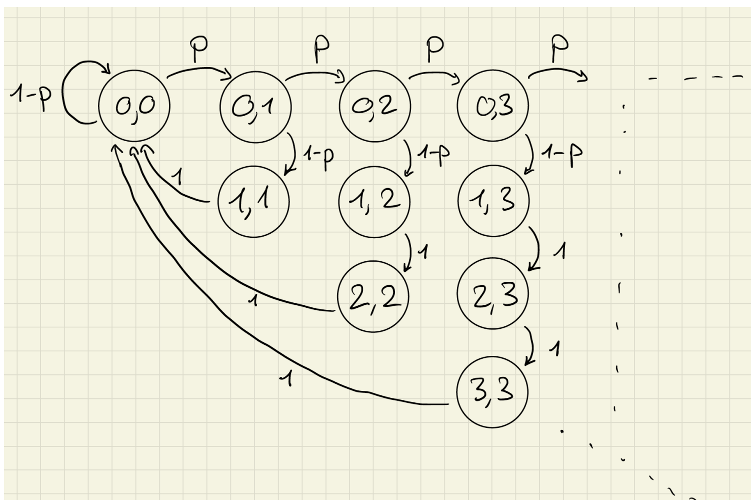
Midterm Exam

Exercise 1. (10 points)

Consider the Markov chain $(X_n,\, n \geq 0)$ with state space $S = \{(j,k),\, j,k \in \mathbb{N},\, j \leq k\}$ and transition probabilities given by

$$\begin{cases} p_{(0,0),(0,0)} = 1-p & p_{(0,k),(0,k+1)} = p \quad \text{for } k \geq 0 \\ p_{(0,k),(1,k)} = 1-p \quad \text{for } k \geq 1 & p_{(j,k),(j+1,k)} = 1 \quad \text{for } 1 \leq j \leq k \\ p_{(k,k),(0,0)} = 1 \quad \text{for } k \geq 1 & \end{cases}$$

where $0 < p < 1$. Here is the corresponding transition graph for the first states:



a) Compute $f_{(0,0),(0,0)}(n) = \mathbb{P}(T_{(0,0)} = n \,|\, X_0 = (0,0))$ for a generic value of $n \geq 1$.

NB: $T_{(0,0)} = \inf\{n \geq 1 : X_n = (0,0)\}$.

b) Prove that the chain X is recurrent.

c) Prove that the chain X is also positive-recurrent.

Hint: Compute $\mathbb{E}(T_{(0,0)} \,|\, X_0 = (0,0))$, using the identity: $\sum_{k \geq 0} k\, p^k = \frac{p}{(1-p)^2}$ valid for $0 < p < 1$.

d) Compute the stationary distribution π of the chain X.

e) Is π also a limiting distribution? Justify.

f) Does detailed balance hold? Justify.

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Exercise 2. (10 points)

Let $N \geq 5$ be an integer and consider two Markov chains $(X_n,\, n \geq 0)$, $(Y_n,\, n \geq 0)$ defined on the same state space $S = \{0, 1, 2, \ldots, N-1\}$ and with the same transition matrix $P = (p_{ij})_{i,j \in S}$, where

$$p_{ij} = \begin{cases} p & \text{if } j = i+1 \,(\text{mod } N) \\ q & \text{if } j = i-1 \,(\text{mod } N) \\ 0 & \text{otherwise} \end{cases}$$

and $p, q > 0$ are such that $p + q = 1$. Let also $(Z_n,\, n \geq 0)$ be the Markov chain on S defined as follows:

$$Z_n = Y_n - X_n \,(\text{mod } N) \quad \text{for } n \geq 0$$

a) Compute the transition matrix $Q = (q_{ij})_{i,j \in S}$ of the chain Z.

b) Compute the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$ of the matrix Q (for a generic value of N).

Hint: If $A = \text{circ}(c_0, c_1, \ldots, c_{N-1})$ is an $N \times N$ circulant matrix, i.e.

$$A = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-2} & c_{N-1} \\ c_{N-1} & c_0 & c_1 & & & c_{N-2} \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ c_2 & & & c_{N-1} & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{N-2} & c_{N-1} & c_0 \end{pmatrix}$$

then its eigenvalues are given by

$$\lambda_k = \sum_{j=0}^{N-1} c_j \, \exp(2\pi i j k / N) \quad k = 0, \ldots, N-1$$

(please note that with this notation, the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$ are not ordered.)

c) For what values of $N \geq 5$ is the Markov chain Z ergodic? For these values, compute the limiting and stationary distribution of the chain. Is detailed balance satisfied? *Justify all your answers.*

d) For the values of N found in part c), compute the spectral gap γ of the chain Z.

e) For large values of N (still satisfying the condition found in part c), how large should n be (approximately) in order to ensure that the distribution of Z_n is ε-close (in total variation distance) to the stationary distribution?

Hint: You may use the approximation $\cos(x) \simeq 1 - \frac{x^2}{2}$, valid for small values of x.

BONUS f) What is the average time between two encounters of the chains X and Y? Does it depend on the values of N, p and q? If yes, how?