Homework 8
Quantum Information Processing

Exercise 1 *Corrupted dense coding*

Alice and Bob share an entangled pair in the state :

$$|\Psi\rangle = (1+\delta^2)^{-1/2}\Big\{|B_{00}\rangle + \delta e^{i\gamma}|01\rangle\Big\}$$

where $\delta \in \mathbb{R}_+$ and $\gamma \in [0, 2\pi]$. Here $|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is the perfect Bell state and we view the additional term proportional to δ as a "corruption". Alice wants to transmit two classical bits of information to Bob. They do not know that the entangled pair is corrupted and they use the *usual dense coding protocol* (seen in class).

(a) Suppose Alice sends message (00). Calculate the probabilities that at the end of the protocol Bob gets $P(00)$, $P(01)$, $P(10)$, $P(11)$.

(b) Suppose Alice sends message (10). Calculate first the global state received by Bob and the 4 probabilities above.

Exercise 2 *Corrupted teleportation*

Consider the same state as above

$$|\Psi\rangle = (1+\delta^2)^{-1/2}\Big\{|B_{00}\rangle + \delta e^{i\gamma}|01\rangle\Big\}$$

shared between Alice and Bob and suppose Alice has the extra state $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$. We want to analyze the teleportation protocol under the corrupted entanglement link.

(a) Alice does a measurement in the *perfect* Bell basis in her lab. Compute the possible outcomes for the global state shared by Alice and Bob and the respective probabilities.

(b) Describe the next steps of the protocol and explain what are the teleported states that Bob gets when the protocol is completed. In particular compare the teleported states with $|\varphi\rangle$.

Exercise 3 *An entanglement criterion for 2 qubits*

The general state of two qubits is of the form

$$|\Psi\rangle = a_{00}\,|0\rangle \otimes |0\rangle + a_{01}\,|0\rangle \otimes |1\rangle + a_{10}\,|1\rangle \otimes |0\rangle + a_{11}\,|1\rangle \otimes |1\rangle\,,$$

(a) Show that $|\Psi\rangle$ is a product state *if and only if* $\det A = 0$, where A is the matrix

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}.$$

(b) Use this criterion to determine when the state (with δ, ϵ reals and $\gamma \in [0, 2\pi]$)

$$|\Psi_1\rangle = \frac{1}{\sqrt{1+\delta^2+\epsilon^2}}\big(|B_{00}\rangle + \delta e^{i\gamma}\,|1\rangle \otimes |0\rangle + \epsilon\,|0\rangle \otimes |1\rangle\big)$$

is entangled.

Exercise 4 $|W\rangle$ *state*

Consider the following state called W in quantum information,

$$|W\rangle = \frac{1}{\sqrt{3}}\big(|100\rangle + |010\rangle + |001\rangle\big)$$

Show that the three qubits are "totally" entangled in the sense :

(a) It is impossible to write the state as a product of three one-qubit states : $|W\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle$ avec $|\psi_i\rangle \in \mathbb{C}^2$.

(b) It is impossible to write the state as a product of a one-qubit state and a two-qubit state (which might itself be entangled) : $|W\rangle \neq |\psi_1\rangle \otimes |\psi_{23}\rangle$ with $|\psi_1\rangle \in \mathbb{C}^2$ et $|\psi_{23}\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$.

Exercise 5 *Entanglement swapping with 3 qubits*

Consider 6 quantum particles 1, 2, 3, 4, 5, 6. Pairs 12, 34, 56 are in the Bell state. Thus the state of the 6 particles is :

$$|\Psi\rangle = |B_{00}\rangle_{12} \otimes |B_{00}\rangle_{34} \otimes |B_{00}\rangle_{56}$$

We imagine that particles 1, 3, 5 are close in space (say on earth) and particles 2, 4, 6 are far away respectively on the moon, the space station and another satellite. We do a local measurement on earth which projects the state of 1, 3, 5 on the state $|GHZ\rangle_{135} = \frac{1}{\sqrt{2}}(|000\rangle_{135} + |111\rangle_{135})$.

(a) The resulting global state after the measurement is proportional to $P|\Psi\rangle$ for a certain projector P. Which is this projector ?

(b) Compute the resulting state of 2, 4, 6 after the measurement.