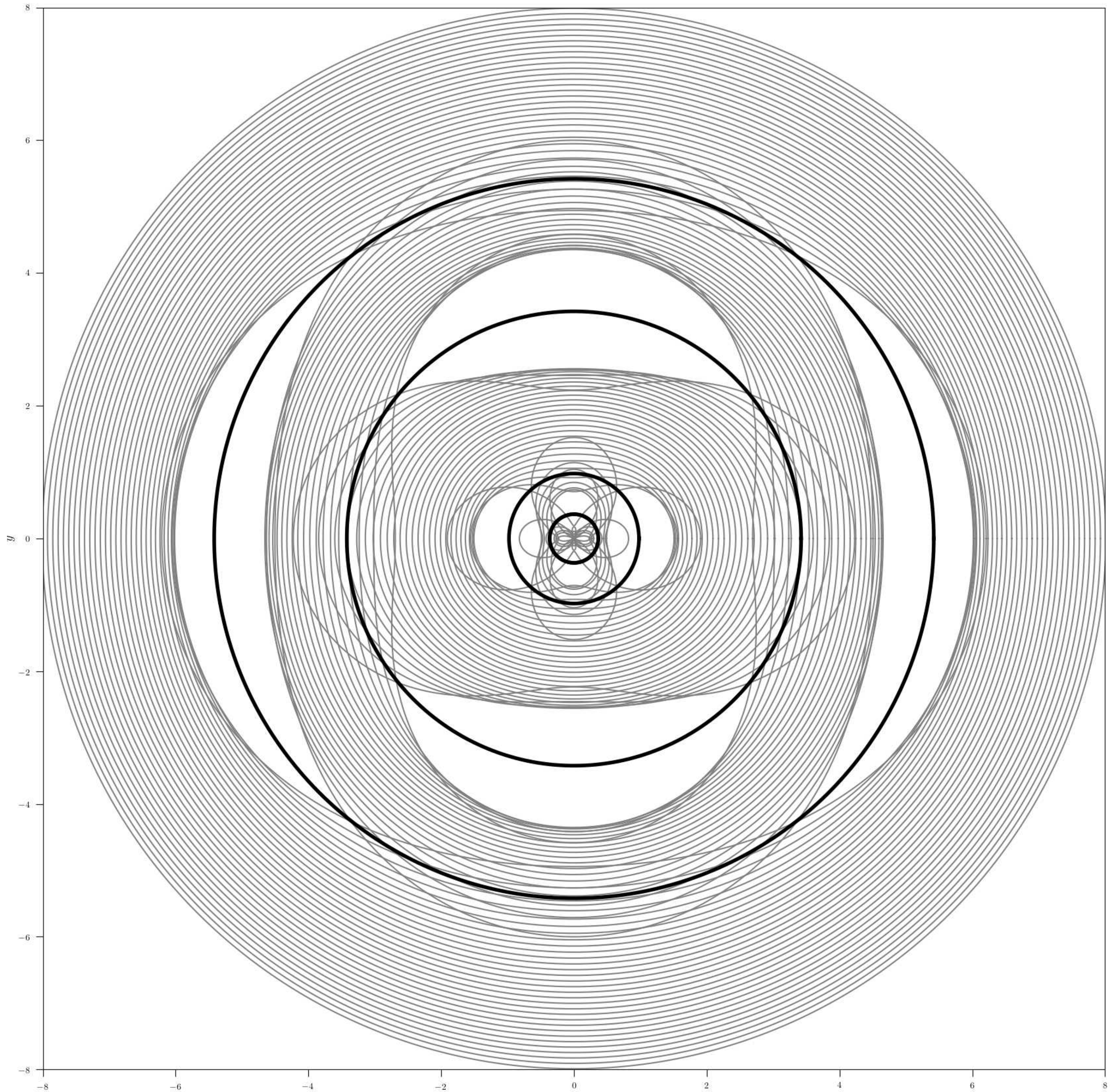


$R = 7.0$
$R_{\mathrm{OLR}} < R$
y
x
8
6
4
2
0
−2
−4
−6
−8

$R = 8.0$

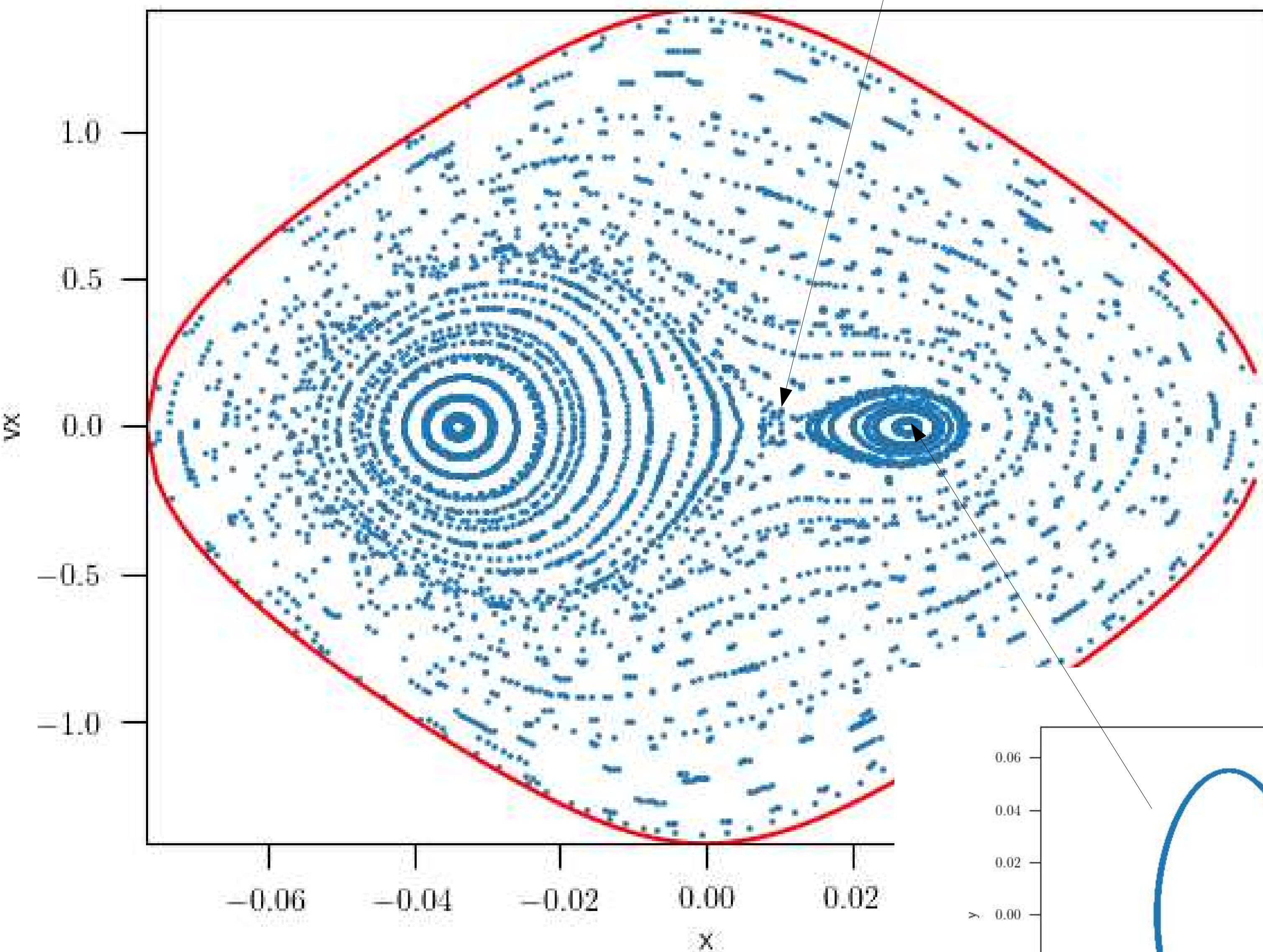
$R_{\mathrm{OLR}} < R$

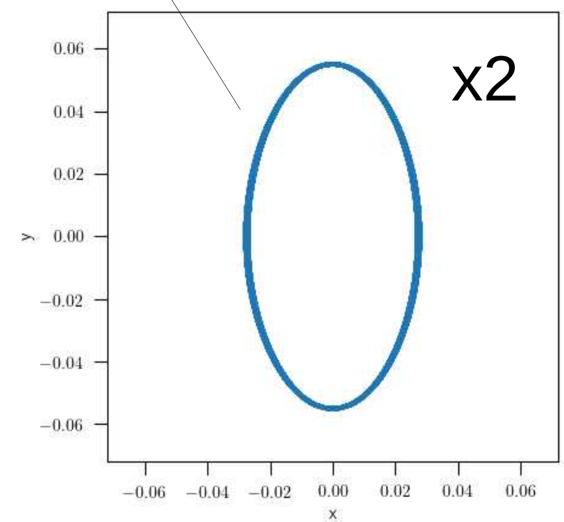




8
6
4
2
0
−2
−4
−6
−8
y
−8
−6
−4
−2
0
2
4
6
8

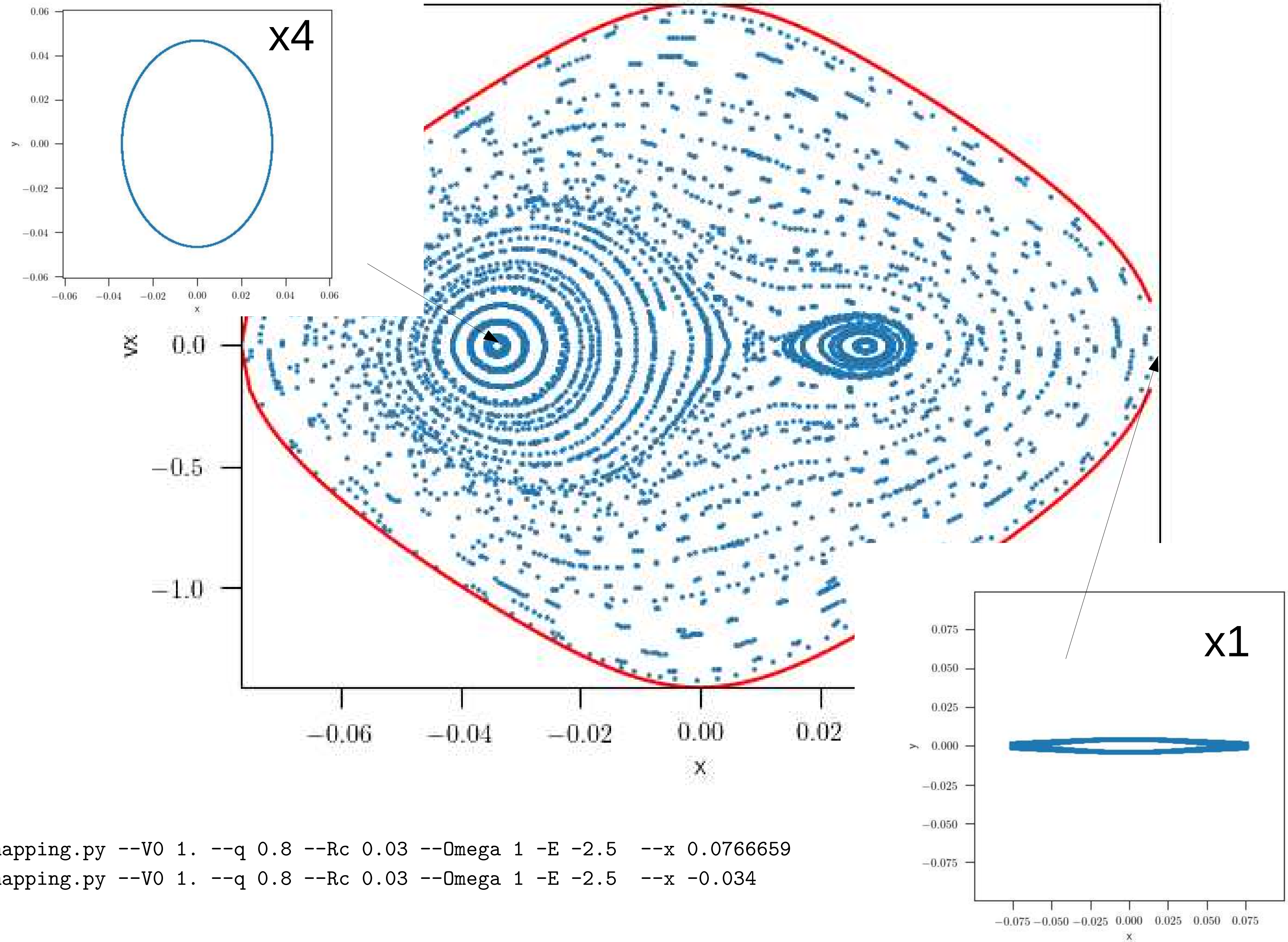
Bifurcation : apparition of x2 (stabe)/x3 (unstable) orbits





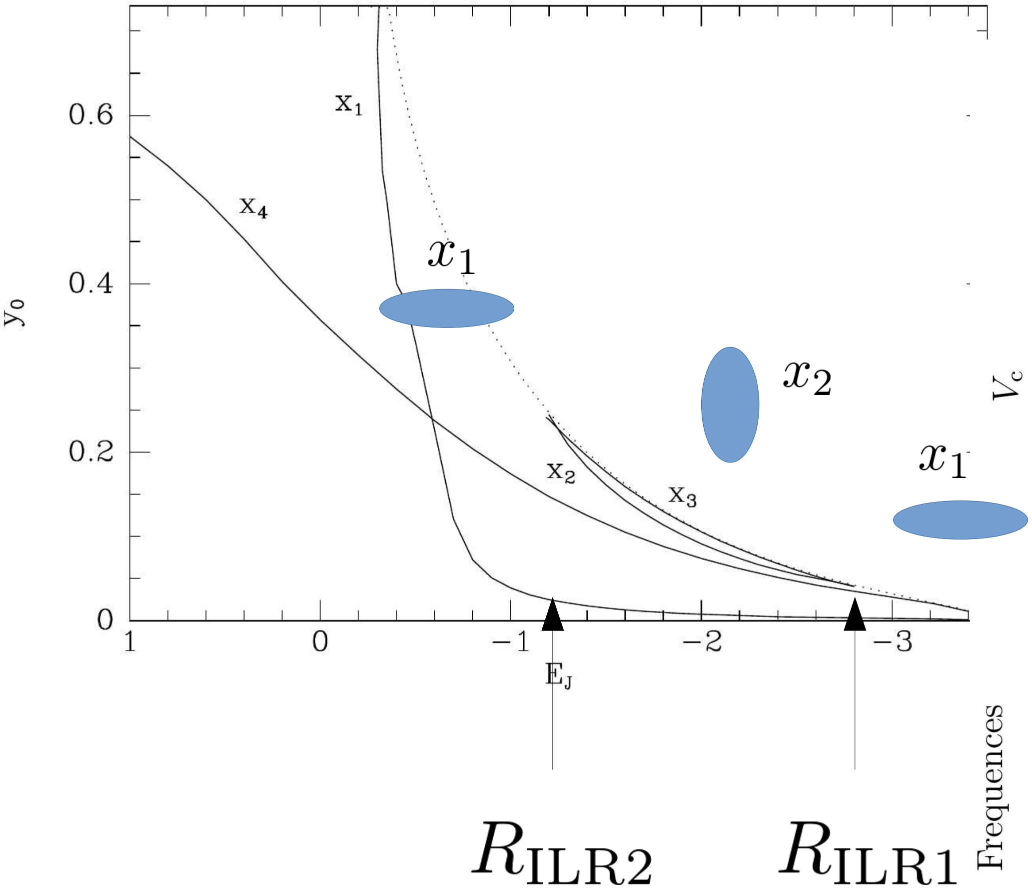

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0268
```

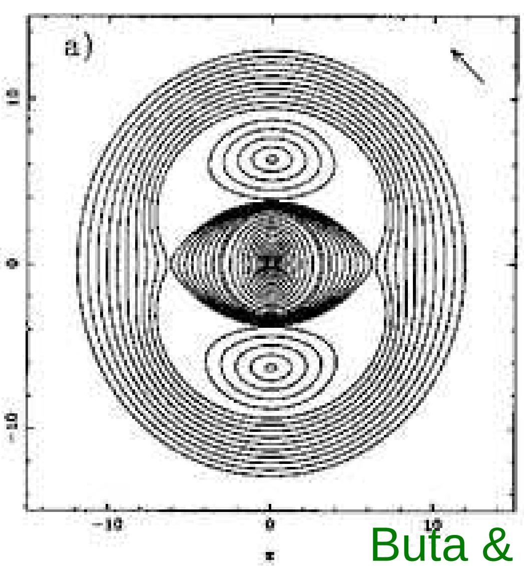
x1 : prograde x4 : retrograde

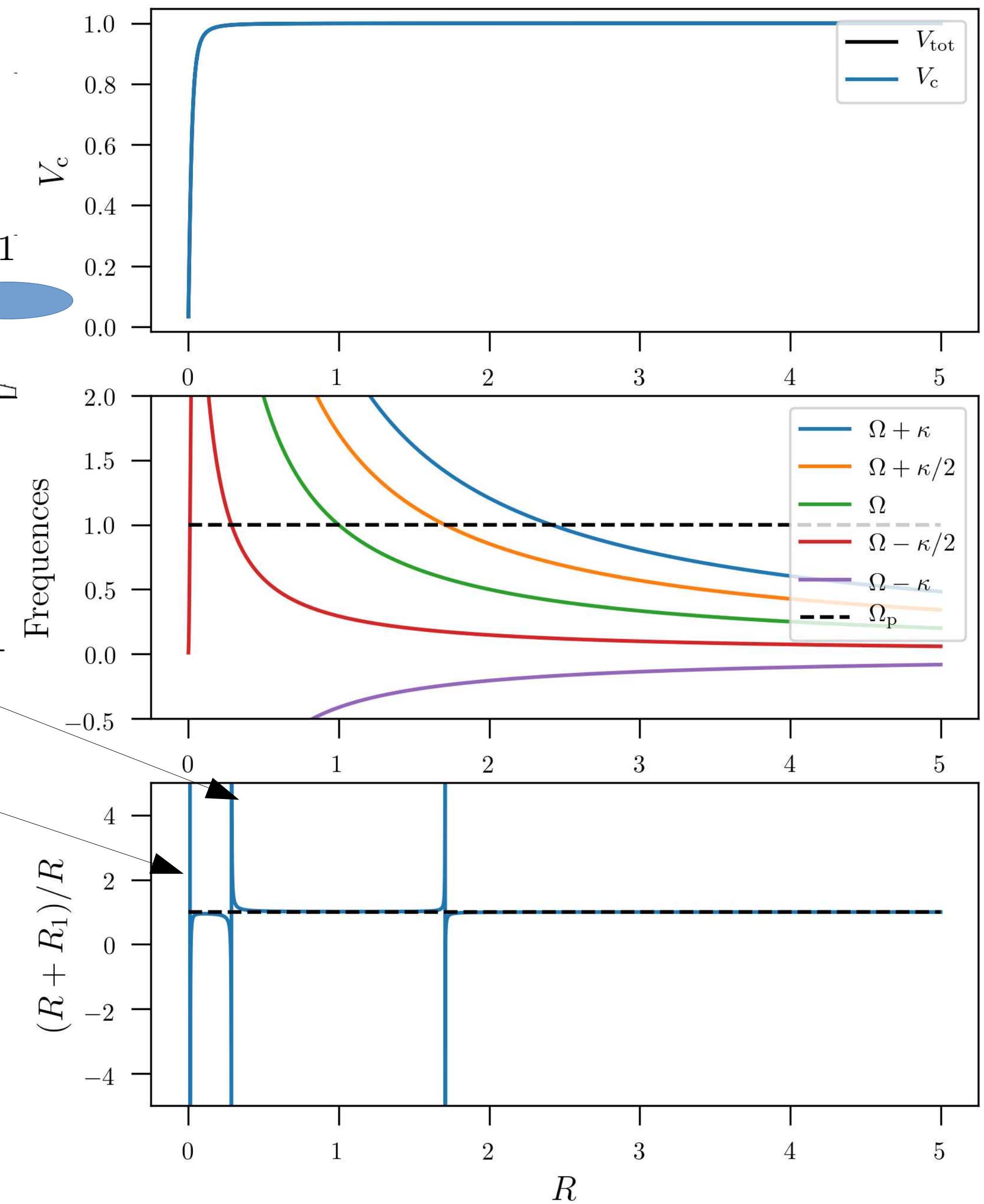



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x -0.034
```

Lindblad frequencies for the Logarithmic potential







Buta & Combes 1998

The End