

# Astrophysics III: Stellar and galactic dynamics

## Solutions

**Problem 1:**

For a given potential in cylindrical coordinates  $\Phi(R, \theta, z)$ , the vertical and epicyclic frequencies read:

$$\Omega^2(R_g) = \frac{1}{R} \left( \frac{\partial \Phi}{\partial R} \right)_{(R_g, z=0)} \quad (1)$$

$$\nu^2(R_g) = \left( \frac{\partial^2 \Phi}{\partial z^2} \right)_{(R_g, z=0)} \quad (2)$$

$$\kappa^2(R_g) = \left( \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \right)_{(R_g, z=0)} = \left( R \frac{\partial(\Omega^2)}{\partial R} + 4\Omega^2 \right)_{(R_g, z=0)} \quad (3)$$

For the point mass, one obtains (dropping the  $g$  subscript for short):

$$\Omega^2(R) = \frac{GM}{R^3} \quad (4)$$

$$\nu^2(R) = \frac{\partial}{\partial z} \left( \frac{\partial \Phi}{\partial z} \right)_{(R, z=0)} = \frac{\partial}{\partial z} \left( \frac{GMz}{(x^2 + y^2 + z^2)^{3/2}} \right)_{(R, z=0)} \quad (5)$$

$$= \frac{GM}{R^3} - 3 \left( \frac{GMz^2}{R^5} \right)_{z=0} = \frac{GM}{R^3} \quad (6)$$

$$\kappa^2(R) = \left[ \frac{\partial^2 \Phi}{\partial R^2} - \frac{1}{R} \frac{\partial \Phi}{\partial R} + \frac{4}{R} \frac{\partial \Phi}{\partial R} \right]_{(R, z=0)} = \frac{GM}{R^3} \quad (7)$$

For the Plummer-Schuster potential:

$$\Omega^2(R) = \frac{GM}{(e^2 + R^2)^{3/2}} \quad (8)$$

$$\nu^2(R) = \frac{GM}{(e^2 + R^2)^{3/2}} \quad (9)$$

$$\kappa^2(R) = 4 \frac{GM}{(e^2 + R^2)^{3/2}} - 3R^2 \frac{GM}{(e^2 + R^2)^{5/2}} \quad (10)$$

For the Miyamoto-Nagai potential:

$$\Omega^2(R) = \frac{GM}{(R^2 + (a+b)^2)^{3/2}} \quad (11)$$

$$\nu^2(R) = \frac{GM(a+b)}{b(R^2 + (a+b)^2)^{3/2}} \quad (12)$$

$$\kappa^2(R) = 4\frac{GM}{(R^2 + (a+b)^2)^{3/2}} - 3R^2\frac{GM}{(R^2 + (a+b)^2)^{5/2}} \quad (13)$$

### **Problem 2:**

The code to implement is given bellow:

#### **For pm.py:**

```
def Potential(x,y,z):
    R2 = x**2+y**2
    R = sqrt(R2)
    return -GM/R
```

```
def Nu2(r):
    return GM/r**3
```

```
def Omega2(r):
    return GM/r**3
```

```
def Kappa2(r):
    return GM/r**3
```

#### **For plummer.py:**

```
def Potential(x,y,z):
    R2 = x**2+y**2
    R = sqrt(R2)
    return -GM/sqrt( R2 + e**2 )
```

```
def Nu2(r):
    return GM/( e**2 + r**2 )**(3./2.)
```

```
def Omega2(r):
    return GM/( e**2 + r**2 )**(3./2.)
```

```
def Kappa2(r):
    return 4* GM/( e**2 + r**2 )**(3./2.) - 3* r**2 * GM/( e**2 + r**2 )
    *(5./2.)
```

#### **For miyamoto.py:**

```
def Potential(x,y,z):
    R2 = x**2+y**2
    R = sqrt(R2)
    return -GM/sqrt( R2 + ( a + sqrt(z**2+b**2) )**2 )
```

```
def Nu2(r):
    ab = a+b
```

```

ab2 = ab**2
return GM*ab / ( b* ( r**2 + ab2)**(3./2.) )

```

```

def Omega2(r):
    ab = a+b
    ab2 = ab**2
    return GM / ( r**2 + ab2 )**(3./2.)

```

```

def Kappa2(r):
    ab = a+b
    ab2 = ab**2
    return 4*GM / ( r**2 + ab2 )**(3./2.) - 3* r**2 * GM/( r**2 + ab2 )
    *(5./2.)

```

### Problem 3:

Using the definition

$$v_c = R\Omega(R)$$

it follows that

$$\frac{d\Omega}{dR} = \frac{1}{R} \frac{dv_c}{dR} - v_c \frac{1}{R^2}$$

Then

$$A(R) \equiv \frac{1}{2} \left( \frac{v_c}{R} - \frac{dv_c}{dR} \right) = \frac{1}{2} \left( -R \left( \frac{1}{R} \frac{dv_c}{dR} - \frac{v_c}{R^2} \right) \right) = -\frac{1}{2} R \frac{d\Omega}{dR}$$

$$B(R) \equiv -\frac{1}{2} \left( \frac{v_c}{R} + \frac{dv_c}{dR} \right) = -\frac{1}{2} \left( \Omega + \left( \frac{1}{R} \frac{dv_c}{dR} - \frac{v_c}{R^2} \right) + \frac{v_c}{R} \right) = -\left( \Omega + \frac{1}{2} R \frac{d\Omega}{dR} \right)$$

$$\Omega = A - B = \frac{1}{2} \left( \frac{v_c}{R} - \frac{dv_c}{dR} \right) + \frac{1}{2} \left( \frac{v_c}{R} + \frac{dv_c}{dR} \right) = \frac{v_c}{R} = \Omega$$

$$\begin{aligned} \kappa^2 &= \left( R \frac{d(\Omega^2)}{dR} + 4\Omega^2 \right) = \left( 2R\Omega \frac{d\Omega}{dR} + 4\Omega^2 \right) = 2\Omega \left( R \frac{d\Omega}{dR} + 2\Omega \right) \\ &= 2\Omega (-2B) = -4B(A - B) \end{aligned}$$

### Problem 4:

We have:

$$R^2 = x^2 + y^2$$

$$\vec{L} = \vec{r} \times \vec{v} = \vec{x} \times \dot{\vec{x}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{pmatrix}$$

Since we're working in the  $z = 0$  plane, and the  $z$  component of  $\vec{L}$  is given by  $L_z = xy - yx$ , inserting this to compute  $L^2$  gives

$$L^2 = (yz - zy)^2 + (zx - xz)^2 + (xy - yx)^2 = (x^2 + y^2)z^2 + L_z^2 = R^2\dot{z}^2 + L_z^2$$

Now we use the energy conservation:

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{eff}(R, z)$$

Now eliminate  $\dot{z}^2$  by using our expression for  $L^2$ :

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\frac{1}{R^2}(L^2 - L_z^2) + \Phi_{eff}(R, z)$$

Solving for  $\dot{R}$  gives and using  $\Phi_{eff} = \frac{1}{2}\frac{L_z^2}{R^2} + \Phi$ :

$$\dot{R} = \pm\sqrt{2\left(E - \frac{1}{2}\frac{L^2 - L_z^2}{R^2} - \Phi_{eff}\right)} = \pm\sqrt{2\left(E - \frac{L^2}{2R^2} - \Phi\right)}$$