Solution 8
Quantum Information Processing

Exercise 1 *Corrupted dense coding*

(a) If the message of Alice is (00) she sends her particle to Bob who measures the global state in the usual Bell basis. The probability that he gets the perfect Bell state is :

$$P(00) = |\langle B_{00}|\Psi\rangle|^2 = \frac{1}{1+\delta^2}$$

since $\langle B_{00}|$ is orthogonal to $|01\rangle$. We also have

$$P(01) = |\langle B_{01}|\Psi\rangle|^2 = \frac{\delta^2}{2(1+\delta^2)}$$

$$P(10) = |\langle B_{10}|\Psi\rangle|^2 = \frac{\delta^2}{2(1+\delta^2)}$$

$$P(11) = |\langle B_{11}|\Psi\rangle|^2 = 0$$

Remark : the 4 probabilities sum to 1.

(b) If the message of Alice is (10) she applies iY on her particle and the global state received by Bob is :

$$|\Psi'\rangle = -(1+\delta^2)^{-1/2}(|B_{10}\rangle - \delta e^{i\gamma}|11\rangle))$$

The 4 probabilities become :

$$P(00) = |\langle B_{00}|\Psi'\rangle|^2 = \frac{\delta^2}{2(1+\delta^2)}$$

$$P(01) = |\langle B_{01}|\Psi'\rangle|^2 = 0$$

$$P(10) = |\langle B_{10}|\Psi'\rangle|^2 = \frac{1}{(1+\delta^2)}$$

$$P(11) = |\langle B_{11}|\Psi\rangle|^2 = \frac{\delta^2}{2(1+\delta^2)}$$

Exercise 2 *Corrupted teleportation*

In the standard teleportation protocol Alice first performs a Bell basis measurement. We label the particles of Alice with $1, 2$ and the one of Bob with 3. The global state prior to the measurement is thus (up to a normalization factor)

$$(\alpha|0\rangle_1 + \beta|1\rangle_1) \otimes \frac{1}{(1+\delta^2)^{1/2}}(|B_{00}\rangle_{23} + \delta e^{i\gamma}|01\rangle_{23})$$

The state after the measurement collapses to the state (up to a normalization factor)

$$\frac{1}{(1+\delta^2)^{1/2}}|B_{ij}\rangle_{12}\langle B_{ij}|_{12} \otimes I_3(\alpha|0\rangle_1 + \beta|1\rangle_1) \otimes (|B_{00}\rangle_{23} + \delta e^{i\gamma}|01\rangle_{23})$$

for $ij = 00, 01, 10, 11$.

Suppose Alice gets : $ij = 00$, that is $|B_{00}\rangle_{12} = \frac{1}{\sqrt{2}}(|00\rangle_{12} + |11\rangle_{12})$. Then Bob gets a state proportional to

$$\frac{1}{(1+\delta^2)^{1/2}}\frac{1}{\sqrt{2}}(\langle 00|_{12} + \langle 11|_{12}) \otimes I(\alpha|0\rangle_1 + \beta|1\rangle_1) \otimes (\frac{1}{\sqrt{2}}|00\rangle_{23} + \frac{1}{\sqrt{2}}|11\rangle_{23} + \delta e^{i\gamma}|01\rangle_{23})$$
$$= \frac{1}{(1+\delta^2)^{1/2}}(\frac{\alpha}{2}|0\rangle_3 + \frac{\alpha}{\sqrt{2}}\delta e^{i\gamma}|1\rangle_3 + \frac{\beta}{2}|1\rangle_3) \quad (1)$$

The probability is $\frac{1}{4(1+\delta^2)}(|\alpha|^2 + |\beta + \sqrt{2}\alpha\delta e^{i\gamma}|^2)$. Alice sends the classical message 00 to Bob who then "knows" that he should not perform any operation on his state (or "apply" I_3). Of course for $\delta \to 0$ this state tends to the ideal state $\alpha|0\rangle_3 + \beta|1\rangle_3$.

Remark : To compute the probability one can use the formula $\langle\phi|P|\phi\rangle = \langle\phi|P^2|\phi\rangle = \|P\phi\|^2$ where $|\phi\rangle$ is the state prior to measurement and P the projector.

Suppose Alice gets : $|B_{01}\rangle_{12} = \frac{1}{\sqrt{2}}(|01\rangle_{12} + |10\rangle_{12})$. Then Bob gets a state proportional to

$$\frac{1}{(1+\delta^2)^{1/2}}\frac{1}{\sqrt{2}}(\langle 01|_{12} + \langle 10|_{12}) \otimes I(\alpha|0\rangle_1 + \beta|1\rangle_1) \otimes (\frac{1}{\sqrt{2}}|00\rangle_{23} + \frac{1}{\sqrt{2}}|11\rangle_{23} + \delta e^{i\gamma}|01\rangle_{23})$$
$$= \frac{1}{(1+\delta^2)^{1/2}}(\frac{\alpha}{2}|1\rangle_3 + \frac{\beta}{\sqrt{2}}\delta e^{i\gamma}|1\rangle + \frac{\beta}{2}|0\rangle_3) \quad (2)$$

The probability is $\frac{1}{4(1+\delta^2)}(|\beta|^2 + |\alpha + \sqrt{2}\beta\delta e^{i\gamma}|^2)$. Alice sends the classical message 01 to Bob who applies then knows he must apply the operation X_3 to the state obtaining :

$$\frac{1}{(1+\delta^2)^{1/2}}(\frac{\alpha}{2}|0\rangle_3 + \frac{\beta}{\sqrt{2}}\delta e^{i\gamma}|0\rangle + \frac{\beta}{2}|1\rangle_3)$$

Again, this state tends to the ideal state for $\delta \to 0$.

Suppose Alice gets : $|B_{10}\rangle_{12} = \frac{1}{\sqrt{2}}(|01\rangle_{12} - |10\rangle_{12})$. Then Bob gets a state proportional to :

$$\frac{1}{(1+\delta^2)^{1/2}}\frac{1}{\sqrt{2}}(\langle 01|_{12} - \langle 10|_{12}) \otimes I(\alpha|0\rangle_1 + \beta|1\rangle_1) \otimes (\frac{1}{\sqrt{2}}|00\rangle_{23} + \frac{1}{\sqrt{2}}|11\rangle_{23} + \delta e^{i\gamma}|01\rangle_{23})$$
$$= \frac{1}{(1+\delta^2)^{1/2}}(\frac{\alpha}{2}|1\rangle_3 - \frac{\beta}{\sqrt{2}}\delta e^{i\gamma}|1\rangle - \frac{\beta}{2}|0\rangle_3) \quad (3)$$

The probability is $\frac{1}{4(1+\delta^2)}(|\beta|^2 + |\alpha - \sqrt{2}\beta\delta e^{i\gamma}|^2)$. Alice sends the classical message 10 to Bob who applies then knows he must apply the operation Z_3X_3 to the state obtaining :

$$\frac{1}{(1+\delta^2)^{1/2}}(\frac{\alpha}{2}|0\rangle_3 - \frac{\beta}{\sqrt{2}}\delta e^{i\gamma}|0\rangle + \frac{\beta}{2}|1\rangle_3)$$

Again, this state tends to the ideal state for $\delta \to 0$.

Suppose Alice gets : $|B_{00}\rangle_{12} = \frac{1}{\sqrt{2}}(|00\rangle_{12} - |11\rangle_{12})$. Then Bob gets a state proportional to

$$\frac{1}{(1+\delta^2)^{1/2}}\frac{1}{\sqrt{2}}(\langle 00|_{12} - \langle 11|_{12}) \otimes I(\alpha|0\rangle_1 + \beta|1\rangle_1) \otimes (\frac{1}{\sqrt{2}}|00\rangle_{23} + \frac{1}{\sqrt{2}}|11\rangle_{23} + \delta e^{i\gamma}|01\rangle_{23})$$
$$= \frac{1}{(1+\delta^2)^{1/2}}(\frac{\alpha}{2}|0\rangle_3 + \frac{\alpha}{\sqrt{2}}\delta e^{i\gamma}|1\rangle_3 - \frac{\beta}{2}|1\rangle_3) \quad (4)$$

The probability is $\frac{1}{4(1+\delta^2)}(|\alpha|^2 + |\beta - \sqrt{2}\alpha\delta e^{i\gamma}|^2)$. Alice sends the classical message 11 to Bob who applies then knows he must apply the operation Z_3 to the state obtaining :

$$\frac{1}{(1+\delta^2)^{1/2}}(\frac{\alpha}{2}|0\rangle_3 - \frac{\alpha}{\sqrt{2}}\delta e^{i\gamma}|1\rangle + \frac{\beta}{2}|1\rangle_3)$$

Again, this state tends to the ideal state for $\delta \to 0$.

Remark : We can check that the four probabilities sum to 1.

Exercise 3 *A criterion for entanglement of two qubits*

(a) For a product state we have $|\Psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$ hence $a_{00} = \alpha\gamma$, $a_{01} = \alpha\delta$, $a_{10} = \beta\gamma$, $a_{11} = \beta\delta$. Thus $\det A = a_{00}a_{11} - a_{01}a_{10} = \alpha\gamma\beta\delta - \alpha\delta\beta\gamma = 0$.

Conversely suppose $\det A = a_{00}a_{11} - a_{01}a_{10} = 0$. Suppose first $a_{00} = 0$. Then we must also have $a_{01} = 0$ or $a_{10} = 0$. In the first case $|\Psi\rangle = a_{11}|11\rangle + a_{10}|10\rangle = |1\rangle \otimes (a_{11}|1\rangle + a_{10}|0\rangle)$ which is product state. In the second case we proceed similarly. Suppose now that $a_{00} \neq 0$. Then we must have $a_{11} = -\frac{a_{01}a_{10}}{a_{00}}$ and

$$|\Psi\rangle = a_{00}\big(|00\rangle + \frac{a_{01}}{a_{00}}|01\rangle + \frac{a_{10}}{a_{00}}|10\rangle - \frac{a_{01}a_{10}}{a_{00}^2}|11\rangle\big)$$
$$= a_{00}\big(|0\rangle + \frac{a_{10}}{a_{00}}|1\rangle\big) \otimes \big(|0\rangle + \frac{a_{01}}{a_{00}}|1\rangle\big)$$

which is a product state.

(b) According to the criterion the state is a *product state* if and only if

$$1 - \delta\epsilon e^{i\gamma} = 0$$

In other words

$$1 = \delta\epsilon\cos\gamma \quad \textit{and} \quad \delta\epsilon\sin\gamma = 0$$

If $\gamma = 0, \pi, 2\pi$ this is the case when $\delta\epsilon = 1$. If on the other hand $\gamma \neq 0, \pi, 2\pi$ it is impossible to satisfy both equalities.

To summarize the state is product when $\gamma = 0, \pi, 2\pi$ and $\delta\epsilon = 1$. Otherwise it is entangled.

Exercise 4 $|W\rangle$ *state*

(a) We proceed by contradiction. We write $|\psi_i\rangle = \alpha_i|1\rangle + \beta_i|0\rangle$ and expand the tensor product which gives a sum of 8 terms. Equating with $|W\rangle$ we find contradictions. For example

$$\alpha_1\beta_2\beta_3 = \frac{1}{\sqrt{3}}, \quad \beta_1\alpha_2\beta_3 = \frac{1}{\sqrt{3}}, \quad \beta_1\beta_2\alpha_3 = \frac{1}{\sqrt{3}}$$

but also

$$\alpha_1\alpha_2\alpha_3 = 0$$

The last equality implies at least one $\alpha_i = 0$ which is impossible because of the above equalities.

(b) Similarly if $|\psi_{23}\rangle = \gamma_{00}|00\rangle + \gamma_{01}|01\rangle + \gamma_{10}|10\rangle + \gamma_{11}|11\rangle$ et $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$ we obtain (for example)

$$\alpha_1\gamma_{00} = 0, \quad \alpha_1\gamma_{01} = \frac{1}{\sqrt{3}}$$

which imples $\gamma_{00} = 0$, but on the other hand we must also have

$$\beta_1\gamma_{00} = \frac{1}{\sqrt{3}}$$

hence a contradiction.

Exercise 5 *Entanglement swapping*

(a) The projector is

$$P = |GHZ\rangle_{135}\langle GHZ|_{135} \otimes I_2 \otimes I_4 \otimes_6$$

since particles with an even index are not observed (measured).

(b) After the measurement the global state is

$$P|\Psi\rangle$$

which gives for the state of particles 246 (after normalization)

$$|GHZ\rangle_{246}$$

Indeed the projector imposes that 135 are in states 000 or 111. But since $|Psi\rangle$ is a product of $|B_{00}\rangle$'s particles 246 must be in the same states as 135.